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AN UPPER ESTIMATE FOR THE LARGEST SINGULAR VALUE OF A SPECIAL MATRIX, II

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Abstract: Our goal is to prove an upper bound for the largest singular value of the following matrix

$$\boldsymbol{A} = \boldsymbol{D}^{-1}\boldsymbol{E}\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right)^{-1}\boldsymbol{E}^t.$$

Here $\cdot^t$ denotes the matrix transpose, $\boldsymbol{D}$ is a non-singular matrix, and $\boldsymbol{E}$ is thin matrix.

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1. Introduction

In many applications, we have to deal with a matrix in the form ($\cdot^t$ is matrix transpose)

$$\boldsymbol{A} = \boldsymbol{D}^{-1}\boldsymbol{E}\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right)^{-1}\boldsymbol{E}^t,$$

where:

1. $\boldsymbol{D}$ is a symmetric, positive definite $(m \times m)$-matrix.
2. $\boldsymbol{E}$ is a full rank $(m \times n)$-matrix.
3. It follows from (2), $n \leq m$ that $\det\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right) \neq 0$.

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Our goal is to establish an upper bound for $\|\boldsymbol{A}\|$.

In many applications $\boldsymbol{D}$ is a diagonal matrix, see for example the results in [1, 2, 4, 5, 6, 7, 10] for simple constructions of moving least squares approximations.

2. Main Result

Lemma 2.1. *Let* $1 < n \leq m$ *and* $\operatorname{rank}\left(\boldsymbol{E}^t\boldsymbol{E}\right) = n$. *Then*

$$\sigma_{\max}(\boldsymbol{A}) = \|\boldsymbol{A}\| \leq k(\boldsymbol{D})k^2(\boldsymbol{E}) = \frac{\sigma_{\max}(\boldsymbol{D})}{\sigma_{\min}(\boldsymbol{D})}\frac{\sigma^2_{\max}(\boldsymbol{E})}{\sigma^2_{\min}(\boldsymbol{E})},$$

where $k(\cdot)$ *(resp.* $\sigma_{\min}(\cdot)$, $\sigma_{\max}(\cdot)$*) is the condition number (resp. lower, largest singular value).*

Proof. The matrix $\boldsymbol{D}$ is a symmetric positive definite matrix. Then there exist two matrxices $\boldsymbol{D}^{\frac{1}{2}}$ and $\boldsymbol{D}^{-\frac{1}{2}}$ such that:

$$\boldsymbol{D}^{-\frac{1}{2}} = \left(\boldsymbol{D}^{\frac{1}{2}}\right)^{-1}, \quad \left(\boldsymbol{D}^{\frac{1}{2}}\right)^2 = \boldsymbol{D}.$$

We will separate the proof in several steps.

1. Let
$$\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^t$$
be the singular value decomposition of $\boldsymbol{E}\boldsymbol{D}^{-\frac{1}{2}}$, where:

   (a) $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary matrices.

   (b) If σ_i are the singular values of $(m \times n)$-matrix $\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}$, $n \leq m$, then

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_1 \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

2. We have (using [3, Chapter 3, P3-SVD and P5-SVD]):

$$\sigma_{\max}\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right) \leq \sigma_{\max}\left(\boldsymbol{D}^{-\frac{1}{2}}\right)\sigma_{\max}\left(\boldsymbol{E}\right) = \frac{\sigma_{\max}\left(\boldsymbol{E}\right)}{\sqrt{\sigma_{\min}\left(\boldsymbol{D}\right)}}.$$

Here $\sigma_{\min}\left(\boldsymbol{D}\right) \neq 0$, because $\boldsymbol{D}$ is non-sigular matrix.

3. We have (using [3, Chapter 3, P3-SVD and P7-SVD]):

$$\sigma_{\min}\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right) \geq \sigma_{\min}\left(\boldsymbol{D}^{-\frac{1}{2}}\right)\sigma_{\min}\left(\boldsymbol{E}\right) = \frac{\sigma_{\min}\left(\boldsymbol{E}\right)}{\sqrt{\sigma_{\max}\left(\boldsymbol{D}\right)}}.$$

Here $\sigma_{\max}\left(\boldsymbol{D}\right) \geq \sigma_{\min}\left(\boldsymbol{D}\right) \neq 0$.

