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GAUSS-WINKLER TYPE INEQUALITY FOR SUGENO INTEGRALS

Dug Hun Hong
Department of Mathematics
Myongji University
Yongin Kyunggido, 449-728, SOUTH KOREA

Abstract: This paper propose a Gauss-Winkler type inequality for Sugeno integrals. Indeed, we find the optimal constant H for which the following Gauss-Winkler type inequality for fuzzy integrals

$$\left((S)\int_0^1 x^2 f(x)d\mu\right)^2 \leq H\left((S)\int_0^1 f(x)d\mu\right)\left((S)\int_0^1 x^4 f(x)d\mu\right)$$

holds where $f:[0,1]\to[0,\infty)$ is a nondecreasing function and μ is the Lebesgue measure on $\mathbb{R}$. Some examples are provided to illustrate the validity of the proposed inequality.

AMS Subject Classification: 26E50
Key Words: fuzzy measure, Sugeno integral, Gauss type inequality

1. Introduction and Preliminaries

A number of studies have examined the Sugeno integral since its introduction in 1974 [16], it has been exhaustively investigated by many authors. Ralescu and Adams [12] generalized a range of fuzzy measures and gave several equivalent definitions of fuzzy integrals. Wang and Klir [17] provided an overview of fuzzy measure theory.

Caballero and Sadarangani [2-4] proved a Hermite-Hadamard type inequality, a Cauchy type inequality, and Fritz Carlson's inequality for fuzzy integrals. Román-Flores et al. [13-15] presented several new types of inequalities for Sugeno integeals, including a Prekopa-Leindler type inequality, a Jensen type inequality, and some convolution type inequalities. Flores-Franulič et al. [5, 6] presented Chebyshev's inequality, Stolarsky's inequality for fuzzy integrals.

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Ouyang and Fang [10] generalized their main results to prove some optimal upper bounds for the Sugeno integral of monotone function in [14]. Ouyang et al. [9] generalized a Chebyshev type inequality for fuzzy integral of monotone functions based on an arbitrary fuzzy measure. Hong [7] improved on previous work presenting a Hardy-type inequality for Sugeno integrals. Hong [8] proposed a Liapunov type inequality for Sugeno integrals and find an optimal constant for which Liapunov type inequality for Sugeno integrals holds for non-increasing concave functions. Recently, Hong [9] proposed a Berwald type inequality and a Favard type inequality for Sugeno integrals.

In this paper, we propose a Gauss-Winkler type inequality for Sugeno integrals and find an optimal constant for which Gauss-Winkler type inequality for Sugeno integrals holds for nondecreasing functions. Some examples are provided to illustrate the validity of the proposed inequality.

Definition 1. Let Σ be a σ-algebra of subsets of $\mathbb{R}$ and let $\mu : \Sigma \to [0, \infty]$ be a non-negative, extended real-valued set function. We say that μ is a fuzzy measure if and only if

(a) $\mu(\emptyset) = 0$.

(b) $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity).

(c) $\{E_p\} \subseteq \Sigma$ and $E_1 \subseteq E_2 \subseteq \cdots$ imply $\lim_{p\to\infty} \mu(E_p) = \mu\left(\bigcup_{p=1}^{\infty} E_p\right)$ (continuity form below).

(d) $\{E_p\} \subseteq \Sigma$, $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_1) < \infty$ imply $\lim_{p\to\infty} \mu(E_p) = \mu\left(\bigcap_{p=1}^{\infty} E_p\right)$ (continuity form above).

If f is a non-negative real-valued function defined on $\mathbb{R}$, then we denote by $F_\alpha = \{x \in X | f(x) \geq \alpha\} = \{f \geq \alpha\}$ the α-level of f, for $\alpha > 0$, and $F_0 = \overline{\{x \in X | f(x) > 0\}} = \operatorname{supp}(f)$ is the support of f.

We note that

$$\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$$

If μ is a fuzzy measure on $A \subset \mathbb{R}$, then we define the following:

$$\mathfrak{F}^\mu(A) = \{f : A \to [0, \infty) | f \text{ is } \mu\text{-measurable}\}.$$

Definition 2. Let μ be a fuzzy measure on $(\mathbb{R}, \Sigma)$. If $f \in \mathfrak{F}^\mu(\mathbb{R})$ and $A \in \Sigma$, then the Sugeno integral(or the fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined as

$$(S)\int_A f d\mu = \sup_{\alpha\in[0,\infty)} [\alpha \wedge \mu(A \cap F_\alpha)].$$

In particular, if $A = X$ then

$$(S)\int_{\mathbb{R}} f d\mu = (S)\int f d\mu = \sup_{\alpha\in[0,\infty)} [\alpha \wedge \mu(F_\alpha)]$$

.

