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SOME PROPERTIES OF $\tau_1\tau_2 - \delta$ SEMI OPEN SETS/CLOSED SETS IN BITOPOLOGICAL SPACES

M. Arunmaran[1 §], K. Kannan[2]
[1,2]Department of Mathematics and Statistics
University of Jaffna
Jaffna, SRI LANKA

Abstract: In this paper, we introduce some properties of $\tau_1\tau_2 - \delta$ semi open sets / closed sets in a bitopological space. With that, we investigate several results in $\tau_1\tau_2 - \delta$ semi open sets / closed sets and $\tau_1\tau_2 - \delta$ semi continuous in bitopological spaces. Further, we prove some results in $\tau_2\tau_1 - \delta$ semi open sets / closed sets in bitopological spaces. Bitopological space does not exist for every metric space. But it exists for special type of metric spaces,called as "asymmetric metric spaces". Bitopological spaces have some applications in various parts in mathematics. Moreover, a high level of modern knowledge of bitopological spaces theory has made it possible to introduce and study algebra of new type the corresponding representation of which brings one to the special class of bitopological spaces.

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1. Introduction

In 1963, J.C. Kelly in [1] initiated the study of bitopological spaces as a natural structure by studying quasimetrics and its conjugate. Besides, he introduced various separation properties into bitopological spaces, and obtained general-

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§Correspondence author

izations of some important classical results. A quasi-pseudo-metric on a set X is a non-negative real valued function $p(,)$ on the product $X \times X$ such that $p(x,x) = 0, \forall\, x \in X$, $p(x,z) \leq p(x,y) + p(y,z), \forall\, x,y,z \in X$. If $p(,)$ satisfies the condition $p(x,y) = 0$ if and only if $x = y, \forall\, x,y \in X$, then $p(,)$ is a quasi-metric. Bitopological spaces arise in a natural way by considering the topologies induced by sets of the form $B_{px\epsilon} = \{y : p(x,y) < \epsilon\}$ and $B_{qx\epsilon} = \{y : q(x,y) < \epsilon\}$; where p and q are quasi metrics on X and $q(x,y) = p(y,x)$. A topological space occurs for every metric space.(i.e. Every metric space is a topological space.) But bitopological spaces naturally occur for quasi metric spaces or asymmetric metric spaces. Quasi-uniform spaces, which are generalizations of quasi-metric spaces, also induce bitopological spaces. This structure is a richer structure than that of a topological space. Considerable effort had been expended in obtaining appropriate generalizations of standard topological properties to bitopological category by various authors. Most of them deal with the theory itself, but very few with applications. S.N.Maheswari and R.Prasad [3] introduced semi open sets in bitopological spaces in 1977. Further properties of this notion were studied by Bose [4] in 1981. Banerjee [11] introduced the notion $\delta-$ open sets in bitopological spaces in 1987. Khedr [12] introduced and studied about $\tau_1\tau_2 - \delta$ open sets. Later, T.Fukutake [8] defined one kind of semi open sets and studied their properties in 1989. Recently, A.Edward Samuel and D.Balan [2] obtained $\tau_1\tau_2 - \delta$ semi open sets in bitopological spaces.

In section 2 of this paper, we introduce some basic definitions and examples of various kind of open sets, semi open sets, closed sets, semi closed sets, continuity and semi continuity in bitopological spaces. In section 3, we discuss some properties of $\tau_1\tau_2 - \delta$ semi open sets and closed sets in bitopological spaces. In addition, we give some examples also. Furthermore, we obtain several results in arbitrary union, finite union, arbitrary intersection and finite intersection of collection of sets. Besides, we have some results in $\tau_2\tau_1 - \delta$ semi open sets in bitopological spaces. In this section, we prove the following results: In a bitopological space (X, τ_1, τ_2),

1. A is $\tau_1\tau_2 - \delta$ semi open if and only if $X \backslash A$ is $\tau_1\tau_2 - \delta$ semi closed.

2. Every $\tau_1 - \delta$ closed set is $\tau_1\tau_2 - \delta$ semi closed set.

3. Every $\tau_1\tau_2 - \delta$ semi closed set is $\tau_1\tau_2-$ semi closed.

4. A subset A is $\tau_1\tau_2 - \delta$ semi closed set if and only if $A \supseteq \tau_2-\text{int}(\tau_1 - \delta\text{cl}(A))$.

