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L^p BOUNDS FOR MARCINKIEWICZ INTEGRALS ALONG SURFACES AND EXTRAPOLATION

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Abstract: In this article, we study the L^p estimates of Marcinkiewicz integral operators when the kernels belong to $L^q\left(\mathbf{S}^{n-1}\right)$ for some $q > 1$. These estimates allow us to use an extrapolation argument to establish the L^p boundedness of Marcinkiewicz integrals when their kernels in $L(\log L)^{1/2}(\mathbf{S}^{n-1}) \cup B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ with $q > 1$. Our results are essential improvements and extensions of some known results on Marcinkiewicz integrals.

AMS Subject Classification: 40B20, 40B15, 40B25
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1. Introduction

Let $n \geq 2$ and $\mathbf{S}^{n-1}$ denote the unit sphere in $\mathbf{R}^n$ which is equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(\cdot)$. Also, let $x' = x/|x|$ for $x \in \mathbf{R}^n \setminus \{0\}$ and p' denote the exponent conjugate to p; that is $1/p + 1/p' = 1$.

For $\rho = a+ib$ ($a, b \in \mathbf{R}$ with $a > 0$), let $K_{\Omega,h}(u) = \Omega(u')h(|u|)\,|u|^{\rho-n}$, where $h : [0,\infty) \to \mathbf{C}$ is a measurable function and Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ with $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(u)d\sigma(u) = 0. \tag{1}$$

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Let $d \neq 0$ and $\mathcal{H}_d$ be the class of all functions $\phi : (0,\infty) \to \mathbf{R}$ which are smooth and satisfy the following growth conditions:

$$|\phi(t)| \leq C_1 t^d, \qquad |\phi''(t)| \leq C_2 t^{d-2}, \qquad C_3 t^{d-1} \leq |\phi'(t)| \leq C_4 t^{d-1} \tag{2}$$

for $t \in (0,\infty)$, where C_1, C_2, C_3 and C_4 are positive constants independent of t.

For $\phi \in \mathcal{H}_d$, h and Ω as above, we define the Marcinkiewicz integral operator $\mathcal{M}^{\rho}_{\Omega,\phi,h}$, initially for C_0^{∞} on $\mathbf{R}^n$, by

$$\mathcal{M}^{\rho}_{\Omega,\phi,h} f(x) = \left(\int_0^{\infty} \left| t^{-\rho} \int_{|u|\leq t} f(x - \phi(|u|)u') K_{\Omega,h}(u) du \right|^2 \frac{dt}{t} \right)^{1/2}.$$

If $\phi(t) = t$, we denote $\mathcal{M}^{\rho}_{\Omega,\phi,h}$ by $\mathcal{M}^{\rho}_{\Omega,h}$. The operators $\mathcal{M}^{\rho}_{\Omega,\phi,h}$ have their roots in the classical Marcinkiewicz integral operators which were introduced by Stein in [16]. When $h = 1$ and $\rho = 1$, Stein established the L^p $(1 < p \leq 2)$ boundedness of $\mathcal{M}^{\rho}_{\Omega,h}$ provided that $\Omega \in Lip_{\alpha}(\mathbf{S}^{n-1})$ with $0 < \alpha \leq 1$.

The Marcinkiewicz integral operators have attracted the attention of many authors for along time due to the powerful role they play in many significant problems arisings in mathematics such as Poisson integrals, singular integrals and singular Radon transforms.

For more information about the importance and the recent advances on the study of such operators, we refer the readers to [1], [2], [4], [6], [7], [8], [9], [10], [16], as well as [18], and the references therein.

