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THE DECYCLING NUMBER OF (n,k)-ARRANGEMENT GRAPHS $A_{n,k}$

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Abstract: A subset $F \subset V(G)$ is called *a decycling set* if the subgraph $G-F$ is acyclic. The minimum cardinality of a decycling set is called *the decycling number* of G, which is proposed by Beineke and Vandell [1]. In this paper, we consider a particular topology graph called the (n,k)-arrangement graph $A_{n,k}$. We use $\nabla(A_{n,k})$ to denote the decycling number of $A_{n,k}$. This paper proves that

$$\nabla(A_{n,2}) = n(n-3)+1, n \geq 4.$$

AMS Subject Classification: 05C85, 05C85, 68R10

Key Words: graph theory, decycling set, decycling number, (n,k)-arrangement graph, acyclic subgraph

1. Introduction

Let $G=(V,E)$ be a simple graph, with vertex set V and edge set E. A subset

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$F \subset V(G)$ is called *a decycling set* if the subgraph $G - F$ is acyclic, that is, if $G - F$ is a forest. The minimum cardinality of a decyling set is called *the decycling number* (or *feedback number*) of G, which is proposed by Beineke and Vandell [1]. A decycling set of this cardinality is called a minimum decycling set (or *feedback set*).

Determining the decycling number of a graph G is equivalent to finding the greatest order of an induced forest of G proposed by Erdős, Saks and Sós [2], since the sum of the two numbers equals the order of G. A review of several results and open problems on the decycling number was provided by Bau and Beineke [3].

In fact, the problem of finding the decycling number is NP-hard for graphs in general [4] (also see[5]). The best known approximation algorithm for this problem has approximation ratio 2 [6]. Most recent research has been devoted to solving the problem for certain special graphs in polynomial time, e.g., reducible graphs [7], cocomparability graphs [8], convex bipartite graphs [8], cyclically reducible graphs [9], interval graphs [10] and hypercubes [11].

The decycling set problem has important application to several fields, for example, deadlock prevention in operation systems. Once a deadlock has been detected, a strategy is needed to break up the deadlock. Usually a deadlock in a system can be described by using a wait-for graph. In a wait-for graph, each vertex represents a process, and the existence of an edge(i, j) indicates that process i is waiting for process j to release a resource requested by process i. A deadlock exists in a system if and only if the corresponding wait-for graph contains a cycle. One of the best known approaches for solving the deadlock processes as possible in the wait-for graph. Using graph-theoretic terminology, the strategy is equivalent to finding a decycling set for such a system.

In this paper, we consider a particular topology graph called the (n, k)-arrangement graph $A_{n,k}$. We use $\nabla(A_{n,k})$ to denote the decycling number of $A_{n,k}$, this paper proves that $\nabla(A_{n,2}) = n(n-3)+1, n \geq 4$. The proof of the result is in Section 4. In Section 3, we obtain a lower bound of decycling number of $A_{n,2}$. In Section 2, we give some properties of $A_{n,k}$.

2. Properties of $A_{n,k}$

Day and Tripathi proposed a generalized star graph, called the arrangement graph, as an attractive interconnection scheme for massively parallel systems. An arrangement graph is specified by two parameters n and k, satisfying $1 \leq k \leq n-1$.

Let $\langle n\rangle = \{1,2,3,...,n\}$ and $\langle k\rangle = \{1,2,3,...,k\}$.

Definition 2.1. The (n,k)-arrangement graph, denoted by $A_{n,k} = (V,E)$, $1 \le k \le n-1$, is defined as follows:

$V = \{p_1p_2\dots p_k | p_i \in \langle n\rangle, and \;\; p_i \neq p_j \;\; for \; i \neq j\}$, and

$E = \{(p,q) | p,q \in \; V \; and \; for \; some \; i \in \; \langle k\rangle, p_i \neq q_i \; and \; p_j = q_j \; for \; j \neq i\}$.

That is, the nodes of $A_{n,k}$ are the arrangements of k elements out of the n symbols $\langle n\rangle$, and the edges of $A_{n,k}$ connect arrangements which differ in exactly one of their k positions. An edge of $A_{n,k}$ connecting two arrangements which differ only in position i is called an i-edge.

