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POLYNOMIAL CONNECTION OF GRIFONE AND ITS ASSOCIATED LIE ALGEBRAS

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Abstract: The aim of our work is to study some properties of a connexion within the sense of Grifone to polynomial coefficients and certain geometrical structures of associated Lie algebras.

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1. Introduction

Several authors have worked on Lie algebras and the corresponding connections, particularly [1], [4], [8], [7]. In our papers [10] and [12] we have studied the Lie algebras of polynomial vector fields on $\mathbb{R}^n$. These studies were extended in [10] by combining the Lie algebra with a connection within the sense of Grifone Γ on a differentiable manifold M of class C^∞ in order to give some properties of these algebras and compute on the one hand the cohomology

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space of Chevalley-Eilenberg of the horizontal part of the Lie algebra of vector fields on the tangent bundle of M whose the corresponding Lie derivative of Γ is zero, and on the other hand, the nullity of horizontal space of the curvature. Lie algebras defined by a 1-vector form on the tangent bundle TM of the differentiable manifold M of class C^∞ are reviewed by [6]. In particular, the nullity space associated to curvature with L and the first cohomology space of Chevalley-Eilenberg of these Lie algebras were addressed. In addition, [11] gave an outline of the linear connection with polynomial coefficients to a given Ricci curvature in $\mathbb{R}^n$ and particularly on Lie algebras attached to this connection withing the sense of Grifone. In this paper, we would like to provide some views on the associated Lie algebra with a connection of Grifone Γ to polynomial coefficients. Since Γ an almost-product structure on the tangent bundle TM, that is to say, $\Gamma^2 = I$, where I denotes the identity map; it defines a connection of Grifone on M see.[10]. We prove that all attached Lie algebra to a connection Γ with polynomial coefficients is a Lie algebra of the polynomial vector fields. In addition, we find that if the connection Γ is to polynomial coefficients of infinite degree on a differentiable manifold of n $(1 \leq n \leq 4)$ dimensional such that $\Gamma^2 = I$ then the associated curvature R is zero. Then we notice that the nullity fields of the curvature R corresponding to a Grifone connection which its coefficients are polynomials, are polynomial vector fields. Considering also that $I^p = \cap_{q \neq p} Ker\Gamma^q$, $1 \leq p \leq n$ and Γ is bi-invariante on M then I^p is an ideal of the Lie algebra A_Γ. In case where elements of this ideal are diagonal polynomial vector fields on $\chi(M)$, we find that I^p is commutative. Wereas, we denote A_X^Γ the Lie algebra of polynomial vector fields associated to the connection Γ, if Γ is bi-invariante on vector fields X, Y of A_X^Γ on M then this algebra is a distribution on M, in other words $[X, Y] \in A_X^\Gamma$. We find that $A_\Gamma \cap \mathfrak{N}_R$ coincides to its normalizer and all derivation of this Lie algebra is inner. In other words, the first cohomology space of Chevalley-Eilenberg of $A_\Gamma \cap \mathfrak{N}_R$ amounts to zero. In addition, we suppose that the connection of Grifone is triangularizable. So the nullity space $\mathfrak{N}_R$ of the curvature R attached to Γ is involutive but the Lie algebra A_Γ is not semisimple. If all eigenvalues of the connection Γ are not multiple roots of A_Γ then this previous algebra is semisimple. Let's suppose that the connection Γ is bi-invariante (or invariante) for any vector fields of TM, so that the nullity space $\mathfrak{N}_R$ is involutive. In the same hypothesis, we find equivalent propositions between elements of A_Γ, the derivation of the intersection of this Lie algebra with the nullity space $\mathfrak{N}_R$, as well as the characteristic of the connection Γ. Examples have been given for a better illustration of our results.

2. Preliminary

Let be M a differentiable manifold $n-$dimensionnal, TM is the tangent fiber of M. Any objects are supposed C^∞ on M or TM. The set $\chi(M)$ (resp. $\chi(TM)$) denotes the Lie algebra of vector fields on M (resp. TM) with the habitual bracket of vector fields.

