International Journal of Pure and Applied Mathematics
Volume 116 No. 2 2017, 385-389
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v116i2.10
ijpam.eu

ON THE WEIGHTED COMPOSITION OPERATORS ON SPECIAL SEQUENCE SPACES

Fatemeh Zangeneh[1], Bahmann Yousefi[2 §]
[1,2]Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, IRAN

Abstract: In this paper we study weighted composition operators on Banach spaces of formal power series and we investigate that when the numerical range of a weighted composition operator is closed.

AMS Subject Classification: 47B37, 47A16
Key Words: weighted composition operator, numerical range, compact operator, weak topology

1. Introduction

Let $\{\beta(n)\}_n$ be a sequence of positive numbers with $\beta(0) = 1$ and let $1 < p < \infty$. Let $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ be such that

$$\|f\|_p = \|f\|_{H^p(\beta)} = (\sum_{n=0}^{+\infty} |\hat{f}(n)|^p \beta(n)^p)^{1/p} < \infty.$$

The notation $f(z) = \sum_{n=0}^{+\infty} \hat{f}(n) z^n$ shall be used weather or not the series converges for any value of z. The space of such formal power series is called the weighted Hardy space, which is denoted by $H^p(\beta)$. In the case $p = 2$, the classical Hardy

Received: April 14, 2017
Revised: July 3, 2017
Published: October 7, 2017

§Correspondence author

space, Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1, \beta(n) = (n+1)^{-\frac{1}{2}}$ and $\beta(n) = (n+1)^{\frac{1}{2}}$, respectively. The spaces $H^p(\beta)$ are reflexive Banach spaces with the norm $\|.\|_p$ and the dual of $H^p(\beta)$ is $H^q(\beta^{\frac{p}{q}})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta^{\frac{p}{q}} = \{\beta(n)^{\frac{p}{q}}\}_n$.

Recall that for $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$ in $H^q(\beta^{p/q})$, note that

$$||g||_q^q = ||g||_{H^q(\beta^{p/q})}^q = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p < \infty.$$

If $\lim \frac{\beta(n+1)}{\beta(n)} = 1$ or $\liminf \beta(n)^{\frac{1}{n}} = 1$, then $H^p(\beta)$ consists of functions analytic on the open unit disk U.

A complex number λ is said to be a bounded point evaluation on $H^p(\beta)$ if the functional of point evaluation at λ, e_λ, is bounded. A complex number λ is a bounded point evaluation on $H^p(\beta)$ if and only if $\left\{ \frac{\lambda^n}{\beta(n)} \right\}_n \in l^q$.

We denote the set of multipliers

$$\{\varphi \in H^p(\beta) : \varphi H^p(\beta) \subseteq H^p(\beta)\}$$

by $H^p_\infty(\beta)$ and the operator of multiplication by φ on $H^p(\beta)$ by M_φ with $\|\varphi\|_\infty = \|M_\varphi\|$.

Let φ be an analytic self map of U and ψ be a multiplier of $H^p(\beta)$. A weighted composition operator $C_{\psi,\varphi}$ maps an analytic function $f \in H^p(\beta)$ into $(C_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z))$. The function φ is called the composition map and the function ψ is called the multiplier map. We will use the notations $H(U)$ and $C(\overline{U})$ to denote the set of analytic functions on U and the set of continuous functions on $\overline{U}$, the closure of U. Some sources on formal power series and composition operators include [1–7].

2. Main Result

In this section we study the numerical range of weighted composition operators acting on weighted Hardy spaces $H^p(\beta)$.

In the following we define some definitions that will be used in the main theorem.

Definition 2.1. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^p(\beta)$ and define $f^{*}(z) = \sum_{n=0}^{\infty} \hat{f}(n)|\hat{f}(n)|^{(p-q)/q} z^n$.