The (n,k)-arrangement graph $A_{n,k}$ is regular of degree $k(n-k)$, the number of nodes $n!/(n-k)!$, and diameter $\lfloor 3k/2 \rfloor$. The $(n,n-1)$-arrangement graph $A_{n,n-1}$ is isomorphic to the n-star graph S_n. The arrangement graph provides more flexibility than the star graph in terms of choosing the major design parameters: degree, diameter, and number of nodes. The arrangement graph has been shown to be node and edge symmetric, strongly hierarchical, maximally fault tolerant and strongly resilient [12].

An example of $A_{n,k}$ for $n=4$ and $k=2$ is given in Figure 1.

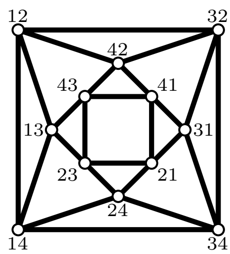

Figure 1: The arrangement graph $A_{4,2}$

There are $n!/(n-k)!$ nodes in $A_{n,k}$ which have element i in position j, for any fixed i and j $(1 \le i \le n, 1 \le j \le k)$. These nodes are interconnected in a manner identical to an $A_{n-1,k-1}$ graph. For a fixed position j, an $A_{n,k}$ can be partitioned into n node-disjoint copies of $A_{n-1,k-1}$. This partitioning of $A_{n,k}$ into n copies of $A_{n-1,k-1}$ can be done in k different ways corresponding to the k possible values of $j (1 \le j \le k)$ and can be carried out recursively. Figure 1 shows that $A_{4,2}$ can be viewed as an interconnection of four $A^{\prime}_{3,1}s$ by fixing the symbol in position 2.

Let $V_{n,k}$ denote the vertex set of $A_{n,k}$, $A_{n-1,k-1}(i)$ denotes a subgraph of $A_{n,k}$ induced by all the vertices with the same last symbol i for $1 \le i \le n$, and $V_{n-1,k-1}(i)$ denotes the vertex set of $A_{n-1,k-1}(i)$.

For simplicity, let $S[V]$ denote the subgraph of $A_{n,k}$ induced by vertex set V, and let $F_{n,k}$ denote the decycling set of $A_{n,k}$. Let $|S|$ denote the number of distinct elements of set S.

Lemma 2.1. *[13] $A_{n,k}$ can be decomposed into n subgraphs $A_{n-1,k-1}(i)$ $(1 \leq i \leq n)$ and each subgraph $A_{n-1,k-1}(i)$ is isomorphic to $A_{n-1,k-1}$.*

Since the $(n, n-1)$-arrangement graph $A_{n,n-1}$ is isomorphic to the n-star graph S_n and the vertex decycling number of S_n [14] is $\nabla(S_n) \geq \frac{(n-3)n!+2}{2(n-2)}$, for $n \geq 3$, then we only consider the situation of $1 \leq k \leq n-2$.

Lemma 2.2. $|F_{n,k}| \geq \frac{n!}{(n-k+1)!} \cdot (n-k-1)$ *for* $1 \leq k \leq n-2$.

Proof. Since $A_{n,1}$ is isomorphic to K_n and the decycling number of K_n is $n-2$, furthermore, $A_{n,k}$ can be decomposed into n subgraphs $A_{n-1,k-1}(i)$ $(1 \leq i \leq n)$ which is isomorphic to $A_{n-1,k-1}$, so we can conclude that

$$\begin{aligned}
&|F_{n,k}| \geq n \cdot |F_{n-1,k-1}| \\
&\geq n \cdot (n-1) \cdot |F_{n-2,k-2}| \geq \cdots \geq n \cdot (n-1) \cdots (n-k+2) \cdot |F_{n-k+1,1}| \\
&= n \cdot (n-1) \cdots (n-k+2) \cdot (n-k-1) \\
&= \frac{n!}{(n-k+1)!} \cdot (n-k-1)
\end{aligned}$$

for $1 \leq k \leq n-2$. □

3. Lower Bound of $\nabla(A_{n,2})$

As one can see the decycling number of K_n is $n-2$, which implies that each acyclic subgraph of $A_{n-1,1}(i)$ should contain at most 2 vertices. We arbitrarily choose 2 vertices, say $x_i i$ and $y_i i$, from each of the subgraph $A_{n-1,1}(i)$, $(1 \leq i \leq n)$, and denote

$R_i = \{x_i i, y_i i\} \subset V_{n-1,1}(i), i, x_i, y_i \in \langle n \rangle$ and i, x_i, y_i differ from each other.