Definition 2.1. [10] We have an exacte sequence of vector fiber on the tangent bundle TM of M.

$$0 \to \pi^*(TM) \xrightarrow{i} TTM \xrightarrow{j} \pi^*(TM) \to 0$$

$\pi : TM \to M$ is a projection from tangent fiber to M; $p : TTM \to TM$ is a projection from tangent bundle to TM; i the natural injection; $j = (p, \pi_*)$ where π_* is the tangent linear mapping of π. The mapping $J = i \circ j$ is the tangent structure on TM.

Definition 2.2. [10] Let Γ be a connection withing the sense of Grifone. By definition, Γ is a vector $1-$forme on the tangent bundle TM, C^∞ on $\tau M = TM - \{0\}$ and such that $J\Gamma = J$ and $\Gamma J = -J$, where J denotes the natural structure on TM. We put $h = \frac{1}{2}(I + \Gamma)$ and $v = \frac{1}{2}(I - \Gamma)$, h is an horizontal projector from the eigen subspace of eigenvalue $+1$, v is the vertical projector from the eigen subspace of eigenvalue -1. That connection Γ permits to obtain a tangent bundle decomposition TTM of TM in direct sum of horizontal and vertical spaces :

$$TTM = H(TM) \oplus V(TM)$$

with $H(TM) = Im(h) = Ker(v)$ and $V(TM) = Im(v) = Ker(h)$. The curvature R attached to Γ is defined by $R = \frac{1}{2}[h, h]$, naturally we have, $R = \frac{1}{2}[\Gamma, \Gamma]$, such that for any $X, Y \in \chi(\tau M)$, $R(X, Y) = [\Gamma[X, Y], Z] + [[X, Y], Z] - \Gamma[\Gamma[X, Y], Z] - \Gamma[[X, Y], \Gamma Z]$. In local coordinates (x^i, y^j) of τM, Γ is written $dx^i \otimes \frac{\partial}{\partial x^i} - 2\Gamma^i_j dx^i \otimes \frac{\partial}{\partial y^j} - dy^i \otimes \frac{\partial}{\partial y^i}$.

$$R = \frac{1}{2} R^i_{jk} dx^j \wedge dx^k \otimes \frac{\partial}{\partial y^i} \text{ where } R^i_{jk} = \frac{\partial \Gamma^i_k}{\partial x^j} - \frac{\partial \Gamma^i_j}{\partial x^k} + \Gamma^l_k \frac{\partial \Gamma^i_j}{\partial y^l} - \Gamma^l_j \frac{\partial \Gamma^i_k}{\partial y^l}.$$

The nullity space of the curvature R of Γ is

$$\mathfrak{N}_R = \{X \in \chi(\tau M) \text{ such that } R(X, Y) = 0, \text{ for all } Y \in \chi(\tau M)\}.$$

This nullity space is a distrubution of τM. In general, $\mathfrak{N}_R$ is not involutive.

3. On the connection of Grifone to polynomial coefficients

In the next, we suppose that the connection Γ is within the sense of Grifone or simply connection of Grifone to polynomial coefficients, except with expresse mention.

Definition 3.1. [10] Let Γ be a connection of Grifone on the differentiable manifold M of class $\mathcal{C}^\infty$ and n- dimensional. We define the Lie algebra associated to the connection Γ by

$$A_\Gamma = \{X \in \chi(M) \text{ such that } [X, \Gamma] = 0\}.$$

Proposition 3.2. *Any Lie algebra associated to a connection within the sense of Grifone with polynomial coefficients is a Lie algebra of polynomial vector fields.*

Proof. Let Γ a connection of Grifone, $X = X^i \frac{\partial}{\partial x^i} \in A_\Gamma$, where X^i with $i = 1, \ldots, n$ are polynomials of class $\mathcal{C}^\infty$ on M. We have $[X, \Gamma] = 0$ amounts to saying that the following system of n^2 partial derivative equations is satisfied:

$$X^i \frac{\partial \Gamma^j_k}{\partial x^i} - \Gamma^j_i \frac{\partial X^i}{\partial x^k} + \Gamma^i_k \frac{\partial X^j}{\partial x^i} = 0 \text{ where } i, j, k \in \{1, \ldots, n\}.$$

By solving this system, we obtain that $X^i,\ i, j, k \in \{1, \ldots, n\}$ are polynomials. Thus X is a polynomial vector fields. □

Example 3.3. Let $M = \mathbb{R}^3$, $\left(x^i\right)_{1 \leq i \leq 3}$ the local coordinates in $T\mathbb{R}^3$, Γ a connection within the sense of Grifone which components are: $\Gamma^1_2 = e^{x^1}$, $\Gamma^2_3 = x^1 e^{x^2}$ and $\Gamma^i_j = 0$ elsewhere.