The study of parametric Marcinkiewicz integral operator $\mathcal{M}^{\rho}_{\Omega,h}$ was initiated by Hörmander in [13] in which he satisfied the $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ boundedness of $\mathcal{M}^{\rho}_{\Omega,1}$ under the conditions $\rho > 0$ and $\Omega \in Lip_{\alpha}(\mathbf{S}^{n-1})$ with $\alpha > 0$. However, the authors of [14] showed that $\mathcal{M}^{\rho}_{\Omega,1}$ is still bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ when $Re(\rho) > 0$ and $\Omega \in Lip_{\alpha}(\mathbf{S}^{n-1})$ with $0 < \alpha \leq 1$. These results were improved in [11]. In fact, the authors proved that $\mathcal{M}^{\rho}_{\Omega,h}$ is of type $(2,2)$ when $\Omega \in L(\log L)(\mathbf{S}^{n-1})$ and $h \in \Delta_{\gamma}(\mathbf{R}^+, \frac{dt}{t})$, where $\Delta_{\gamma}(\mathbf{R}^+, \frac{dt}{t})$ is the collection of all measurable functions $h : [0,\infty) \to \mathbf{C}$ satisfying $\|h\|_{\Delta_{\gamma}(\mathbf{R}^+, \frac{dt}{t})} = \sup_{k\in\mathbf{Z}} \left(\int_{\theta^k}^{\theta^{k+1}} |h(t)|^{\gamma} \frac{dt}{t} \right)^{1/\gamma} < \infty$ for any $\theta \geq 2$.

On the other hand, Al-Qassem and Al-Salman obtained in [1] that $\mathcal{M}^{1}_{\Omega,1}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ if $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ with $q > 1$. Further, they established the optimality of the condition $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ in the sense that $-1/2$ in $B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ cannot be replaced by any smaller number. Subsequently, the study of the L^p boundedness of $\mathcal{M}^{1}_{\Omega,1}$ under various conditions on the kernels has received a large amount of attention of many authors. For example, Walsh in [18] found that $\mathcal{M}^1_{\Omega,1}$ is bounded on $L^2(\mathbf{R}^n)$ if $\Omega \in L(logL)^{1/2}(\mathbf{S}^{n-1})$. Moreover, he pointed out that the condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is optimal in the sense that the operator $\mathcal{M}^1_{\Omega,1}$ may lose the L^2 boundedness if Ω is assumed to be in the space $L(\log L)^{\varepsilon}(\mathbf{S}^{n-1})$ for some $\varepsilon < 1/2$. Later on, under the same above conditions, Al-Salman et al. in [4] improve the result of [18]. Precisely, he showed the same result for any $1 < p < \infty$.

Recently, it was proved in [5] that if $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $1 < \gamma \leq 2$, then $\mathcal{M}^1_{\Omega,\phi,h}$ is bounded on $L^p(\mathbf{R}^n)$ for any p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$. Very recently, Ali established in [6] that if $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1}) \cup B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in L^\gamma\left(\mathbf{R}^+, \frac{dt}{t}\right)$ for some $\gamma > 1$, then $\mathcal{M}^\rho_{\Omega,\varphi,h}$ is bounded on $L^p(\mathbf{R}^n)$ for any p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, where φ is a $\mathcal{C}^2([0,\infty))$, convex increasing function with $\varphi(0) = 0$.

In view of the results in [5] and [6], a question arises naturally. Does the L^p boundedness of the operators $\mathcal{M}^\rho_{\Omega,\phi,h}$ holds under the conditions $\phi \in \mathcal{H}_d$, Ω belongs to the space $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1}) \cup B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $\gamma > 1$. We shall obtain an answer to this question in the affirmative as described in the following theorems.

Theorem 1. *Let $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $\gamma > 1$, $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1 < q \leq 2$ and $\phi \in \mathcal{H}_d$ for some $d \neq 0$. Then for any $f \in L^p(\mathbf{R}^n)$ with p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, a constant C_p (independent of Ω, h, γ, and q) exists such that*

$$\left\|\mathcal{M}^\rho_{\Omega,\phi,h} f\right\|_{L^p(\mathbf{R}^n)} \leq C_p A(\gamma)(q-1)^{-1/2} \|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)},$$

where $A(\gamma) = \begin{cases} \gamma^{1/2} & \text{if } \gamma > 2, \\ (\gamma-1)^{-1/2} & \text{if } 1 < \gamma \leq 2. \end{cases}$

The conclusion from Theorem 1 and applying an extrapolation method as in [3] and [15] lead to the following theorem.