Each subgraph $G[R_i]$ just has one edge called 1-edge (cf.Figure 2).

Denote $R = \bigcup_{i=1}^{n} R_i$, we give a useful lemma as follows.

Lemma 3.1. *$G[R]$ contains cycle.*

Proof. We porve the result by the following three cases.

Case 1. $x_1, x_2, \ldots, x_n$ differ from each other.

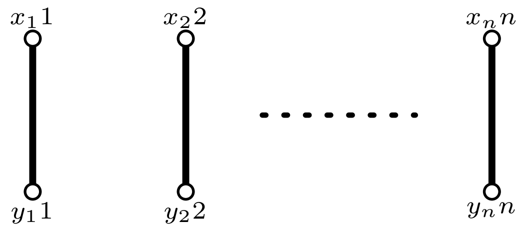

Figure 2: Each subgraph $G[R_i]$ of $A_{n,2}$

(a) If there exists $k(k \geq 3)$ identical elements of $y_1, y_2, \dots, y_n$, for example, $y_i = y_j = y_m$, then $G[R]$ contains the cycle $C(y_i i, y_j j, y_m m, y_i i)$, each edge of the cycle is 2-edge.

(b) If there exists 2 identical elements of $y_1, y_2, \dots, y_n$, for example, $y_i = y_j$, then in $\{x_1, x_2, \dots, x_n\} = \langle n \rangle$, there is exactly one element x_m such that $x_m = y_i = y_j$. So $G[R]$ contains the cycle $C(x_m m, y_i i, y_j j, x_m m)$.

(c) If $y_1, y_2, \dots, y_n$ are all different from each other, then $\langle n \rangle = \{x_1, x_2, \dots, x_n\} = \{y_1, y_2, \dots, y_n\}$ and $x_i \neq y_i$. For each $x_i (1 \leq i \leq n)$, there is exactly one element $y_j (1 \leq j \leq n, j \neq i)$ such that $y_j = x_i$. So $G[R]$ has 2-edge $(x_i i, y_j j)$, $(1 \leq i \leq n)$. $G[R]$ has n 2-edges and n 1-edges, so $G[R]$ has $2n$ edges. Whereas $G[R]$ has only $2n$ vertices, so we know $G[R]$ has at least one cycle.

Case 2. There exists k pair equal elements and there are not more than 2 elements equal to each other in $\{x_1, x_2, \dots, x_n\}$.

Without loss of generality, assume $x_1 = x_2, x_3 = x_4, \dots, x_{2k-1} = x_{2k}$. So, the $n - k$ elements $x_1, x_3, \dots, x_{2k-1}$, $x_{2k+1}, x_{2k+2}, \dots, x_n$ are all different from each other. The following is a subgraph of $G[R]$ (cf.Figure 3.), where the vertical edge is 1-edge and the horizontal edge is 2-edge.

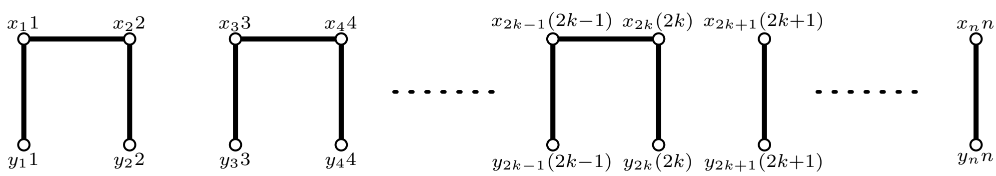

Figure 3: The subgraph $G[R]$ in Case 2.