Let be $X = X^i \frac{\partial}{\partial x^i}$ where $i = 1, 2, 3$. We know that $X \in A_\Gamma \Leftrightarrow X^i \frac{\partial \Gamma^j_k}{\partial x^i} - \Gamma^j_i \frac{\partial X^i}{\partial x^k} + \Gamma^i_k \frac{\partial X^j}{\partial x^i}$. We put

$$\begin{cases} G^1_{jk} = X^1 \frac{\partial \Gamma^j_k}{\partial x^1} - \Gamma^j_1 \frac{\partial X^1}{\partial x^k} + \Gamma^1_k \frac{\partial X^j}{\partial x^1} \\ G^2_{jk} = X^2 \frac{\partial \Gamma^j_k}{\partial x^2} - \Gamma^j_2 \frac{\partial X^2}{\partial x^k} + \Gamma^2_k \frac{\partial X^j}{\partial x^2} \\ G^3_{jk} = X^3 \frac{\partial \Gamma^j_k}{\partial x^3} - \Gamma^j_3 \frac{\partial X^3}{\partial x^k} + \Gamma^3_k \frac{\partial X^j}{\partial x^3} \end{cases}$$

Thus

$$[X,\Gamma]=0\Leftrightarrow\begin{cases} X^1e^{x^1}+e^{x^1}\frac{\partial X^1}{\partial x^1}=0 & (1)\\ e^{x^1}\frac{\partial X^2}{\partial x^1}+X^1e^{x^1}=0 & (2)\\ e^{x^1}\frac{\partial X^3}{\partial x^1}=0 & (3)\\ -e^{x^1}\frac{\partial X^2}{\partial x^1}-e^{x^1}\frac{\partial X^2}{\partial x^2}-e^{x^1}\frac{\partial X^2}{\partial x^3}+x^1e^{x^2}\frac{\partial X^1}{\partial x^2}=0 & (4)\\ X^2x^1e^{x^2}+x^1e^{x^2}\frac{\partial X^1}{\partial x^2}=0 & (5)\\ x^1e^{x^2}\frac{\partial X^3}{\partial x^2}=0 & (6)\\ x^1e^{x^2}\left(\frac{\partial X^3}{\partial x^1}+\frac{\partial X^3}{\partial x^2}+\frac{\partial X^3}{\partial x^3}\right)=0 & (7)\end{cases}$$

By (3), we have $X^3=X^3\left(x^2,x^3\right)$. With (7), we obtain $\frac{\partial X^3}{\partial x^3}=0$ implies that $X^3=k=C\left(x^1,x^2\right)$. Consequently $X^3=X^3\left(x^2\right)$. The equation (1) gives us $\frac{\partial X^1}{\partial x^1}=-X^1$, it amounts to saying that $\frac{\partial X^1}{\partial X^1}=-\partial x^1$. By member to member integration, we have $X^2=e^{-x^1}e^{C\left(x^2,x^3\right)}+K\left(x^2,x^3\right)$ where $X^2=X^1+K\left(x^2,x^3\right)$. Thus X is of the form $X=e^{-x^1}e^{C\left(x^2,x^3\right)}\frac{\partial}{\partial x^1}+\left(e^{-x^1}e^{C\left(x^2,x^3\right)}+Q\left(x^2,x^3\right)\right)\frac{\partial}{\partial x^2}+$ $k\frac{\partial}{\partial x^3}+\ldots$.
Hence A_Γ is a Lie algebra of the polynomial fields.

Proposition 3.4. *If Γ is a connection within the sense of Grifone whose coefficients are all homogeneous polynomials of infinite degree on the differentiable manifold of n $(1\le n\le 4)$ such that $\Gamma^2=I$ then the curvature R attached to Γ is zero $(R=0)$.*

Proof. Obvious. □

Example 3.5. Let $M=\mathbb{R}^3$, $\left(x^i\right)_{1\le i\le 3}$ be the local coordinates on $T\mathbb{R}^3$, Γ a connection such that its components are: $\Gamma_2^1=e^{x^1}$, $\Gamma_1^2=e^{-x^1}$, $\Gamma_3^3=1$ and $\Gamma_j^i=0$ elsewhere. We have $\Gamma^2=I$. Any components of the curvature R associated to Γ are null. Hence $R=0$.