Note that $||f^{*}||_q^q = ||f^{*}||_{H^q(\beta^{p/q})}^q = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p = ||f||_p^p$ and obviously one can see that $< f, f^{*} >= ||f||_p^p$. Also,

Definition 2.2. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$ belongs to $H^q(\beta^{p/q})$ and define ${}^{*}g(z) = \sum_{n=0}^{\infty} \hat{g}(n)|\hat{g}(n)|^{(q-p)/p} z^n$.

Notice that $||{}^{*}g||_p^p = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p = ||g||_q^q < \infty$ and so ${}^{*}g \in H^p(\beta)$. Obviously, one can see that ${}^{*}(f^{*}) = f$ for all f in $H^p(\beta)$ and $({}^{*}g)^{*} = g$ for all g in $(H^p(\beta))^{*}$. Also, clearly $< g, {}^{*}g >= ||g||_q^q$.

Definition 2.3. If T is a bounded linear operator on $H^p(\beta)$, the numerical range of T is denoted by $W(T)$ that is defined by

$$W(T) = co\{< Tf, f^{*} >: f \in H^p(\beta) \text{ and } ||f||_p = 1\}.$$

Note that clearly $W(T) = co\{< T({}^{*}g), g >: g \in (H^p(\beta))^{*} \text{ and } ||g||_q = 1\}$.

Theorem 2.4. *Let* $\liminf \beta(n)^{\frac{1}{n}} = 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} = \infty$. *Suppose that* φ *is an analytic self map of* U *and there exists* ξ_0 *in* ∂U *such that the limit of* ${}^{*}e_\lambda(\varphi(\lambda))$ *exists and is finite as* $\lambda \to \xi_0$. *If* $\psi \in C(\overline{U}) \cap H_\infty^p(\beta)$, *then* $0 \in \overline{W}(C_{\psi,\varphi})$.

Proof. First note that since $\liminf \beta(n)^{\frac{1}{n}} = 1$ then for each λ in the open unit disk, the functional of evaluation at λ, e_λ, is a bounded linear functional and we have

$$e_\lambda(z) = \sum_{n=0}^{\infty} \frac{\bar{\lambda}^n z^n}{\beta(n)^p},$$

and

$$||e_\lambda||^q = \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q}.$$

Let $\lambda \in U$, then $e_\lambda \in (H^p(\beta))^{*}$ and so ${}^{*}e_\lambda \in H^p(\beta)$. Also, note that $||{}^{*}e_\lambda||_p^p = ||e_\lambda||_q^q = {}^{*} e_\lambda(\lambda)$. Put $E_\lambda = \frac{e_\lambda}{||e_\lambda||_q}$ and ${}^{*}E_\lambda = \frac{{}^{*}e_\lambda}{||{}^{*}e_\lambda||_p}$. Then $||{}^{*}E_\lambda||_p = ||E_\lambda||_q = 1$

and we have

$$\begin{aligned}
< C^{\square}_{\psi,\varphi} E_\lambda, E_\lambda > &= \frac{1}{||{}^{\square}e_\lambda||_p \, ||e_\lambda||_q} <^{\square} e_\lambda, C^{\square}_{\psi,\varphi} e_\lambda > \\
&= \frac{1}{||e_\lambda||_q^q} \psi(\lambda) <^{\square} e_\lambda, e_{\varphi(\lambda)} > \\
&= \psi(\lambda) ||e_\lambda||_q^{-q} \, {}^{\square}e_\lambda(\varphi(\lambda)),
\end{aligned}$$

which tends to 0 as $\lambda \to \xi_0$. So we get $0 \in \overline{W}(C_{\psi,\varphi})$. □

Theorem 2.5. *Let* $\frac{1}{p} + \frac{1}{q} = 1$, $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} = \infty$, $H^p(\beta) \subset H(U)$, *and* $C_{\psi,\varphi}$ *be compact on* $H^p(\beta)$. *Also suppose that* $\psi \in C(\overline{U}) \cap H^2_{\infty}(\beta)$ *and there exists* ξ_0 *in* ∂U *such that the limit of* ${}^{\square}e_\lambda(\varphi(\lambda))$ *is finite as* $\lambda \to \xi_0$. *Then* $0 \in W(C_{\psi,\varphi})$ *if and only if* $W(C_{\psi,\varphi})$ *is closed.*