Theorem 2. *Suppose that $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $\gamma > 1$, Ω satisfies* (1) *and $\phi \in \mathcal{H}_d$ for some $d \neq 0$.*
(i) If $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$, then

$$\left\|\mathcal{M}^\rho_{\Omega,\phi,h} f\right\|_{L^p(\mathbf{R}^n)} \leq C_p A(\gamma) \|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})} \|f\|_{L^p(\mathbf{R}^n)} \left(1 + \|\Omega\|_{B_q^{(0,-1/2)}(\mathbf{S}^{n-1})}\right)$$

for $|1/p-1/2|<\min\{1/2,1/\gamma'\}$;
(ii) If $\Omega\in L(logL)^{1/2}(\mathbf{S}^{n-1})$, *then*

$$\left\|\mathcal{M}^{\rho}_{\Omega,\phi,h}f\right\|_{L^p(\mathbf{R}^n)}\le C_pA(\gamma)\left\|h\right\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\left\|f\right\|_{L^p(\mathbf{R}^n)}\left(1+\left\|\Omega\right\|_{L(logL)^{1/2}(\mathbf{S}^{n-1})}\right)$$

for $|1/p-1/2|<\min\{1/2,1/\gamma'\}$.

Here and henceforth, the letter C denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables

2. Preparation

In this section, we present some definitions and establish some lemmas used in the sequel. Let us start this section by introducing the following definition.

Definition 3. Let $\theta\ge 2$. For a suitable function ϕ defined on $\mathbf{R}^+$, a measurable function $h:\mathbf{R}^+\to\mathbf{C}$ and $\Omega:\mathbf{S}^{n-1}\to\mathbf{R}$, we define the family of measures $\{\sigma_{\Omega,\phi,h,t}:t\in\mathbf{R}^+\}$ and the corresponding maximal operators $\sigma^*_{\Omega,\phi,h}$ and $M_{\Omega,\phi,h,\theta}$ on $\mathbf{R}^n$ by

$$\begin{aligned}\int_{\mathbf{R}^n} f d\sigma_{\Omega,\phi,h,t} &= t^{-\rho}\int_{1/2t\le|u|\le t} f(\phi(|u|)u')h(|u|)\frac{\Omega(u')}{|u|^{n-\rho}}du,\\ \sigma^*_{\Omega,\phi,h}f(x) &= \sup_{t\in\mathbf{R}^+}||\sigma_{\Omega,\phi,h,t}|*f(x)|,\\ M_{\Omega,\phi,h,\theta}f(x) &= \sup_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}||\sigma_{\Omega,\phi,h,t}|*|f(x)|\frac{dt}{t},\end{aligned}$$

where $|\sigma_{\Omega,\phi,h,t}|$ is defined in the same way as $\sigma_{\Omega,\phi,h,t}$, but with replacing Ωh by $|\Omega h|$. We write $t^{\pm\alpha}=\inf\{t^{+\alpha},t^{-\alpha}\}$ and $\|\sigma\|$ for the total variation of σ.

In order to prove Theorem 1, it suffices to satisfy the following lemmas.

Lemma 4. *Let* $\theta=2^{q'\gamma'}$, $h\in\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})$ *for some* $\gamma>1$, $\Omega\in L^q\left(\mathbf{S}^{n-1}\right)$ *for some* $q>1$, *and* $\phi\in\mathcal{H}_d$ *for some* $d\neq 0$. *Then there exist constants* C *and* α *with* $0<2\alpha q'<1$ *such that*

$$\|\sigma_{\Omega,\phi,h,t}\|\le\ln 2; \tag{3}$$

$$\int_{\theta^k}^{\theta^{k+1}}\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right|^2\frac{dt}{t}\le C(\ln\theta)\left|\xi\theta^{kd}\right|^{\pm\frac{2\alpha}{q'}}\|h\|^2_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} \tag{4}$$

hold for all $k\in\mathbf{Z}$. *The constant* C *is independent of* k, ξ *and* ϕ.

Proof. We prove this lemma only for the case $d > 0$ because the proof for $d < 0$ is essentially the same and requires only minor modifications. Also, we prove this lemma for $1 < q \leq 2$, since $L^q\left(\mathbf{S}^{n-1}\right) \subseteq L^2\left(\mathbf{S}^{n-1}\right)$ for $q \geq 2$. We reach (3) directly by using the definition of $\sigma_{\Omega,\phi,h,t}$. By Hölder's inequality, we obtain that

$$\begin{aligned}
\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right| &\leq \int_{\frac{1}{2}t}^{t} |h(s)| \left| \int_{\mathbf{S}^{n-1}} e^{-i\phi(s)x\cdot\xi}\, \Omega(x)\, d\sigma(x) \right| \frac{ds}{s} \\
&\leq \|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})} \left(\int_{\frac{1}{2}t}^{t} \left| \int_{\mathbf{S}^{n-1}} e^{-i\phi(s)\xi\cdot x} \Omega(x)\, d\sigma(x) \right|^{\gamma'} \frac{ds}{s} \right)^{1/\gamma'}.
\end{aligned}$$