Definition 3.6. We call mixed polynomial fields the vector fields $P\left(x^i\right)\frac{\partial}{\partial x^j}$ with $i,j\in\mathbb{N}^*$, $j\ne i$ where P is a polynomial function depending on $\left(x^i\right)$, $1\le i\le n$.

Proposition 3.7. *Let Γ be a connection within the sense of Grifone whose coefficients are mixed polynomials of finite (resp. infinite) degree on M. The curvature R associated to Γ is then of polynomial coefficients of finite (resp. infinite) degree.*

Proof. In local coordinates (x^i, y^i), $1 \le i \le n$, of the tangent bundle TM of M, the curvature R is written

$$R^i_{jk} = \frac{\partial \Gamma^i_k}{\partial x^j} - \frac{\partial \Gamma^i_j}{\partial x^k} + \Gamma^l_k \frac{\partial \Gamma^i_j}{\partial y^l} - \Gamma^l_j \frac{\partial \Gamma^i_k}{\partial y^l}. \tag{3.1}$$

Considering that coefficients of Γ are mixed polynomials of finite degree on M, we obtain that partial derivatives $\frac{\partial \Gamma^i_k}{\partial x^j}, \frac{\partial \Gamma^i_j}{\partial x^k}$ are either zeros or polynomials of degree less than or more than that of the Γ. For expressions $\Gamma^l_k \frac{\partial \Gamma^i_j}{\partial y^l}$ and $\Gamma^l_j \frac{\partial \Gamma^i_k}{\partial y^l}$. After having computed these derivations we found that sometimes the terms become all zeros. Otherwise, they are constants or polynomials of less or more degree than the one of Γ. by adding these different terms up, we always have the degree corresponding to that of Γ. □

Example 3.8. Let $M = \mathbb{R}^3$ a differentiable manifold of finite dimensional, (x^j) where $1 \le j \le 3$ the natural components, Γ^i_j with $1 \le i, j \le 3$ the components of the connection within the sense of Grifone defined by $\Gamma^1_3 = 1$, $\Gamma^2_1 = x^1x^2x^3$, $\Gamma^2_2 = 1$, $\Gamma^3_1 = x^1x^2x^3$, $\Gamma^3_3 = -1$ and $\Gamma^i_j = 0$ elsewhere.
We know that the coefficients of the curvature R are given by the relation (3.1) thus we have an almost product structure, that is $\Gamma^2 = I$. The components of the curvature R corresponding to Γ are: $R^2_{12} = R^3_{12} = x^1x^2 = -R^2_{21} = -R^3_{21}$ and $d\left(R^2_{12}\right) = d\left(R^3_{12}\right) = \cdots = 1$, $R^2_{13} = R^3_{13} = x^1x^2 = -R^2_{31} = -R^3_{31}$ and $R^i_{jk} = 0$ elsewhere. Thus any coefficients of R are polynomials.

Example 3.9. Let's consider a differentiable manifold of finite dimension $M = \mathbb{R}^3$, (x^i, y^i) where $1 \le j \le 3$ the local coordinates. Let Γ be a connection which the coefficients are $\Gamma^1_1 = 1$, $\Gamma^2_1 = x^2y^2y^3$, $\Gamma^2_2 = 1$, $\Gamma^3_1 = x^1y^2y^3$, $\Gamma^3_3 = 1$ and $\Gamma^i_j = 0$ elsewhere.
According to computation, we find that coefficients of the curvature R are given by: $R^2_{12} = R^3_{12} = \left(y^2 - x^2\right) y^3 = -R^2_{21} = -R^3_{21}$, $R^2_{13} = -x^2y^2 = -R^2_{31}$ and $R^3_{13} = -x^2y^2 = -R^3_{31}$ and $R^i_{jk} = 0$ elsewhere. Then R is to polynomial coefficients.