5. Let A and B be subsets in (X, τ_1, τ_2) and $A \cup B$ is a $\tau_1\tau_2 - \delta$ semi open set. Then, A,B need not be $\tau_1\tau_2 - \delta$ semi open sets.

6. Let A and B be subsets in (X, τ_1, τ_2) and $A \cap B$ is a $\tau_1\tau_2 - \delta$ semi closed set. Then, A,B need not be $\tau_1\tau_2 - \delta$ semi closed sets.

7. Let A, B be the subsets in (X, τ_1, τ_2) and (Y, σ_1, σ_2) respectively. If $A \in \tau_2\tau_1 - \delta$ semi open in X and $B \in \sigma_1\sigma_2 - \delta$ semi open set in Y, then $A \times B \in \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2 - \delta$ semi open set in $(X \times Y, \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2)$.

8. Let A, B be the subsets in (X, τ_1, τ_2) and (Y, σ_1, σ_2) respectively. If $A \in \tau_2\tau_1 - \delta$ semi closed in X and $B \in \sigma_1\sigma_2 - \delta$ semi closed set in Y, then $A \times B \in \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2 - \delta$ semi closed set in $(X \times Y, \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2)$.

9. A subset A is $\tau_2\tau_1 - \delta$ semi open in (X, τ_1, τ_2) if and only if A is $\tau_{2s}\tau_{1s}-$ semi open in $(X, \tau_{1s}, \tau_{2s)}$.

In Section 4, we discuss the properties of $\tau_1\tau_2 - \delta$ semi continuous functions in (X, τ_1, τ_2). In this section, we mention some important results which were proved by some authors. Finally, we prove that any constant function in a bitopological space is $\tau_1\tau_2 - \delta$ semi continuous. With that, we show that the following result holds:

Let $f : (X, \tau_1, \tau_2) \longrightarrow (X, \sigma_1, \sigma_2)$ be $\tau_1\tau_2 - \delta$ semi continuous. Then, $\forall$ σ_1-open set V in Y, $\exists$ $\tau_1\tau_2 - \delta$ semi open set P in X such that $f(P) \subseteq V$.

2. Preliminaries

Let τ_i-int(A), τ_i-cl(A), $\tau_i - \delta$int(A) and $\tau_i - \delta$cl(A) be the interior, closure, $\delta-$interior and $\delta-$closure of A with respect to the topology τ_i respectively, i=1,2. Let $\tau_j - \delta$int(A) and $\tau_j - \delta$cl(A) are the $\delta-$interior and $\delta-$closure of A with respect to the topology τ_j ; $j = 1s, 2s$.

Definition 2.1. [1] Let X be a nonempty set. Let τ_1 and τ_2 be two topologies on X.Then, (X, τ_1, τ_2) is called bitopological space.

Example 2.2. Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{b\}, \{a, b\}\}$. Then, (X, τ_1, τ_2) is a bitopological space.

Example 2.3. Let τ_1, τ_2 be discrete and trivial topologies respectively on X. Then, (X, τ_1, τ_2) is a bitopological space.

Definition 2.4. Let A be subset of (X, τ_1, τ_2). Then, A is said to be open, if $A \in \tau_1 \cap \tau_2$. In (X, τ_1, τ_2), complement of open set is called closed set.

Example 2.5. Let $X = \{a, b, c, d\}, \tau_1 = \{\phi, X, \{a, c, d\}, \{a\}\}$ and $\tau_2 = \{\phi, X, \{b\}, \{a\}, \{a, b\}\}$.

Then, $\phi, X, \{a\}$ are open sets in (X, τ_1, τ_2).
$X, \phi, \{b, c, d\}$ are closed sets in (X, τ_1, τ_2).

Definition 2.6. [1] Let A be subset of (X, τ_1, τ_2). Then, A is said to be $\tau_1\tau_2-$open, if $A \in \tau_1 \cup \tau_2$. In (X, τ_1, τ_2), complement of $\tau_1\tau_2-$open set is called $\tau_1\tau_2-$closed set.