The following properties of the Sugeno integral are well known and can be found in [17]:

Proposition 1. (see [17]) *If μ is a fuzzy measure on $\mathbb{R}$ and $f, g \in \mathfrak{F}^\mu(\mathbb{R})$, then*

(i) $(S)\int_A f d\mu \leq \mu(A)$;

(ii) $(S)\int_A K d\mu = K \wedge \mu(A)$ *for any constant* $K \in [0,\infty)$;

(iii) $(S)\int_A f d\mu \leq (S)\int_A g d\mu$, *if* $f \leq g$ *on* A;

(iv) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Rightarrow (S)\int_A f d\mu \geq \alpha$;

(v) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (S)\int_A f d\mu \leq \alpha$;

(vi) $(S)\int_A f d\mu < \alpha \Leftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) < \alpha$;

(vii) $(S)\int_A f d\mu > \alpha \Leftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

Note 1. Let $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$, then by Proposition 1, (v), (vi),

$$F(\alpha) = \alpha \Rightarrow (S)\int_0^1 f(x) d\mu = \alpha.$$

Theorem 1. (see [10]) Let $f : [0,\infty) \to [0,\infty)$ be continuous and non-increasing or non-decreasing functions and μ be the Lebesgue measure on $\mathbb{R}$. Let

$$(S)\int_0^a f(x) d\mu = p.$$

If $0 < p < a$, then $f(p) = p$, $f(a-p) = p$, respectively.

2. Gauss-Winkler Type Inequality for Sugeno Integrals

The classical Gauss-Winkler inequality provides the following inequality [1, p 94]:

$$\left(\int_0^1 x^2 f(x)d\mu\right)^2 \leq \frac{5}{9}\left(\int_0^1 f(x)d\mu\right)\left(\int_0^1 x^4 f(x)d\mu\right) \tag{1}$$

where $f : [0,1] \to [0,\infty)$ is nondecreasing.

However, this inequality is not valid for the Sugeno integral as is shown in the following example.

Example 1. Let $f(x) = x$ for $x \in [0, 1/2]$ and $1/2$ for $x \in (1/2, 1]$. Then, a straightforward calculus shows that

$$(S)\int_0^1 f(x)d\mu = \frac{1}{2}.$$

By Theorem 1, by solving the equation $x = \frac{1}{2}(1-x)^2$, we obtain

$$(S)\int_0^1 x^2 f(x)d\mu = 2-\sqrt{3} \approx 0.2679,$$

and by solving the equation $x = \frac{1}{2}(1-x)^4$, we obtain

$$(S)\int_0^1 x^4 f(x)d\mu \approx 0.2024.$$

Consequently

$$\begin{aligned} 0.0718 \approx \left((S)\int_0^1 x^2 f(x)d\mu\right)^2 &> 0.0562 = (\frac{5}{9})(\frac{1}{2})(0.2024) \\ &\approx \frac{5}{9}\left((S)\int_0^1 f(x)d\mu\right)\left((S)\int_0^1 x^4 f(x)d\mu\right). \end{aligned}$$

Therefore, inequalities of (1) does not hold for Sugeno integrals.

The following result shows a Gauss-Winkler type inequality for Sugeno integrals.

Theorem 2. (Fuzzy Gauss-Winkler Inequality) Let $f : [0,1] \to [0,\infty)$ be a nondecreasing function and that μ the Lebesgue measure on $\mathbb{R}$. Then

$$\left((S)\int_0^1 x^2 f(x)d\mu\right)^2 \leq (2-\alpha)\left((S)\int_0^1 f(x)d\mu\right)\left((S)\int_0^1 x^4 f(x)d\mu\right) \tag{2}$$

where $0 < \alpha \leq 1$ satisfies the following equation

$$16\alpha(1-\alpha)^2 - \alpha(2-\alpha)^3 = 0.$$

Proof. Let

$$H = \sup\left\{ \frac{\left((S)\int_0^1 x^2 f(x)d\mu\right)^2}{\left((S)\int_0^1 f(x)d\mu\right)\left((S)\int_0^1 x^4 f(x)d\mu\right)} \middle| f : \text{nondecreasing on}[0,1] \right\}.$$

and let

$$H_\alpha = \sup\left\{ \frac{\left((S)\int_0^1 x^2 f(x)d\mu\right)^2}{\left((S)\int_0^1 f(x)d\mu\right)\left((S)\int_0^1 x^4 f(x)d\mu\right)} \middle| f : \text{nondecreasing}, \right.$$
$$\left. (S)\int_0^1 x^2 f(x)d\mu = \alpha \right\}.$$

Then

$$H = \sup_{0<\alpha<1} H_\alpha.$$

We consider H_α. Let

$$f_0(x) = \begin{cases} 0, & \text{if } x \in [0, 1-\alpha]), \\ \frac{\alpha}{(1-\alpha)^2}, & \text{if } x \in (1-\alpha, 1] \end{cases}$$

Then it is easy to check that

$$f_0 = \inf\left\{ f : \text{nondecreasing on}[0,1] \middle| (S)\int_0^1 x^2 f(x)d\mu = \alpha \right\}.$$

Hence we have

$$H_\alpha = \frac{\alpha^2}{\left((S)\int_0^1 f_0(x)d\mu\right)\left((S)\int_0^1 x^4 f_0(x)d\mu\right)}.$$