Proposition 3.10. *Let Γ be a connection on M. Then the nullity fields of the curvature R associated to Γ are polynomial fields.*

Proof. Let Γ be a connection on M, X a nullity field of the curvature R, we have $X^iR^k_{ij} = 0$ where $k \in \{1, \dots, n\}$. According to Proposition 3.7, components R^k_{ij} of R are polynomials. By resolving the equation systems $X^iR^k_{ij} = 0$ with $k \in \{1, \dots, n\}$, we find that X^i, $\quad i \in \{1, \dots, n\}$ are polynomials may be null on M. Hence the proof. □

Example 3.11. Let $M=\mathbb{R}^3$ be a differentiable manifold of finite dimension, $\left(x^i,y^i\right)$ where $1\leq j\leq 3$ the local coordinates. Consider Γ a connection defined by : $\Gamma^2_1=x^2$, $\Gamma^2_3=x^2y^2$ and $\Gamma^i_j=0$ elsewhere.
After computation, we find that coefficients of the curvature R are given by: $R^2_{12}=-1=-R^2_{21}$, $R^2_{13}=-\left(x^2\right)^2=-R^2_{31}$, $R^2_{23}=y^2=-R^2_{32}$ and $R^i_{jk}=0$ elsewhere. Let $X\in\mathfrak{N}_R$ be, so $X^jR^2_{jk}=0$ with $k=1,2,3$. This system is equivalent to

$$\left\{\begin{array}{l} X^2R^2_{21}+X^3R^2_{31}=0\\ X^1R^2_{12}+X^3R^2_{32}=0\\ X^1R^2_{13}+X^2R^2_{23}=0\end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} X^2+\left(x^2\right)^2X^3=0\\ -X^1-y^2X^3=0\\ -\left(x^2\right)^2X^1+y^2X^2=0\end{array}\right.$$

We obtain $X^1=-y^2X^3$, $\quad X^2=-\left(x^2\right)^2X^3$ where X^3 is a polynomial on M. A nullity field is then written: $X=-y^2X^3\frac{\partial}{\partial x^1}-\left(x^2\right)^2X^3\frac{\partial}{\partial x^2}+X^3\frac{\partial}{\partial x^3}+Y^i\frac{\partial}{\partial y^i}$.

4. On the invariance of connection Γ

Always supposing that coefficients of the connection Γ are polynomial vector fields.

Definition 4.1. A connection Γ is said invariante of A_Γ on M if for all $X\in A_\Gamma$,
$\Gamma^k\left[X,Y\right]=\left[\Gamma^kX,Y\right]$, $k=1,2,...$ for $Y\in\chi(M)$.

Proposition 4.2. *The connection Γ is bi-invariante of A_Γ on M. In other words,*
$\Gamma^k\left[X,Y\right]=\left[\Gamma^kX,Y\right]=\left[X,\Gamma^kY\right]$, $k=1,2,...$ *for* $Y\in\chi(M)$.

Proof. Obvious. □

Let be $I^p=\cap_{q\neq p}Ker\Gamma^q$ with $1\leq p\leq n$.

Proposition 4.3. *If Γ is bi-invariante on M then $I^p(1\leq p\leq n)$ is an ideal of A_Γ.*

Proof. Immediate. Because, let $X\in I^p$ be, we have $\Gamma^qX=0$ with $q\neq p$. Thus, $\Gamma^q\left[X,Y\right]=\left[\Gamma^qX,Y\right]=0$ for $Y\in A_\Gamma$ and I^p is ideal. Hence the result. □

Proposition 4.4. *If any elements of I^p are diagonal polynomial vector fields on $\chi(M)$ then I^p is a commutative ideal.*

Proof. Suppose that elements of I^p are diagonal fields on $\chi(M)$, we have $I^p \subset H^d_k$ where H^d_k denotes the set of diagonal polynomial fields of k degree. Now, according to the Theorem 3.10 of [12], H^d_k is a Lie algebra of commutative polynomial fields . Thus I^p is a commutative ideal. □

Proposition 4.5. *Let A^Γ_X be the associated Lie algebra of polynomial vector fields to the connection Γ. If Γ is bi-invariante on vector fields X, Y of A^Γ_X on M, then this algebra is a distribution on M. In other words, $[X, Y] \in A^\Gamma_X$.*