4. $\boldsymbol{E}$ is thin (i.e. $n \leq m$) matrix and rank $\left(\boldsymbol{E}\right) = n$. Then

$$\text{rank }\left(\boldsymbol{E}\right) = \text{rank }\left(\boldsymbol{E}^t\boldsymbol{E}\right) = n.$$

Indeed, let $\boldsymbol{E} = \boldsymbol{U}_E\boldsymbol{\Sigma}_E\boldsymbol{V}_E^t$ be the singular value decomposition of $(m\times n)$-matrix $\boldsymbol{E}$, $n \leq m$. Then

$$\boldsymbol{E}^t\boldsymbol{E} = \left(\boldsymbol{U}_E\boldsymbol{\Sigma}_E\boldsymbol{V}_E^t\right)^t\left(\boldsymbol{U}_E\boldsymbol{\Sigma}_E\boldsymbol{V}_E^t\right) = \boldsymbol{V}_E\boldsymbol{\Sigma}_E^t\boldsymbol{\Sigma}_E\boldsymbol{V}.$$

Therefore rank $\left(\boldsymbol{E}^t\boldsymbol{E}\right) = n$, because rank $\left(\boldsymbol{\Sigma}_E^t\boldsymbol{\Sigma}_E\right) = n$. Moreover (using the definition of singular value), the matrix $\boldsymbol{E}$ has $\min\{m,n\} = n$ singular values, and all non-zero singular values of $\boldsymbol{E}$ are the the square roots of the non-zero eigenvalues of both $\boldsymbol{E}^t\boldsymbol{E}$ and $\boldsymbol{E}\boldsymbol{E}^t$. But all eigenvalues of $\boldsymbol{E}^t\boldsymbol{E}$ are strictly positive. Hence $0 < \sigma_{\min}\left(\boldsymbol{E}\right) \leq \sigma_{\max}\left(\boldsymbol{E}\right)$.

5. Obviously ($\boldsymbol{D}$ and $\boldsymbol{D}^{-\frac{1}{2}}$ are symmetric matrices)

$$\begin{aligned}
\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E} &= \left(\boldsymbol{E}^t\boldsymbol{D}^{-\frac{1}{2}}\right)\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right)\\
&= \left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right)^t\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right)\\
&= \left(\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^t\right)^t\left(\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^t\right)\\
&=\boldsymbol{V}\boldsymbol{\Sigma}^t\boldsymbol{U}^t\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^t\\
&=\boldsymbol{V}\begin{pmatrix}\boldsymbol{\Sigma}_1 & \boldsymbol{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Sigma}_1\\ \boldsymbol{0}\end{pmatrix}\boldsymbol{V}^t\\
&=\boldsymbol{V}\boldsymbol{\Sigma}_1^2\boldsymbol{V}^t.
\end{aligned}$$

is the singular value decomposition of $\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}$.

6. Using the previous item and the unitaty property of $\boldsymbol{V}$, we have

$$\begin{aligned}\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right)^{-1} &= \left(\boldsymbol{V}\boldsymbol{\Sigma}_1^2\boldsymbol{V}^t\right)^{-1}\\ &= \left(\boldsymbol{V}^t\right)^{-1}\boldsymbol{\Sigma}_1^{-2}\left(\boldsymbol{V}\right)^{-1}\\ &= \boldsymbol{V}\boldsymbol{\Sigma}_1^{-2}\boldsymbol{V}^t,\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{D}^{-1}\boldsymbol{E}\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right)^{-1}\boldsymbol{E}^t\\ &= \boldsymbol{D}^{-1}\boldsymbol{E}\boldsymbol{V}^t\boldsymbol{\Sigma}_1^{-2}\boldsymbol{V}\boldsymbol{E}^t.\end{aligned}$$

7. Using the previous item, we have

$$\begin{aligned}\sigma_{\max}\left(\boldsymbol{A}\right) &\leq \sigma_{\max}\left(\boldsymbol{D}^{-1}\right)\sigma_{\max}^2\left(\boldsymbol{E}\right)\sigma_{\max}\left(\boldsymbol{\Sigma}_1^{-2}\right)\\ &= \frac{\sigma_{\max}^2\left(\boldsymbol{E}\right)}{\sigma_{\min}\left(\boldsymbol{D}\right)\sigma_{\min}\left(\boldsymbol{\Sigma}_1^2\right)},\end{aligned}$$

because all singular values of an unitary matrix are 1 (indeed, the singular value decomposition of the unitaty matrix $\boldsymbol{V}$ is $\boldsymbol{V} = \boldsymbol{V}\boldsymbol{I}\boldsymbol{I}$) and

$$\sigma_{\min}\left(\boldsymbol{\Sigma}_1^2\right) = \sigma_{\min}^2\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{E}\right) > 0.$$

8. Therefore

$$\begin{aligned}\sigma_{\max}\left(\boldsymbol{A}\right) &\leq \frac{\sigma_{\max}^2\left(\boldsymbol{E}\right)}{\sigma_{\min}\left(\boldsymbol{D}\right)\sigma_{\min}\left(\boldsymbol{\Sigma}_1^2\right)}\\ &\leq \frac{\sigma_{\max}^2\left(\boldsymbol{E}\right)}{\sigma_{\min}\left(\boldsymbol{D}\right)\left(\frac{\sigma_{\min}\left(\boldsymbol{E}\right)}{\sqrt{\sigma_{\max}\left(\boldsymbol{D}\right)}}\right)^2}\\ &= \frac{\sigma_{\max}\left(\boldsymbol{D}\right)}{\sigma_{\min}\left(\boldsymbol{D}\right)}\frac{\sigma_{\max}^2\left(\boldsymbol{E}\right)}{\sigma_{\min}^2\left(\boldsymbol{E}\right)}.\end{aligned}$$