Example 2.7. Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}$ and

$$\tau_2 = \{\phi, X, \{b\}, \{a\}, \{a, b\}\}.$$

Then, $\phi, X, \{a\}, \{b\}, \{a, b\}$ are $\tau_1\tau_2-$open sets in (X, τ_1, τ_2).
$X, \phi, \{b, c\}, \{a, c\}, \{c\}$ are $\tau_1\tau_2-$closed sets in (X, τ_1, τ_2).

Definition 2.8. [1] In a bitopological space (X, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2, if for each point x in X, there is a τ_1-neighbourhood base of τ_2-closed sets, or, as is easily seen to be equivalent, if for each point x in X and each τ_1-closed set P such that $x \in X$ and each τ_1-closed set P such that $x \notin P$, there are a τ_1-open set U and a τ_2-open set V such that $x \in U$, $P \subseteq V$ and $U \cap V = \phi$. (X, τ_1, τ_2) is, or τ_1 and τ_2 are pairwise regular, if τ_1 is regular with respect to τ_2 and vice versa.

Definition 2.9. [5] Let A be subset of bitopological space (X, τ_1, τ_2). Then, A is called

1. $\tau_{12}-$regular open, if $A = \tau_1-\text{int}(\tau_2-\text{cl}(A))$.
2. $\tau_{21}-$regular open, if $A = \tau_2-\text{int}(\tau_1-\text{cl}(A))$.
3. $\tau_1\tau_2-$semi open, if $A \subseteq \tau_2-\text{cl}(\tau_1-\text{int}(A))$.
4. $\tau_1\tau_2-$semi closed, if $A \supseteq \tau_2-\text{int}(\tau_1-\text{cl}(A))$.

Remark 2.10. Let A be subset of bitopological space (X, τ_1, τ_2). Then, A is called

1. $\tau_{12}-$regular closed, if $A = \tau_1-\text{cl}(\tau_2-\text{int}(A))$.
2. $\tau_{21}-$regular closed, if $A = \tau_2-\text{cl}(\tau_1-\text{int}(A))$.

Definition 2.11. [5] Let A be subset of bitopological space (X, τ_1, τ_2). Then:

1. A is said to be $\tau_1 - \delta$ open set, if for $x \in A$, there exists $\tau_{12}-$ regular open set G such that $x \in G \subset A$. Complement of $\tau_1 - \delta$ open set is called $\tau_1 - \delta$ closed set.

2. A is said to be $\tau_2 - \delta$ open set, if for $x \in A$, there exists $\tau_{21}-$ regular open set G such that $x \in G \subset A$. Complement of $\tau_2 - \delta$ open set is called $\tau_2 - \delta$ closed set.

3. Collection of all $\tau_1 - \delta$ open sets and $\tau_2 - \delta$ open sets are denoted by τ_{1s} and τ_{2s} respectively. And also $\tau_{1s} \subset \tau_1$ and $\tau_{2s} \subset \tau_2$.

Definition 2.12. [1] A bitopological space (X, τ_1, τ_2) is said to be pairwise Hausdorff, if for each two distinct points x and y, there are a τ_1-neighbourhood U of x and a τ_2-neighbourhood V of y such that $U \cap V = \phi$.

Definition 2.13. [1] A bitopological space (X, τ_1, τ_2) is said to be pairwise normal, if given a τ_1-closed set A and a τ_2-closed set B with $A \cap B = \phi$, there exist a τ_2-open set U and a τ_1-open set V such that $A \subseteq U, B \subseteq V$ and $U \cap V = \phi$. Equivalently, (X, τ_1, τ_2) is pairwise normal, if given a τ_2-closed set C and a τ_1-open set D such that $C \subseteq D$,there are a τ_1-open set G and a τ_2-closed set F such that $C \subseteq G \subseteq F \subseteq D$.

Definition 2.14. [2] Let A be subset of bitopological space (X, τ_1, τ_2). Then, A is called $\tau_1\tau_2 - \delta$ semi open set, if there exists an $\tau_1 - \delta$ open set U such that $U \subseteq A \subseteq \tau_2-\text{cl}(U)$.

Definition 2.15. [2] Let A be subset of bitopological space (X, τ_1, τ_2). Then, A is called $\tau_1\tau_2 - \delta$ semi closed set, if there exists an $\tau_1 - \delta$ closed set F such that $\tau_2-\text{int}(F) \subseteq A \subseteq F$.