(a) If there exists $k(k \geq 3)$ identical elements of $y_1, y_2, \dots, y_n$, for example, $y_i = y_j = y_m$, then $G[R]$ has the cycle $C(y_i i, y_j j, y_m m, y_i i)$. Each edge of the cycle is 2-edge.

(b) If $y_1, y_2, \ldots, y_n$ differ from each other, then $\langle n \rangle = \{y_1, y_2, \ldots, y_n\}$. For each element $x_{2i-1} \in \langle n \rangle$, $(1 \le i \le k)$, there is exactly one element $y_j \in \langle n \rangle$ such that $y_j = x_{2i-1} = x_{2i}$. So $G[R]$ contains the cycle $C(y_j j, x_{2i-1}(2i-1), x_{2i}(2i), y_j j)$.

(c) If there exists m pair equal elements in $\{y_1, y_2, \ldots, y_n\}$, and there are not more than 2 elements equal to each other in $\{y_1, y_2, \ldots, y_n\}$. We denote $y'_1 = y'_2, y'_3 = y'_4, \ldots, y'_{2m-1} = y'_{2m}$, and denote $\{z_1, z_2, \ldots, z_n\} = \langle n \rangle$, $\{y'_1, y'_2, \ldots, y'_n\} = \{y_1, y_2, \ldots, y_n\}$, and the vertex $y'_i z_i \in R(1 \le i \le n)$. So, the $n-m$ elements $y'_1, y'_3, \ldots, y'_{2m-1}, y'_{2m+1}, y'_{2m+2}, \ldots, y'_n$ are all different from each other(cf.Figure 4.).

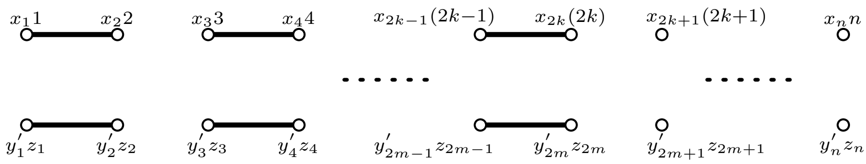

Figure 4: The subgraph $G[R]$ in Case 2(c)

Denote

$$A = \{x_1, x_3, \ldots, x_{2k-1}\}, \quad C = \{x_{2k+1}, x_{2k+2}, \ldots, x_n\},$$

and

$$B = \{y'_1, y'_3, \ldots, y'_{2m-1}\}, \quad D = \{y'_{2m+1}, y'_{2m+2}, \ldots, y'_n\}.$$

Obviously, $A \bigcap C = \emptyset$ and $B \bigcap D = \emptyset$.

(c.1) If $A \bigcap B \ne \emptyset$ or $A \bigcap D \ne \emptyset$ or $B \bigcap C \ne \emptyset$.

For $x_{2i-1} \in A$, $1 \le i \le k$, if x_{2i-1} is equal to an arbitrary element of $B \cup D = \{y'_1, y'_3, \ldots, y'_{2m-1}, y'_{2m+1}, y'_{2m+2}, \ldots, y'_n\}$, for example $x_{2i-1} = y'_j \in B \cup D$, then $G[R]$ contains the cycle $C(x_{2i-1}(2i-1), x_{2i}(2i), y'_j z_j, x_{2i-1}(2i-1))$.

Similarly, for $y'_{2i-1} \in B$, $1 \le i \le m$, if y'_{2i-1} is equal to an arbitrary element of $A \cup C = \{x_1, x_3, \ldots, x_{2k-1}, x_{2k+1}, x_{2k+2}, \ldots, x_n\}$, then $G[R]$ contains cycle too.

(c.2) If $A \bigcap B = \emptyset$, $A \bigcap D = \emptyset$ and $B \bigcap C = \emptyset$.

We first claim that $m = k$. Otherwise, assume $m < k$, then $|A| = k$, $|B| = m$, $|A \cup B| = m + k$. As $A \cup B \cup C \cup D \subseteq \langle n \rangle$, and A, B, C, D are disjoint sets with each other, we can easily know that $C \subseteq \langle n \rangle - A \cup B$, $D \subseteq \langle n \rangle - A \cup B$.