Because $\mu\{f_0 \geq \alpha\} = \alpha$, noting that $\frac{\alpha}{(1-\alpha)^2} \geq \alpha$, we have by Note 1

$$(S)\int_0^1 f_0(x)d\mu = \alpha.$$

Now, let

$$(S)\int_0^1 x^4 f_0(x)d\mu = x_0.$$

Since f_0 is continuous and increasing on $(1-\alpha, 1]$ and right limit of f_0 at $1-\alpha$ is less than α, then by Theorem 1,

$$x_0 = (1-x_0)^4 f_0((1-x_0)) = (1-x_0)^4 \frac{\alpha}{(1-\alpha)^2}$$

and hence

$$H_\alpha = \frac{(1-\alpha)^2}{(1-x_0)^4} = \frac{\alpha}{x_0}.$$

To find H we now consider the optimization problem:

$$\begin{aligned} H &= \sup_{0<\alpha<1} H_\alpha \\ &= \text{Maximize } \frac{\alpha}{x} \\ \text{subject to} \quad & x = (1-x)^4 \frac{\alpha}{(1-\alpha)^2}, \quad 0<\alpha<1. \end{aligned}$$

Let

$$f(x,\alpha) = \frac{\alpha}{x}, \quad g(x,\alpha) = \frac{\alpha}{(1-\alpha)^2} - \frac{x}{(1-x)^4}.$$

Then we have

$$\begin{aligned} \nabla f(x,\alpha) &= \left(-\frac{\alpha}{x^2}, \frac{1}{x}\right) \\ \nabla g(x,\alpha) &= \left(-\frac{1+3x}{(1-x)^5}, \frac{1+\alpha}{(1-\alpha)^3}\right). \end{aligned}$$

By using the Lagrange multiplier method, we have

$$f(x,\alpha) = \lambda \nabla g(x,\alpha), \quad g(x,\alpha) = 0$$

which imply that

$$-\frac{\alpha}{x^2} = \lambda \frac{1+3x}{(1-x)^5}, \quad \frac{1}{x} = \lambda \frac{1+\alpha}{(1-\alpha)^3}.$$

By solving these equations, we obtain the solution

$$x = \frac{\alpha}{2-\alpha}$$

where α satisfies the following equation.

$$g(x,\alpha) = \frac{\alpha}{(1-\alpha)^2} - \frac{x}{(1-x)^4} = \frac{16\alpha(1-\alpha)^2 - \alpha(2-\alpha)^3}{16(1-\alpha)^4} = 0$$

and the optimal value is

$$H = 2 - \alpha.$$

Note 2. If we solve the equation

$$16\alpha(1-\alpha)^2 - \alpha(2-\alpha)^3 = 0, \ 0 < \alpha < 1$$

with the aid of computer work, then we obtain

$$\alpha^* \approx 0.5745, \quad x^* \approx 0.4030.$$

Hence we can conclude that $f(x,\alpha)$ assumes maximum at (x^*,α^*) and the optimal value is

$$H = \frac{\alpha^*}{x^*} = 2 - \alpha^* \approx 1.4255.$$

Example 2. In Example 1, if we substitute 1.4255 to $\frac{5}{9}$, then the inequality (1) holds since

$$0.0718 \approx \left((S)\int_0^1 x^2 f(x) d\mu\right)^2$$
$$\leq (1.4255)\left(\int_0^1 f(x)d\mu\right)\left(\int_0^1 x^4 f(x)d\mu\right) \approx 0.1443.$$

The following example shows that the constant $H \approx 1.4255$ in Theorem 2 is optimal for Inequality (2).

Example 3. Let $f(x) = 0$ for $x \in [0, 0.4255]$ and 3.1731 for $x \in (0.4255, 1]$. Then, a straightforward calculus shows that

$$(S)\int_0^1 f(x)d\mu = 0.5745.$$

By Theorem 1, by solving the equation $x = 0.5745(1-x)^2$, we obtain

$$(S)\int_0^1 x^2 f(x)d\mu \approx 0.5745,$$

and by solving the equation $x = 0.5746(1-x)^4$, we obtain

$$(S)\int_0^1 x^4 f(x)d\mu \approx 0.4030.$$

Consequently

$$\left((S)\int_0^1 x^2 f(x)d\mu\right)^2 \approx 0.3300,$$

$$1.4255\left((S)\int_0^1 f(x)d\mu\right)\left((S)\int_0^1 x^4 f(x)d\mu\right) \approx (1.4255)(0.5745)(0.4030) = 0.3300.$$

Therefore, the constant $H \approx 1.4255$ in Theorem 2 is optimal.

The case for non-increasing function is similar.

Theorem 3. (Fuzzy Gauss-Winkler Inequality) Let $f : [0,1] \to [0,\infty)$ be a non-increasing function and that μ the Lebesgue measure on $\mathbb{R}$. Then

$$\left((S)\int_0^1 (1-x)^2 f(x)d\mu\right)^2 \leq 1.4255\left((S)\int_0^1 f(x)d\mu\right)\left((S)\int_0^1 (1-x)^4 f(x)d\mu\right).$$

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