Example 2.16. Let $X = \{a, b, c, d\}, \tau_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{d\}, \{a, d\}\}$. Then, $\phi, X, \{a, b\}, \{a, b, c\}$ are $\tau_1\tau_2 - \delta$ semi open sets in (X, τ_1, τ_2). $X, \phi, \{c, d\}, \{d\}$ are $\tau_1\tau_2 - \delta$ semi closed sets in (X, τ_1, τ_2).

Definition 2.17. [7] A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be pairwise continuous if and only if the induced functions $f : (X, \tau_1) \to (Y, \sigma_1)$ and $f : (X, \tau_2) \to (Y, \sigma_2)$ are continuous.

Example 2.18. Let $f : (X, \tau_1) \to (Y, \sigma_1)$; where τ_1 be discrete topology and σ_1 be any topology. Let $f : (X, \tau_2) \to (Y, \sigma_2)$; where τ_2 be any topology, σ_2 be trivial topology. Then, $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise continuous function.

Example 2.19. Let $f : (X, \tau_1) \to (Y, \sigma_1)$; where τ_1, σ_1 be standard and lower limit topologies respectively. Let $f : (X, \tau_2) \to (Y, \sigma_2)$; where τ_2 be any topology, σ_2 be trivial topology. Then, $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not a pairwise continuous function.

3. Some Properties of $\tau_1\tau_2 - \delta$ Semi Open Sets/Closed Sets

Theorem 3.1. *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then, A is $\tau_1\tau_2 - \delta$ semi open if and only if $X \backslash A$ is $\tau_1\tau_2 - \delta$ semi closed in (X, τ_1, τ_2).*

Proof. Let A be $\tau_1\tau_2 - \delta$ semi open. Then, there exists a $\tau_1 - \delta$ openset U such that $U \subseteq A \subseteq \tau_2-\text{cl}(U)$. This implies, $\tau_2-\text{int}(U^c) \subseteq A^c \subseteq U^c$. i.e. $\tau_2-\text{int}(V) \subseteq X\backslash A \subseteq V$; where $U^c = V$ is a $\tau_1 - \delta$ closed set. Thus, $X\backslash A$ is $\tau_1\tau_2 - \delta$ semi closed. Conversely, let $X\backslash A$ is $\tau_1\tau_2 - \delta$ semi closed. Then, $\tau_2-\text{int}(F) \subseteq X\backslash A \subseteq F$, forsome $\tau_1 - \delta$ closed set F. This implies, $F^c \subseteq A \subseteq \tau_2-\text{cl}(F^c)$ and F^c is $\tau_1 - \delta$ open set. Thus, A is $\tau_1\tau_2 - \delta$ semi open. □

Proposition 3.2. *Every $\tau_1 - \delta$ closed set is $\tau_1\tau_2 - \delta$ semi closed set in (X, τ_1, τ_2).*

Proof. Let A be $\tau_1 - \delta$ closed set. Then $A = \tau_1 - \delta\text{cl}(A)$. But $\tau_2-\text{int}(A) = \tau_2-\text{int}(\tau_1 - \delta\text{cl}(A)) \subseteq A$.Hence, A is $\tau_1\tau_2 - \delta$ semi closed. □

Remark 3.3. The converse part of the above proposition need not be true.

Example 3.4. Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$.
Then, $\{b\}$ is a $\tau_1\tau_2 - \delta$ semi closed set. But it is not a $\tau_1 - \delta$ closed set.

Proposition 3.5. *Every $\tau_1\tau_2 - \delta$ semi closed set is $\tau_1\tau_2-$ semi closed in (X, τ_1, τ_2).*

Proof. Let A be $\tau_1\tau_2 - \delta$ semi closed set. Then, $\exists\ U$ be $\tau_1 - \delta$ closed set such that $\tau_2-\text{int}(U) \subseteq A \subseteq U$. Let $U = \tau_1-\text{cl}(A)$. i.e. $\tau_2-\text{int}(\tau_1-\text{cl}(A)) \subseteq A$. Hence, A is $\tau_1\tau_2-$ semi closed. □

Remark 3.6. The converse part of the above proposition need not be true.

Example 3.7. Let $X = \{a, b, c, d\}, \tau_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{d\}, \{a, d\}\}$.
Then, $\{b, d\}$ is a $\tau_1\tau_2-$ semi closed set. But it is not a $\tau_1\tau_2 - \delta$ semi closed set.