On one hand, if $1 < \gamma \leq 2$, then by a change of variable we get that

$$\left|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)\right| \leq \|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')} \left(\int_{\frac{1}{2}t}^{t} \left| \int_{\mathbf{S}^{n-1}} e^{-i\phi(s)\xi\cdot x} \Omega(x) d\sigma(x) \right|^{2} \frac{ds}{s} \right)^{1/\gamma'}$$

$$\leq \|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)} J(\xi,x,y) d\sigma(x)\, d\sigma(y) \right)^{1/\gamma'},$$

where $J(\xi,x,y) = \int_{1/2}^{1} e^{-i\phi(ts)\xi\cdot(x-y)} \frac{ds}{s}$. As in [7]; write $J(\xi,x,y) = \int_{1/2}^{1} Y_t'(s)\frac{ds}{s}$, where

$$Y_t(s) = \int_{1/2}^{s} e^{-i\phi(tw)\xi\cdot(x-y)} dw,\ 1/2 \leq w \leq s \leq 1.$$

By Van der Corput's lemma, the assumptions on ϕ and integration by parts, we conclude

$$|J(\xi,x,y)| \leq C(\ln 2) \left|\xi\cdot(x-y)t^d\right|^{-1},$$

which when combined with the trivial estimate $|J(\xi,x,y)| \leq \ln 2$ gives

$$|J(\xi,x,y)| \leq C(\ln 2) \left|\xi t^d\right|^{-\alpha} \left|\xi'\cdot(x-y)\right|^{-\alpha} \tag{5}$$

for any $0<\alpha<1$. This yields to

$$\begin{aligned}|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)| &\leq C\left|\xi t^{d}\right|^{\frac{-\alpha}{\gamma'}}\|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{(2/\gamma')}\\ &\quad\times\left(\int_{\mathbf{S}^{n-1}}\left|\xi'\cdot(x-y)\right|^{-\alpha q'}d\sigma(x)d\sigma(y)\right)^{1/q'\gamma'}.\end{aligned}$$

By choosing $0<2\alpha q'<1$, we get that the last integral is finite, and hence

$$\int_{\theta^k}^{\theta^{k+1}}|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)|^2\frac{dt}{t}\leq C(\ln\theta)\left|\xi\theta^{kd}\right|^{\frac{-2\alpha}{q'\gamma'}}\|h\|^2_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}.$$

On the other hand, if $\gamma>2$, then by using Hölder's inequality, we get

$$\begin{aligned}|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)| &\leq \|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\left(\int_{\frac{1}{2}t}^{t}\left|\int_{\mathbf{S}^{n-1}}e^{-i\phi(s)\xi\cdot x}\Omega(x)d\sigma(x)\right|^2\frac{ds}{s}\right)^{1/2}\\ &\leq \|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\Omega(x)\overline{\Omega(y)}\right.\\ &\quad\times\left.\left(\int_{\frac{1}{2}}^{1}e^{-i\phi(st)\xi\cdot x}e^{i\phi(st)\xi\cdot y}\frac{ds}{s}\right)d\sigma(x)\,d\sigma(y)\right)^{1/2}.\end{aligned}$$

Using Van der Corput's lemma and the above procedure gives

$$|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)| \leq C\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\left|\xi t^d\right|^{\frac{-\alpha}{q'\gamma'}},$$

and therefore,

$$\int_{\theta^k}^{\theta^{k+1}}|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)|^2\frac{dt}{t}\leq C(\ln\theta)\left|\xi\theta^{kd}\right|^{\frac{-2\alpha}{q'\gamma'}}\|h\|^2_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}. \tag{6}$$

To prove the other estimate in (4), we use the cancelation property of Ω;

$$|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)| \leq \int_{\frac{1}{2}}^{1}\int_{\mathbf{S}^{n-1}}|e^{-i\phi(ts)\xi\cdot x}-1||\Omega(x)||h(st)|d\sigma(x)\frac{ds}{s}$$

$$\leq \ |\xi| \, \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \int\limits_{\frac{1}{2}}^{1} |h(st)||\phi(ts)|\frac{ds}{s}.$$