□

Lemma 2.2. *Let* $1 < n \leq m$ *and* $\operatorname{rank}\left(\boldsymbol{E}^t\boldsymbol{E}\right) = n$. *Then*

$$\sigma_{\min}\left(\boldsymbol{A}\right) \leq 1 \leq \sigma_{\max}\left(\boldsymbol{A}\right) = \left\|\boldsymbol{A}\right\|.$$

Proof. Obviously

$$\boldsymbol{E}^t\boldsymbol{A} = \boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\left(\boldsymbol{E}^t\boldsymbol{D}^{-1}\boldsymbol{E}\right)^{-1}\boldsymbol{E}^t = \boldsymbol{E}^t.$$

It follows from [3, Chapter 3, P15-SVD (a)] that

$$\sigma_{\max}\left(\boldsymbol{E}^t\right)\sigma_{\min}\left(\boldsymbol{A}\right) \leq \sigma_{\max}\left(\boldsymbol{E}^t\boldsymbol{A}\right) = \sigma_{\max}\left(\boldsymbol{E}^t\right).$$

Indeed $\boldsymbol{E}^t$ is $(n \times m)$-matrix and $n \leq m$.

Hence $\sigma_{\min}\left(\boldsymbol{A}\right) \leq 1$.

On the other hand, using [3, Chapter 3, P16-SVD], we recceive

$$\sigma_{\min}\left(\boldsymbol{E}^t\right) = \sigma_{\min}\left(\boldsymbol{E}^t\boldsymbol{A}\right) \leq \sigma_{\max}\left(\boldsymbol{A}\right)\sigma_{\min}\left(\boldsymbol{E}^t\right)$$

Hence $1 \leq \sigma_{\max}\left(\boldsymbol{A}\right)$. □

Combining Lemma 2.1 and Lemma 2.2, we receive the following result.

Theorem 2.1. *Let $1 < n \leq m$ and* rank $\left(\boldsymbol{E}^t\boldsymbol{E}\right) = n$. *Then*

$$1 \leq \|\boldsymbol{A}\| \leq k(\boldsymbol{D})k^2(\boldsymbol{E}).$$

Remark 2.1. If $m = n$, then $\det\left(\boldsymbol{E}\right) \neq 0$ and $\boldsymbol{A} = \boldsymbol{I}$. In this case, we receive the following obvious inequality

$$\frac{\sigma_{\max}\left(\boldsymbol{D}\right)}{\sigma_{\min}\left(\boldsymbol{D}\right)}\frac{\sigma^2_{\max}(\boldsymbol{E}^t)}{\sigma^2_{\min}(\boldsymbol{E}^t)} \geq 1.$$

Remark 2.2. If $n = 1$, then the matrix $\boldsymbol{E}$ has only one singular value. Hence, in this case

$$1 \leq \sigma_{\max}\left(\boldsymbol{A}\right) = \|\boldsymbol{A}\| \leq k(\boldsymbol{D}).$$

Remark 2.3. We cannot omit the term $\frac{\sigma_{\max}(\boldsymbol{D})}{\sigma_{\min}(\boldsymbol{D})}$ (even supposed $\boldsymbol{D}$ to be a diagonal matrix).

A simple example: Let

$$\boldsymbol{D}(x) = \begin{pmatrix} 3 & 0 \\ 0 & x \end{pmatrix}, \; x \in \mathbb{R}_+, \quad \boldsymbol{E} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Then

$$\boldsymbol{A}(x) = \frac{1}{x+27}\begin{pmatrix} x & 3x \\ 9 & 27 \end{pmatrix} \quad \text{and} \quad \sigma_{\max}\left(\boldsymbol{A}(x)\right) = \frac{\sqrt{10}\sqrt{x^2+81}}{|x+27|} > 1.$$

But (the matrix $\boldsymbol{E}$ has only one singular value $\sqrt{10}$) and

$$\frac{\sigma_{\max}(\boldsymbol{E})}{\sigma_{\min}(\boldsymbol{E})} = 1.$$

Hence

$$\sigma_{\max}\left(\boldsymbol{A}(x)\right) \leq \frac{\sigma_{\max}\left(\boldsymbol{D}(x)\right)}{\sigma_{\min}\left(\boldsymbol{D}(x)\right)} = \frac{\max\{3, x\}}{\min\{3, x\}}.$$

Let us mark

$$\min\left\{\sigma_{\max}\left(\boldsymbol{A}(x)\right) : x \in [0, \infty)\right\} = \sigma_{\max}\left(\boldsymbol{A}(3)\right) = 1 = \frac{\sigma_{\max}\left(\boldsymbol{D}(3)\right)}{\sigma_{\min}\left(\boldsymbol{D}(3)\right)}.$$

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