Theorem 3.8. *Let A be subset of a bitopological space (X, τ_1, τ_2). Then, A is $\tau_1\tau_2 - \delta$ semi closed if and only if $A \supseteq \tau_2-$int$(\tau_1 - \delta$cl$(A))$.*

Proof. Let $A \supseteq \tau_2-\text{int}(\tau_1 - \delta\text{cl}(A))$. Then, there exists a $\tau_1 - \delta$ closed set $U(= \tau_1 - \delta\text{cl}(A))$ such that $\tau_2-\text{int}(U) \subseteq A \subseteq U$. Thus, A is $\tau_1\tau_2 - \delta$ semi closed set in (X, τ_1, τ_2). Conversely, let A be $\tau_1\tau_2 - \delta$ semi closed set. Let U be $\tau_1 - \delta$ closed set. Then, $\tau_2-\text{int}(U) \subseteq A \subseteq U$. Thus, $\tau_2-\text{int}(\tau_1 - \delta\text{cl}(A)) \subseteq A$. □

Theorem 3.9. *[7] If A and B are $\tau_1\tau_2 - \delta$ semi open sets in a bitopological space (X, τ_1, τ_2). Then, $A \cup B$ is also $\tau_1\tau_2 - \delta$ semi open set.*

Remark 3.10. Let A and B be subsets of a bitopological space (X, τ_1, τ_2) and $A \cup B$ is a $\tau_1\tau_2 - \delta$ semi open set. Then, A,B need not be $\tau_1\tau_2 - \delta$ semi open sets.

Example 3.11. Let $X = \{a, b, c, d\}, \tau_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{d\}, \{a, d\}\}$.

Then, $\tau_1\tau_2 - \delta$ semi open sets are $\phi, X, \{a, b\}, \{a, b, c\}$. We have $\{a\} \cup \{b\} = \{a, b\}$. But $\{a\}, \{b\}$ are not $\tau_1\tau_2 - \delta$ semi open sets.

Theorem 3.12. *Let any subset $\{F_\alpha\}_{\alpha \in I}$ be a collection of $\tau_1\tau_2 - \delta$ semi closed sets in a bitopological space (X, τ_1, τ_2). Then, $\bigcap_{\alpha \in I} F_\alpha$ is $\tau_1\tau_2 - \delta$ semi closed.*

Proof. Since F_α is $\tau_1\tau_2 - \delta$ semi closed, there exists a $\tau_1 - \delta$ closed set U_α such that $U_\alpha \supseteq F_\alpha \supseteq \tau_2-\text{int}(U_\alpha), \forall\ \alpha \in I$. This implies, $\bigcap_{\alpha \in I}(\tau_2-\text{int}(U_\alpha)) \subseteq \bigcap_{\alpha \in I} F_\alpha \subseteq \bigcap_{\alpha \in I} U_\alpha$. $\tau_2-\text{int}(\bigcap_{\alpha \in I} U_\alpha) \subseteq \bigcap_{\alpha \in I} F_\alpha \subseteq \bigcap_{\alpha \in I} U_\alpha$. Thus, $\bigcap_{\alpha \in I} F_\alpha$ is $\tau_1\tau_2 - \delta$ semi closed. □

Theorem 3.13. *If A and B are $\tau_1\tau_2 - \delta$ semi closed sets in a bitopological space (X, τ_1, τ_2), then, so is $A \cap B$.*

Proof. Since A and B are $\tau_1\tau_2 - \delta$ semi closed sets, $\tau_2-\text{int}(\tau_1 - \delta\text{cl}(A)) \subseteq A$ and $\tau_2-\text{int}(\tau_1 - \delta\text{cl}(B)) \subseteq B$. This implies, $\tau_2-\text{int}(\tau_1 - \delta\text{cl}(A)) \cap \tau_2-\text{int}(\tau_1 - \delta\text{cl}(B)) \subseteq A \cap B$. So, $\tau_2-\text{int}(\tau_1 - \delta\text{cl}(A \cap B)) \subseteq A \cap B$. Hence, the result follows. □

Remark 3.14. If A and B are $\tau_1\tau_2 - \delta$ semi closed sets in a bitopological space (X, τ_1, τ_2). Then, $A \cup B$ need not be $\tau_1\tau_2 - \delta$ semi closed set.