Proof. Let X, Y be two polynomial vector fields on $\chi(M)$ such that $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ with $1 \leq i, j \leq n$. According to an involutive distribution hypothesis in [9] and the Jacobi identity on X, Y, Z of $\chi(M)$, $A^\Gamma_{X,Y}$ is a distribution on M if for any X, Y of $A^\Gamma_{X,Y}$, so we obtain $[X, Y] \in A^\Gamma_{X,Y}$. The annulation of the curvature associated to the connection Γ can be written $R(X, Y) = 0$, then according to calculation we have $[X, Y] = -\frac{1}{3}[\Gamma X, \Gamma Y]$. As X and Y are polynomial vector fields then ΓX and ΓY also are. By applying the invariance of Γ on M, $[\Gamma X, \Gamma Y]$ is a polynomial vector fields. Thus $[\Gamma X, \Gamma Y] \in A^\Gamma_{X,Y}$. Hence the proof. □

Let Γ be a connection which all components are polynomes of degree finite on M; let $(x^i)_{1 \leq n \leq n}$ be the locale coordinates of tangent bundle τM of M.

Proposition 4.6. *The normalizer of Lie algebra $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ coincides to $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$. In other words, $\mathcal{N}(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R) \equiv \mathfrak{A}_\Gamma \cap \mathfrak{N}_R$.*

Proof. We know that $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R \subset \mathcal{N}(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$. What's left to bedone is to show the reverse inclusion. So let $X \in \mathcal{N}(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$ be and $Y = X^j \frac{\partial}{\partial x^j}, j = 1 \ldots n, \in \mathfrak{A}_\Gamma \cap \mathfrak{N}_R$. According to invariance of Γ, it is easy to find that $[X, Y] = Z^i \frac{\partial}{\partial x^j}, 1 \leq i, j \leq n$. This ends the proof. □

Theorem 4.7. *Any derivation of Lie algebra $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is inner. Its first space of Chevalley-Eilenberg cohomology is zero.*

Proof. We only have to apply the Lie derivation of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ by taking a vector fields of $\mathcal{N}(\mathfrak{A}_\Gamma \cap \mathfrak{N}_R)$. According to Theorem 2.10 of 3.4, the derivation of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is inner to a vector fields of its normalizer. According to the Proposition 4, the derivation of $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ is inner. It's immediate to find that the first space of Chevalley-Eilenberg cohomology is zero. □

5. On the diagonalization of the connection Γ

Definition 5.1. Let Γ be a connection within the sense of Grifone which coefficients are polynomial functions. We say that Γ is diagonalizable (As a vector 1−form) to eigenvalues $\lambda_i, 1 \leq i \leq p$ if there exists a vector fields $X_i \in \chi(M)$ such that $(\Gamma - \lambda_i.I)^{n_i}.X_i = 0$ with $1 \leq i \leq p$.

Definition 5.2. The $X_i, 1 \leq i \leq p$ are called eigenfields corresponding to eigenvalues $\lambda_i, 1 \leq i \leq p$ of connection Γ.

Theorem 5.3. *We suppose that the connection Γ is invariance and triangularizable. The nullity space of curvature R associated to Γ is involutive.*

Proof. We suppose that Γ is triangular inferior (respectively superior), that is, $\Gamma_i^j = 0$ for $i \gneq j$ (respectively, $i \lneq j$). Let X, Y be two nullity fields of curvature R attached to Γ. By resolving the system of partial derivative equations having the following expression

$$R([X,Y],Z) = [\Gamma[X,Y], \Gamma Z] + [[X,Y],Z] \\ - \Gamma[\Gamma[X,Y],Z] - \Gamma[[X,Y],\Gamma Z], \quad (5.1)$$

for all $Z \in \chi(M)$ and given the triangularization of Γ, we find that the expression (5.1) becomes $R([X,Y],Z) = 0$. We end up in the involutivity of the nullity space $\mathfrak{N}_R$ of R. □