Since $|\phi(ts)| \leq C(ts)^d$, $\gamma > 1$ and $\frac{1}{2} < s < 1$, we get that

$$|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)| \ \leq \ C \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})} \left|\xi t^d\right|.$$

By this, and since $|\hat{\sigma}_{\Omega,\phi,h,t}(\xi)| \leq (\ln 2)$, we obtain that

$$\int\limits_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,\phi,h,t}(\xi)|^2 \frac{dt}{t} \leq C(\ln\theta) \left|\xi\theta^{kd}\right|^{\frac{2\alpha}{q'\gamma'}} \|h\|^2_{\Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}. \tag{7}$$

Therefore, by (6) and (7), we get (4). The proof is complete. □

By [[7], Lemma 2.4] and using the same arguments as in [[6], Lemma 2.4], we immediately get the following lemma.

Lemma 5. *Let $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $\gamma > 1$ and $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1 < q \leq 2$. Let $\sigma^*_{\Omega,\phi,h}$ be given as in Definition 3. Then for any $f \in L^p(\mathbf{R}^n)$ with $\gamma' < p \leq \infty$, there exists a constant C_p (independent of Ω, h and f) such that*

$$\|\sigma^*_{\Omega,\phi,h} f(x)\|_{L^p(\mathbf{R}^n)} \leq C_p \|h\|_{\Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})} \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)}.$$

The following lemma can be obtained by applying the procedures (with only minor modifications) used in [[6], Lemmas 2.5] which have their roots in [1], [3] as well as [12] (We omit the details).

Lemma 6. *Let $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $1 < \gamma \leq 2$, $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1 < q \leq 2$ and $\theta = 2^{q'\gamma'}$. Let $\phi \in \mathcal{H}_d$ for some $d \neq 0$ and $\{\sigma_{\Omega,\phi,h,t}, t \in \mathbf{R}^+\}$ be given as in Definition 3. Then for any p satisfying $|1/p - 1/2| < 1/\gamma'$, there is a positive constant C_p such that*

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int\limits_{\theta^k}^{\theta^{k+1}} |\sigma_{\Omega,\phi,h,t} * g_k|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)}$$

$$\leq C_p A(\gamma) \frac{\|h\|_{\Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{(q-1)^{1/2}} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|^2_{L^p(\mathbf{R}^n)}$$

holds for arbitrary functions $\{g_k(\cdot), k \in \mathbf{Z}\}$ on $\mathbf{R}^n$.

3. Proof of Theorem 1

The proof of Theorem 1 mainly depends on the approaches that the authors of in [1] and [6] used (see also [2] as well as [12]). Let us first assume that $h \in \Delta_\gamma(\mathbf{R}^+, \frac{dt}{t})$ for some $\gamma > 1$, $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1 < q \leq 2$, $\theta = 2^{q'\gamma'}$, and $\phi \in \mathcal{H}_d$ for some $d > 0$. By Minkowski's inequality, we get that

$$\begin{aligned}
&\mathcal{M}^\rho_{\Omega,\phi,h} f(x) \\
&\quad\leq \sum_{k=0}^{\infty} \left(\int_0^\infty \left| t^{-\rho} \int_{2^{-k-1}t<|u|\leq 2^{-k}t} f(x-\phi(|u|)u')K_{\Omega,h}(u)du \right|^2 \frac{dt}{t} \right)^{1/2} \\
&\quad = \frac{2^a}{2^a-1} \left(\int_0^\infty \left|\sigma_{\Omega,\phi,h,t} * f(x)\right|^2 \frac{dt}{t} \right)^{1/2}.
\end{aligned}$$

For $k \in \mathbf{Z}$, let $\{\Phi_{k,\theta}\}_{-\infty}^{\infty}$ be a collection of $C^\infty(0,\infty)$ functions satisfying the following conditions:

$$\begin{aligned}
&0 \leq \Phi_{k,\theta} \leq 1, \quad \sum_k \Phi_{k,\theta}(t) = 1, \\
&\operatorname{supp} \Phi_{k,\theta} \subseteq \mathcal{I}_{k,\theta} = [\theta^{-kd-|d|}, \theta^{-kd+|d|}], \quad \left| \frac{d^s \Phi_{k,\theta}(t)}{dt^s} \right| \leq \frac{C_s}{t^s}.
\end{aligned}$$