Example 3.15. Let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. $\{b\}$ and $\{a\}$ are $\tau_1\tau_2 - \delta$ semi closed sets. But $\{a, b\}$ is not a $\tau_1\tau_2 - \delta$ semi closed set.

Remark 3.16. Let A and B be subsets of a bitopological space (X, τ_1, τ_2) and $A \cap B$ is a $\tau_1\tau_2 - \delta$ semi closed set. Then, A,B need not be $\tau_1\tau_2 - \delta$ semi closed sets.

Example 3.17. Let $X = \{a,b,c,d\}, \tau_1 = \{\phi, X, \{a\}, \{a,b\}, \{a,c,d\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{d\}, \{a,d\}\}$.

Then, $\tau_1\tau_2 - \delta$ semi closed sets are $\phi, X, \{c,d\}, \{d\}$. We have $\{a,d\} \cup \{b,d\} = \{d\}$. But $\{a,d\}, \{b,d\}$ are not $\tau_1\tau_2 - \delta$ semi closed sets.

Theorem 3.18. *If A and B are $\tau_2\tau_1 - \delta$ semi open sets in a bitopological space (X, τ_1, τ_2). Then, so is $A \cup B$.*

Proof. Since A and B are $\tau_2\tau_1 - \delta$ semi open sets, $\tau_1 - \mathrm{cl}(\tau_2 - \delta\mathrm{int}(A)) \supseteq A$ and $\tau_1 - \mathrm{cl}(\tau_2 - \delta\mathrm{int}(B)) \supseteq B$. This implies, $\tau_1 - \mathrm{cl}(\tau_2 - \delta\mathrm{int}(A)) \cup \tau_1 - \mathrm{cl}(\tau_2 - \delta\mathrm{int}(B)) \supseteq A \cup B$. So, $\tau_1 - \mathrm{cl}(\tau_2 - \delta\mathrm{int}(A \cup B)) \supseteq A \cup B$. Hence, the result follows. □

Theorem 3.19. *Let A, B be the subsets of bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) respectively. If $A \in \tau_2\tau_1 - \delta$ semi open in X and $B \in \sigma_1\sigma_2 - \delta$ semi open set in Y, then, $A \times B \in \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2 - \delta$ semi open set in $(X \times Y, \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2)$.*

Proof. Let $(x, y) \in A \times B$. Then, $x \in A$ and $y \in B$. Since $A \in \tau_2\tau_1 - \delta$ semi open in X and $B \in \sigma_1\sigma_2 - \delta$ semi open in Y, there exists $\tau_{2s} - \delta$ open set U in X and $\sigma_{1s} - \delta$ open set V in Y such that $U \subseteq A$ and $V \subseteq B$. Therefore, $(x, y) \in U \times V \subseteq A \times B$. Since $A \times B = \tau_{2s} - \delta\mathrm{int}_x(A) \times \sigma_{1s} - \delta\mathrm{int}_y(B) = \tau_2\sigma_1 - \delta\mathrm{int}_{x\times y}(A \times B)$. Hence, $A \times B \in \tau_2\sigma_1 - \delta$ open set in $X \times Y$. Since every $\tau_{2s} - \delta$ open set is $\tau_2\tau_1 - \delta$ semi open set. Thus, $A \times B \in \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2 - \delta$ semi open in $(X \times Y, \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2)$. □

Theorem 3.20. *Let A, B be the subsets of bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) respectively. If $A \in \tau_2\tau_1 - \delta$ semi closed in X and $B \in \sigma_1\sigma_2 - \delta$ semi closed set in Y, then, $A \times B \in \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2 - \delta$ semi closed set in $(X \times Y, \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2)$.*