Example 5.4. Let $M = \mathbb{R}^3$ be a smooth manifold, (x^i, y^i) where $1 \leq j \leq 3$ the local coordinates. Consider Γ a connection defined by : $\Gamma_1^1 = (x^1)^2 y^2$, $\Gamma_1^2 = \frac{y^2}{2}$, $\Gamma_2^2 = x^2$, $\Gamma_1^3 = x^2 y^2$, $\Gamma_2^3 = y^2$, $\Gamma_3^3 = 1$ and $\Gamma_j^i = 0$ elsewhere.
According to computation, we find that the coefficients of the curvature R are given by: $R_{12}^1 = -(x^1)^2 x^2 = -R_{21}^1$, $R_{12}^3 = \frac{y^1}{2} - (x^2)^2 = -R_{21}^3$ and $R_{jk}^i = 0$ elsewhere.
Let $X \in \mathfrak{N}_R$ be, so $X^i R_{ij}^k = 0$ with $k = 1, 3$. This system is equivalent to

$$\begin{cases} X^2 R_{21}^1 = 0 \\ X^1 R_{12}^1 = 0 \\ X^2 R_{21}^3 = 0 \\ X^1 R_{12}^3 = 0 \end{cases} \iff \begin{cases} X^1 = 0 \\ X^2 = 0 \end{cases}$$

So, a nullity field is then written: $X = X^3(x^i, y^j)\frac{\partial}{\partial x^3} + Y^i(x^i, y^j)\frac{\partial}{\partial y^i}, 1 \leq i, j, k \leq 3$. Thus $\left[x^2 y^1 \frac{\partial}{\partial y^2}, (x^1)^2 y^1 y^2 \frac{\partial}{\partial x^3} + x^1 x^2 x^3 y^3 \frac{\partial}{\partial y^1}\right] = (x^1)^2 x^2 (y^1)^2 \frac{\partial}{\partial x^3} + x^1 (x^2)^2 x^3 y^3 \frac{\partial}{\partial y^2} \in \mathfrak{N}_R$. So the nullity space $\mathfrak{N}_R$ associated to R is involutive.

Example 5.5. Let $M = \mathbb{R}^3$ be a smooth manifold, $\left(x^i, y^i\right)$ where $1 \le j \le 3$ the local coordinates. Consider Γ a connection defined by : $\Gamma_1^1 = (x^1)^2 y^2$, $\Gamma_1^2 = \frac{y^2}{2}$, $\Gamma_2^2 = x^2$, $\Gamma_1^3 = x^2 y^2$, $\Gamma_2^3 = y^2$, $\Gamma_1^3 = 1$ and $\Gamma_j^i = 0$ elsewhere. According to computation, we find that the coefficients of the curvature R are given by: $R_{12}^1 = -(x^1)^2 x^2 = -R_{21}^1$, $\quad R_{12}^3 = \frac{y^1}{2} - (x^2)^2 = -R_{21}^3$ and $R_{jk}^i = 0$ elsewhere. Let $X \in \mathfrak{N}_R$ be, so $X^i R_{ij}^k = 0$ with $k = 1, 3$. This system is equivalent to

$$\begin{cases} X^3 R_{32}^1 = 0 \\ X^2 R_{23}^1 = 0 \\ X^3 R_{31}^3 = 0 \\ X^1 R_{13}^3 = 0 \end{cases} \Longleftrightarrow \begin{cases} X^1 = 0 \\ X^2 = 0 \\ X^3 = 0 \end{cases}$$

Then a nullity field is written $X = Y^k \left(x^i, y^j\right) \frac{\partial}{\partial y^k}, 1 \le i, j, k \le 3$. Thus the x nullity space $\mathfrak{N}_R = \{Y^k \left(x^i, y^j\right) \frac{\partial}{\partial y^k}, 1 \le i, j, k \le 3\} = \mathfrak{v}(\mathcal{T}M)$ associated to R is involutive.

Remark 5.6. We notice that if Γ is a triangularizable connection then the associated Lie algebra $\mathfrak{A}_\Gamma$ of vector fields to Γ is not semisimple.

Example 5.7. Let $M = \mathbb{R}^3$ be a smooth manifold, $\left(x^i, y^i\right)$ where $1 \le i \le 3$ the local coordinates. Consider the matrix expression of the connection Γ defined by: $\Gamma_1^1 = 2 = \Gamma_2^2$, $\Gamma_2^1 = 1 = \Gamma_3^1 = \Gamma_3^2 = \Gamma_3^3$ and $\Gamma_j^i = 0$ otherwise. We find that $R = 0$ and the nullity space of R is $\mathfrak{N}_R = \chi(\tau M)$ involutive. The eigen subspaces corresponding to eigenvalues $1, 2$ are

$$\mathfrak{A}_{\Gamma,1} = \{X \in \mathbb{R}^3 \text{ such that } X(0, -1, -1)\}$$

and

$$\mathfrak{A}_{\Gamma,2} = \{Y \in \mathbb{R}^3 \text{ such that } Y(1, 0, 0)\}.$$

Each of these submodules does not lend itself to direct factor so according to [3] the Lie algebra $\mathfrak{A}_\Gamma$ is not semisimple.

