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ON THE WEIGHTED COMPOSITION OPERATORS ON HILBERT SPACES OF FORMAL POWER SERIES

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Abstract: In this paper we consider composition operators on the weighted Hardy spaces and we investigate that when the numerical range of a compact composition operator is closed.

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1. Introduction

Let $\{\beta(n)\}_n$ be a sequence of positive numbers with $\beta(0) = 1$. Let $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ be such that

$$\|f\|^2 = \|f\|^2_{H^2(\beta)} = \sum_{n=0}^{+\infty} |\hat{f}(n)|^2 \beta(n)^2 < \infty.$$

The notation $f(z) = \sum_{n=0}^{+\infty} \hat{f}(n) z^n$ shall be used weather or not the series converges for any value of z. The space of such formal power series is called the weighted Hardy space, which is denoted by $H^2(\beta)$. The classical Hardy space, Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1, \beta(n) = (n+1)^{-\frac{1}{2}}$ and $\beta(n) = (n+1)^{\frac{1}{2}}$, respectively. The space $H^2(\beta)$ becomes a

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Hilbert space with inner product

$$\langle f, g\rangle = \sum_{n=0}^{+\infty} a_n \overline{b_n} \beta(n)^2$$

where $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{+\infty} b_n z^n$ are the elements of $H^2(\beta)$. If $\lim \frac{\beta(n+1)}{\beta(n)} = 1$ or $\liminf \beta(n)^{\frac{1}{n}} = 1$, then $H^2(\beta)$ consists of functions analytic on the open unit disk U.

A complex number λ is said to be a bounded point evaluation on $H^2(\beta)$ if the functional of point evaluation at λ, e_λ, is bounded. A complex number λ is a bounded point evaluation on $H^2(\beta)$ if and only if $\left\{\frac{\lambda^n}{\beta(n)}\right\}_n \in l^2$. We denote the set of multipliers

$$\{\varphi \in H^2(\beta) : \varphi H^2(\beta) \subseteq H^2(\beta)\}$$

by $H^2_\infty(\beta)$ and the operator of multiplication by φ on $H^2(\beta)$ by M_φ with $\|\varphi\|_\infty = \|M_\varphi\|$.

Let φ be an analytic self map of U and ψ be a multiplier of $H^2(\beta)$. A weighted composition operator $C_{\psi,\varphi}$ maps an analytic function $f \in H^2(\beta)$ into

$$(C_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z)).$$

The function φ is called the composition map and the function ψ is called the multiplier map. We will use the notations $H(U)$ and $C(\overline{U})$ to denote the set of analytic functions on U and the set of continuous functions on $\overline{U}$, the closure of U. Some sources on formal power series and composition operators include [1–7].

2. Main Result

This work represents the necessary and sufficient conditions for the closedness of the numerical range of a compact composition operator acting on Banach spaces $H^2(\beta)$.

Definition 2.1. The numerical range of $C_{\psi,\varphi}$ acting on $H^2(\beta)$, is denoted by $W(C_{\psi,\varphi})$ that is defined by

$$W(C_{\psi,\varphi}) = \{< C_{\psi,\varphi} f, f >: f \in H^p(\beta) \quad \text{and} \quad ||f||_p = 1\}.$$

In the following we suppose that $\liminf \beta(n)^{\frac{1}{n}} = 1$ and this implies that $H^2(\beta)$ consists of functions analytic on the open unit disk U.

Theorem 2.2. *Let $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} = \infty$, $\varphi : U \to U$ be analytic and $\psi \in C(\overline{U}) \cap H^2_{\infty}(\beta)$. Also, suppose that $C_{\psi,\varphi}$ is bounded on $H^2(\beta)$ and there exists $\xi_0 \in \partial U$ such that φ is defined on ξ_0 and $\sum_{n\geq 0} \frac{\overline{\xi_0}^n \varphi(\xi_0)^n}{\beta(n)^2}$ is finite. Then $0 \in \overline{W}(C_{\psi,\varphi})$.*