Proof. Let $(x, y) \in A \times B$. Then, $x \in A$ and $y \in B$. Since $A \in \tau_2\tau_1 - \delta$ semi closed in X and $B \in \sigma_1\sigma_2 - \delta$ semi closed in Y, there exists $\tau_{2s} - \delta$ closed set U in X and $\sigma_{1s} - \delta$ closed set V in Y such that $A \subseteq U$ and $B \subseteq V$. Therefore, $(x, y) \in A \times B \subseteq U \times V$. Since $A \times B = \tau_{2s} - \delta\mathrm{cl}_x(A) \times \sigma_{1s} - \delta\mathrm{cl}_y(B) = \tau_2\sigma_1 - \delta\mathrm{cl}_{x\times y}(A \times B)$. Hence, $A \times B \in \tau_2\sigma_1 - \delta$ closed set in $X \times Y$. Since every $\tau_{2s} - \delta$ closed set is $\tau_2\tau_1 - \delta$ semi closed set. Thus, $A \times B \in \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2 - \delta$ semi closed in $(X \times Y, \tau_2 \times \sigma_1\ \tau_1 \times \sigma_2)$. □

Theorem 3.21. *Let A be subset of a bitopological space (X, τ_1, τ_2). Then, A is $\tau_2\tau_1 - \delta$ semi open in if and only if A is $\tau_{2s}\tau_{1s} -$ semi open in $(X, \tau_{1s}, \tau_{2s})$.*

Proof. Let A be $\tau_2\tau_1 - \delta$ semi open in (X, τ_1, τ_2). Then, $A \subseteq \tau_1-$cl$(\tau_{2s} - \delta$int $(A))$. Since $\tau_{1s} - \delta$ cl$(A) = \tau_1-$cl(A), $A \subseteq \tau_{1s}-$cl$(\tau_{2s}-$int $(A))$. Conversely, let A be $\tau_2\tau_1-$semi open in $(X, \tau_{1s}, \tau_{2s})$. Then, $A \subseteq \tau_1-$cl$(\tau_{1s}-$cl$(\tau_{2s}-$int$(A)))$. Since $\tau_{1s} \subseteq \tau_1$ and $\tau_{2s} \subseteq \tau_2$, we have $A \subseteq \tau_1-$cl$(\tau_{2s} - \delta$ int$(A))$. Hence, the result follows. □

4. Properties of $\tau_1\tau_2 - \delta$ Semi Continuous Functions

Definition 4.1. [7] Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. Then, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2-$continuous, if the inverse image of each σ_1- open set in Y is $\tau_1\tau_2-$ open set in X.

Example 4.2. Let $X = Y = \{a, b, c\}, \tau_1 = \{\phi, X, \{b\}, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{b, c\}\}$,$\sigma_1 = \{\phi, Y, \{b\}, \{a, b\}\}$ and $\sigma_2 = \{\phi, Y, \{a, c\}\}$.

Now, we consider $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as an identity function, then f is $\tau_1\tau_2-$continuous.

Definition 4.3. [2] Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces.Then, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2 - \delta$ continuous, if the inverse image of each σ_1- open set in Y is $\tau_1\tau_2 - \delta$ open set in X.

Example 4.4. Let $X = Y = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{b\}, \{a, b\}\}$, $\sigma_1 = \{\phi, Y, \{a\}, \{b, c\}\}$ and $\sigma_2 = \{\phi, Y, \{a, b\}\}$.

Then, $\tau_1\tau_2 - \delta$ open sets are $\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}$. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as $f(a) = a, f(b) = b, f(c) = b$, then f is $\tau_1\tau_2 - \delta$ continuous.

Definition 4.5. [2] Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces.Then, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\tau_1\tau_2 - \delta$ semi continuous, if $f^{-1}(V)$ is $\tau_1\tau_2 - \delta$ semi open set in X, for every $\sigma_1 - \delta$ open set V in Y.

Example 4.6. Let $X = Y = \{a, b, c\}, \tau_1 = \{\phi, X, \{b\}, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{b, c\}\}$,$\sigma_1 = \{\phi, Y, \{b\}, \{a, b\}\}$ and $\sigma_2 = \{\phi, Y, \{a, c\}\}$.

Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as $f(a) = a, f(b) = c, f(c) = b$, then f is $\tau_1\tau_2 - \delta$ semi continuous.

Remark 4.7. [7] If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ be two $\tau_1\tau_2 - \delta$ semi continuous functions, then $g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ need not be a $\tau_1\tau_2 - \delta$ semi continuous.