So, $|D| \leq |A \cup B| = n-(m+k)$. But $|D| = n-2m > n-(m+k)$, a contradiction, the same to $m > k$. Hence, $m = k$.

Since $|A| = k$, $|B| = k$, then $|A \cup B| = 2k$. As $|C| = |D| = n - 2k$ and $|\langle n \rangle - A \cup B| = n - 2k$, we can deduce that $C = D = \langle n \rangle - A \cup B$. So, $\{x_{2k+1}, x_{2k+2}, \dots, x_n\} = \{y'_{2k+1}, y'_{2k+2}, \dots, y'_n\}$. For each element $y'_j \in D$, there is exactly one element $x_i \in C$ such that $y'_j = x_i$. So there is 2-edge $(x_i i, y'_j z_j)$ in $G[R]$ $(2k+1 \leq j \leq n)$. Combined with Figure 4, the number of 2-edges in $G[R]$ is $(n-2k)+k+k = n$. From Figure 3, the number of 1-edges in $G[R]$ is n. So, $G[R]$ has $2n$ edges and $G[R]$ has $2n$ vertices. Thus we can deduce that $G[R]$ has at least one cycle.

Case 3. There exists $k (k \geq 3)$ identical elements of $\{x_1, x_2, \dots, x_n\}$.

If $x_i = x_j = x_k$, then $G[R]$ has the cycle $C(x_i i, x_j j, x_k k, x_i i)$, each edge of the cycle is 2-edge.

From Case 1,2 and 3, we conclude that $G[R]$ contains cycle. The Lemma holds. □

Lemma 3.2. $\nabla(A_{n,2}) \geq n(n-3)+1$.

Proof. From Lemma 3.1, the induced subgraph containing $2n$ vertices of $A_{n,2}$ contains cycle. So, in order to ensure no cycles of induced subgraph of $A_{n,2}$, we should remove at least $n(n-1) - 2n + 1 = n(n-3)+1$ vertices from $A_{n,2}$. Thus, $\nabla(A_{n,2}) \geq n(n-3)+1$.

Thus the lemma follows. □

4. Decycling Number of $A_{n,2}$

Let $T_{n-1,1}(i) = \{[i\%n+1]i, [(i+1)\%n+1]i\}$, $for\ 1 \leq i \leq n-1$ and $T_{n-1,1}(n) = \{1n\}$. Obviously, $T_{n-1,1}(i)$ is a subset of $V_{n-1,1}(i)$.

Denote $T = \bigcup_{i=1}^{n} T_{n-1,1}(i)$.

Theorem 4.1. $\nabla(A_{n,2}) = n(n-3)+1$ *for* $n \geq 4$.

Proof. We can easily prove that $G[T]$ is acyclic(cf. Figure.5.).

As $G[T]$ has exactly one path $P(21, 31, 32, 42, 43, 53, \cdots, (n-1)(n-2), n(n-2), n(n-1), 1(n-1), 1n)$, which is formed of alternating edges of 1-edge and 2-edge, it implies that $V_{n,2} \setminus T$ is the decycling set of $A_{n,2}$.

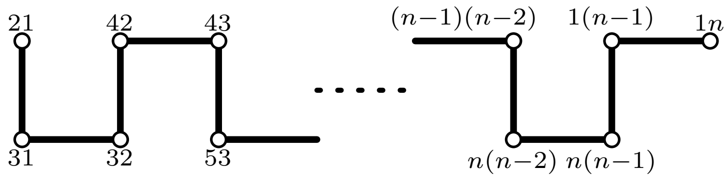

Figure 5: The acyclic subgraph $G[T]$ of $A_{n,2}$

So, we can deduce that

$$\begin{aligned}\nabla(A_{n,2}) &\leq |V_{n,2}\backslash T| \\ &= |V_{n,2}| - |T| \\ &= n(n-1) - (2n-1) \\ &= n(n-3) + 1\end{aligned}$$

Combining with Lemma 3.2, we obtain the decycling number of $A_{n,2}$ is

$$\nabla(A_{n,2}) = n(n-3) + 1.$$

This completes the proof of the theorem. □

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