Proof. Let $W(C_{\psi,\varphi})$ be closed, then by Theorem 2.4, $0 \in W(C_{\psi,\varphi})$. For the converse part, we show that $\overline{W}(C_{\psi,\varphi}) = W(C_{\psi,\varphi})$. Let $\alpha \in \overline{W}(C_{\psi,\varphi})$, then there exists a sequence $\{< C_{\psi,\varphi} h_n, h^{\square}_n\}_n$ converges to α where $||h_n||_p = ||h^{\square}_n||_q = 1$. Since ball$(H^p(\beta))$ is weakly compact, there exists a subsequence $\{h_{n_k}\}_k$ of $\{h_n\}_n$ that converges weakly to an element h in ball$(H^p(\beta))$. Now we have

$$\begin{aligned}
| < C_{\psi,\varphi} h_{n_k}, h^{\square}_{n_k} > - < C_{\psi,\varphi} h, h^{\square} > | \leq & < C_{\psi,\varphi} h_{n_k}, h^{\square}_{n_k} > - < C_{\psi,\varphi} h, h^{\square}_{n_k} > | \\
& + | < C_{\psi,\varphi} h, h^{\square}_{n_k} > - < C_{\psi,\varphi} h, h^{\square} > | \\
= & | < C_{\psi,\varphi}(h_{n_k} - h), h^{\square}_{n_k} > | \\
& + | < C_{\psi,\varphi} h, (h^{\square}_{n_k} - h^{\square}) > | \\
\leq & ||C_{\psi,\varphi}(h_{n_k} - h)|| \;\; ||h^{\square}_{n_k}|| \\
& + | < C_{\psi,\varphi} h, (h^{\square}_{n_k} - h^{\square}) > |.
\end{aligned}$$

Note that $||h^{\square}_{n_k}|| = ||h_{n_k}|| = 1$ and $h_{n_k} \to h$ weakly. Also, since $C_{\psi,\varphi}$ is a compact operator on $H^p(\beta)$, $||C_{\psi,\varphi}(h_{n_k} - h)|| \to 0$. Therefore, the sequence $\{< C_{\psi,\varphi} h_{n_k}, h^{\square}_{n_k} >\}_k$ converges to $< C_{\psi,\varphi} h, h^{\square} >$ and so $\alpha =< C_{\psi,\varphi} h, h^{\square} >$. Hence $\alpha \in cW(C_{\psi,\varphi})$ for some $c \in [0, 1]$. Without loss of generality let $c \neq 0$. Since $W(C_{\psi,\varphi})$ is convex and $0 \in W(C_{\psi,\varphi})$, thus $\alpha \in W(C_{\psi,\varphi})$. This implies that $W(C_{\psi,\varphi})$ is closed and so the proof is complete. □

References

[1] C.C. Cowen and B.D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC press, Boca Raton, 1995.

[2] J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York Inc. 1993.

[3] J.H. Shapiro, Compact composition operators on spaces of boundary-regular holomorphic functions, *Proc. Amer. Math. Soc.*, **100** (1987), 49-57.

[4] B. Yousefi, Bounded analytic structure of the Banach space of formal power series, *Rendiconti Del Circolo Matematico Di Palermo*, Serie II, **Tomo LI** (2002), 403-410.

[5] B. Yousefi, On the eighteenth question of Allen Shields, *International Journal of Mathematics*, **16**, No. 1 (2005), 37-42.

[6] B. Yousefi, Z. Kamali and L. Bagheri, Isometries of weighted Hardy spaces among composition operators, *Korean Annals of Math.*, **23**, No. 2 (2006), 119-125.

[7] B. Yousefi and Sh. Khoshdel, Reflexivity of powers of the multiplication operator on special function spaces, *Acta Mathematica Scientia*, **32B**, No. 6 (2012), 2279-2284.