Define $\widehat{\Psi_{k,\theta}}(\xi) = \Phi_{k,\theta}(|\xi|)$ and $\mathcal{B}_{k,\theta} = \{\xi \in \mathbf{R}^n : |\xi| \in \mathcal{I}_{k,\theta}\}$. Hence, for $f \in \mathcal{S}(\mathbf{R}^n)$, we have

$$\mathcal{M}^\rho_{\Omega,\phi,h} f(x) \leq \frac{2^a}{2^a-1} \sum_{j\in\mathbf{Z}} \mathcal{G}_{\Omega,\phi,h,j,\theta}(f), \tag{8}$$

where

$$\begin{aligned}
\mathcal{G}_{\Omega,\phi,h,j,\theta} f(x) &= \left(\int_0^\infty \left|\mathcal{N}_{\Omega,\phi,h,j,\theta}(x,t)\right|^2 \frac{dt}{t} \right)^{1/2}, \\
\mathcal{N}_{\Omega,\phi,h,j,\theta}(x,t) &= \sum_{k\in\mathbf{Z}} \sigma_{\Omega,\phi,h,t} * \Psi_{k+j,\theta} * f(x) \chi_{[\theta^k,\theta^{k+1})}(t).
\end{aligned}$$

Let us estimate the L^p-norm of $\mathcal{G}_{\Omega,\phi,h,\theta,j}(f)$. On the other hand, by Plancherel's theorem, Fubinis theorem, and Lemma 4, we deduce that there is $0 < \varepsilon < 1$ such that

$$\left\|\mathcal{G}_{\Omega,\phi,h,\theta,j}(f)\right\|^2_{L^2(\mathbf{R}^n)} \leq \sum_{k\in\mathbf{Z}} \int_{\mathcal{B}_{k+j,\theta}} \left(\int_{\theta^k}^{\theta^{k+1}} \left|\hat{\sigma}_{\Omega,\phi,h,\theta,t}(\xi)\right|^2 \frac{dt}{t} \right) \left|\hat{f}(\xi)\right|^2 d\xi$$

$$\leq C(\ln\theta)\,\|h\|^2_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\,\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}\sum_{k\in\mathbf{Z}}\int_{\mathcal{B}_{k+j,\theta}}\left(\left|\theta^{kd}\xi\right|^{\pm\frac{2\alpha}{q'\gamma'}}\right)\left|\hat{f}(\xi)\right|^2 d\xi$$

$$\leq C(\ln\theta)\,\|h\|^2_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\,\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}\,2^{-\varepsilon|j|}\sum_{k\in\mathbf{Z}}\int_{\mathcal{B}_{k+j,\theta}}\left|\hat{f}(\xi)\right|^2 d\xi$$

$$\leq C_p(\ln\theta)\,\|h\|^2_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\,\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}\,2^{-\varepsilon|j|}\,\|f\|^2_{L^2(\mathbf{R}^m)}\,.$$

Thus,

$$\|\mathcal{G}_{\Omega,\phi,h,\theta,j}(f)\|_{L^2(\mathbf{R}^n)}\leq C_pA(\gamma)\frac{\|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{(q-1)^{1/2}}2^{\frac{-\varepsilon|j|}{2}}\,\|f\|_{L^2(\mathbf{R}^n)}\,. \quad (9)$$

On the other hand, by using Lemma 6 and applying the Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [[17], p. 96], we obtain that for $|1/p-1/2|<1/\gamma'$ with $p\neq 2$,

$$\|\mathcal{G}_{\Omega,\phi,h,\theta,j}(f)\|_{L^p(\mathbf{R}^n)}\leq C_pA(\gamma)\frac{\|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{(q-1)^{1/2}}\,\|f\|_{L^p(\mathbf{R}^n)}\,. \quad (10)$$

By interpolation between (10) and (11) we reach

$$\|\mathcal{G}_{\Omega,\phi,h,\theta,j}(f)\|_{L^p(\mathbf{R}^m)}\leq C_p2^{-\varepsilon_p|j|}\frac{\|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{[(q-1)(\gamma-1)]^{1/2}}\,\|f\|_{L^p(\mathbf{R}^m)} \quad (11)$$

holds for $|1/p-1/2|<1/\gamma'$. Consequently, by (9) and (12), we finish the prrof of Theorem 1.

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