Theorem 4.8. *[2] Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ Then, f is $\tau_1\tau_2 - \delta$ semi continuous if and only if $f^{-1}(U)$ is $\tau_1\tau_2 - \delta$ semi closed in X, for each σ_1-closed* *set U in Y.*

Theorem 4.9. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $\tau_1\tau_2{-}\delta$ semi continuous. Then, $\forall\ \sigma_1{-}$open set V in Y, $\exists\ \tau_1\tau_2{-}\delta$ semi open set P in X such that $f(P) \subseteq V$.*

Proof. Assume that f is $\tau_1\tau_2 - \delta$ semi continuous. Then, by the previous theorem (theorem 4.6), $f^{-1}(U)$ is $\tau_1\tau_2 - \delta$ semi closed set in X, $\forall\ \sigma_1-$ closed set U in Y. Let V be σ_1- open set Y. Then, $Y\backslash V$ is σ_1- closed set in Y. This implies $f^{-1}(Y\backslash V)$ is $\tau_1\tau_2 - \delta$ semi closed set in X. But $f^{-1}(Y\backslash V) = f^{-1}(Y)\backslash f^{-1}(V) = X\backslash f^{-1}(V)$ is $\tau_1\tau_2 - \delta$ semi closed in X. So, $f^{-1}(V)$ is $\tau_1\tau_2 - \delta$ semi open in X. Let $P = f^{-1}(V)$. Thus, $f(P) = f(f^{-1}(V)) \subseteq V$. □

Proposition 4.10. *A constant function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\tau_1\tau_2 - \delta$ semi continuous.*

Proof. Let V be $\sigma_1{-}$open set in Y. If $c \in V$, then $f^{-1}(V) = X$ is $\tau_1\tau_2 - \delta$ semi open. If $c \notin V$, then $f^{-1}(V) = \phi$ is $\tau_1\tau_2 - \delta$ semi open. Since X, ϕ are $\tau_1\tau_2 - \delta$ semi open sets in X, f is $\tau_1\tau_2 - \delta$ semi continuous. □

5. Concluding Remarks

In this paper, Some results of $\tau_1\tau_2{-}\delta$ semi open sets/closed sets and $\tau_1\tau_2{-}\delta$ semi continuous functions in bitopological spaces have been discussed. We plan to extend our research work to $\tau_1\tau_2{-}$connectedness and $\tau_1\tau_2{-}$compactness. Further, we are interested to find some interesting results in bitopological spaces.

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References

[1] J.C. Kelly, Bitopological spaces, *Proceeding London Mathematical Society*, **13**, No. 3 (1963), 71-89.

[2] A. Edward Samuel and D. Balan, *$\delta-$interior and $\delta-$closure in Bitopological Spaces*, PhD Thesis, Bharath University, Chennai, Tamilnadu, India, 2015.

[3] S.N. Maheshwari, R. Prasad, Semi open sets and semi continuous functions in bitopological spaces, *Mathematical Not.*, **26** (1977-1978), 29-37.

[4] S. Bose, Semi open sets, semi continuity and semi open mappings in bitopological spaces, *Bulletins of Calcutta Mathematical Society*, **73** (1981), 237-246.

[5] K. Chandrasekhara Rao and V. Subha, A note on the associated bitopological space (X, τ_1^*, τ_2^*), *International Journal of Mathematical Analysis*, **7** (2003), 323-330.

[6] T. Noiri, V. Popa, Some properties of weakly open functions in bitopological spaces, *Novi Sad Journal Mathematics*, **36**, No. 1 (2006), 47-54.

[7] B. Bhattacharya and A. Paul, A new approach of $\delta-$open sets in bitopological spaces, *Gen. Math. Notes*, **20**, No. 2 (2014), 95-110.

[8] T. Fukutake, Semi open sets on bitopological spaces, *Bulletins of Fukuoka University Education*, **38**, No. 3 (1989), 1-7.

[9] Ivan L.Reilly, On pairwise connected bitopological spaces, *Kyungpook Mathematical Journal*, **11**, No. 1 (1971).

[10] Archana Pawar, Contra pairwise continuity in bitopological spaces, *International journal of Mathematics and Statistics Invention*, **1**, No. 2 (2013), 2321-4759.

[11] G.K. Banerjee, On pairwise almost strongly $\theta-$ continuous mapping, *Bulletin of Calcutta Mathematical Society*, **79** (1987), 314-320.

[12] F.H. Khedr, Properties of $\tau_1\tau_2-$delta open sets, *Fasciculi Mathematici*, **52** (2014), 65-81.