Proof. First note that ξ_0 can not be a fixed point of φ. Let $\lambda \in U$, since e_λ is bounded on the Hilbert space $H^2(\beta)$, there exists a function $k_\lambda \in H^2(\beta)$ such that $e_\lambda(f) =< f, k_\lambda >$ for all $f \in H^2(\beta)$, and $||e_\lambda|| = ||k_\lambda||$. Put

$$K_\lambda = \frac{k_\lambda}{||k_\lambda||}, \quad E_\lambda = \frac{e_\lambda}{||e_\lambda||}$$

where $\lambda \in U$. Let

$$\alpha_\lambda =< C_{\psi,\varphi} K_\lambda, E_\lambda >,$$

so $\alpha_\lambda \in W(C_{\psi,\varphi})$. Now we have

$$\begin{aligned} \alpha_\lambda &= < C_{\psi,\varphi} K_\lambda, E_\lambda > \\ &= < K_\lambda, C^*_{\psi,\varphi} E_\lambda > \\ &= \frac{\psi(\lambda)}{||k_\lambda||^2} < k_\lambda, \overline{\psi(\lambda)} e_{\varphi(\lambda)} > \\ &= ||k_\lambda||^{-2} \psi(\lambda) \overline{e_{\varphi(\lambda)}(\lambda)} \\ &= \psi(\lambda) ||e_\lambda||^{-2} \sum_{n\geq 0} \frac{\varphi(\lambda)^n}{\beta(n)^2} \overline{\lambda}^n. \end{aligned}$$

Let $\lambda \to \xi_0$ and note that $\psi(\lambda) \to \psi(\xi_0)$, so we can see that $\alpha_\lambda \to 0$. Hence $0 \in \overline{W}(C_{\psi,\varphi})$ and the proof is complete. □

From the proof of Theorem 2.2, we have the following corollary.

Corollary 2.3. *Suppose that $C_{\psi,\varphi}$ is bounded on $H^2(\beta)$. If φ has a fixed point ξ in U, then $\psi(\xi)$ belongs to $W(C_{\psi,\varphi})$.*

Proof. Put

$$K_\xi = \frac{k_\xi}{||k_\xi||}, \quad E_\xi = \frac{e_\xi}{||e_\xi||}.$$

Then

$$< C_{\psi,\varphi} K_\xi, E_\xi > \; = \; < K_\xi, C^*_{\psi,\varphi} E_\xi >$$

$$\begin{aligned} &= \frac{1}{||k_\xi||^2} < k_\xi, \overline{\psi(\xi)} e_{\varphi(\xi)} > \\ &= ||k_\xi||^{-2} \psi(\xi) < k_\xi, e_\xi > \\ &= ||k_\xi||^{-2} \psi(\xi) k_\xi(\xi) \\ &= \psi(\xi) \end{aligned}$$

belongs to $W(C_{\psi,\varphi})$ and so the proof is complete. □

Recall that we say an operator T is completely continuous on a Banach space X, if weakly convergence of $x_i \to x$ implies the convergence $Tx_i \to Tx$ in norm.

Proposition 2.4. *Let $C_{\psi,\varphi}$ be completely continuous on $H^2(\beta)$. Then under the conditions of Theorem 2.2, $W(C_{\psi,\varphi})$ is closed if and only if $0 \in W(C_{\psi,\varphi})$.*

Proof. If $W(C_{\psi,\varphi})$ is closed, then clearly $0 \in W(C_{\psi,\varphi})$. Conversely, let $w \in \overline{W}(C_{\psi,\varphi})$, then there exists a sequence $\{h_n\}_n$ in ball$H^2(\beta)$ and $h \in H^2(\beta)$ such that $h_n \to h$ weakly and $\{< C_{\psi,\varphi} h_n, h_n >\}_n$ converges to w. This implies that $w =< C_{\psi,\varphi} h, h >$ and so

$$w \in ||h||^2 W(C_{\psi,\varphi})$$

where $h \in ballH^2(\beta)$. Suppose that $h \neq 0$, hence

$$\frac{w}{||h||^2} \in W(C_{\psi,\varphi}).$$

But $W(C_{\psi,\varphi})$ is convex and $0 \in W(C_{\psi,\varphi})$, hence $w \in W(C_{\psi,\varphi})$. So $W(C_{\psi,\varphi}$ is closed whenever $0 \in W(C_{\psi,\varphi})$. □

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