We suppose now that Γ is a connection of polynomial coefficients which eigenvalues are polynomial functions in M.

Definition 5.8. Let f be a function, C^∞ on the manifold M. We say that f is integrable to all vector fields of M if for all $X \in \chi(M)$, we have $X.f = 0$.

Proposition 5.9. *The eigenvalues $\lambda, \mu, \beta, \ldots$ of Γ are integrable to any vector fields of Lie algebra $\mathfrak{A}_\Gamma$.*

Proof. Suppose that the eigenvalue λ of the connection Γ is a function on M. Let $X \in \mathfrak{A}_\Gamma$ be, we have $\Gamma[X, Y] = [X, \Gamma Y]$ for all vector fields Y. If Y is a eigenfield of the connection Γ to eigenvalue λ, we obtain

$$X(\lambda).Y = (\Gamma - \lambda I)[X, Y]. \tag{5.2}$$

By applying $\Gamma - \lambda$ to the relation (5.2), we have $(\Gamma - \lambda I) X(\lambda).Y = 0 = (\Gamma - \lambda I)^2 [X, Y]$, we obtain

$$(\Gamma - \lambda I)^2 [X, Y] = 0. \tag{5.3}$$

As Γ is diagonalizable the equation (5.3) implies

$$(\Gamma - \lambda I)[X, Y] = 0 \tag{5.4}$$

The two relations (5.2) and (5.4) entail $X(\lambda) = 0$. In other words, the function λ is integrable. □

Proposition 5.10. *If eigenvalues $\lambda, \mu, \beta, \ldots$ of the connection Γ are not multiple roots to vector fields of $\mathfrak{A}_\Gamma$ then Lie algebra $\mathfrak{A}_\Gamma$ is semisimple.*

Proof. Suppose that Γ is a diagonalizable which every eigenvalue is a function of 1-multiplicity order. In accordance with Theorem 5.7 of [2] we show $\mathfrak{A}_\Gamma = \mathfrak{A}_{\Gamma,\lambda} \otimes \mathfrak{A}_{\Gamma,\mu} \otimes \mathfrak{A}_{\Gamma,\beta} \ldots$. Hence the proof. □

Remark 5.11. If eigenvalues $\lambda, \mu, \beta, \ldots$ are multiples to vector fields of $\mathfrak{A}_\Gamma$ then the Lie algebra $\mathfrak{A}_\Gamma$ is not necessarily semisimple.

Proposition 5.12. *We denote by θ_X the Lie derivative compared with X vector fields. A vector fields $X \in \mathfrak{A}_\Gamma$ if and only if*

- $\theta_X \lambda = 0$, *for all eigenvalue λ of Γ*
- θ_X *leaves invariante the generalized distributions defined by eigen subspaces of Γ.*

Proof. Let X be an element of A_Γ. According to the definition $\theta_X \lambda = X\lambda$. Yet all eigenvalue λ of Γ is integrable comparing to vector fields of A_Γ we have so $\theta_X \lambda = X\lambda = 0$. Taking a vector fields Y such that $\Gamma Y = \lambda Y$. As $X \in A_\Gamma$ we obtain $\Gamma[X, Y] = [X, \Gamma Y] = [X, \lambda Y] = \lambda [X, Y] - X\lambda Y = \lambda [X, Y]$. So $(\Gamma - \lambda I)[X, Y] = 0$ and that $(\Gamma - \lambda I)^n [X, Y] = 0$ for all $n \in \mathbb{N}^*$ with I denotes one identity matrix. In other words, θ_X preserves the eigen subspace of Γ. Conversely, let X be a vector fields who verifying the two above conditions. Because θ_X preserves the eigen subspace thus we have for all vector fields Y,

$$\Gamma[X, Y] - [X, \Gamma Y] = \Gamma[X, Y] - \lambda [X, Y] + X\lambda Y$$

$$\begin{aligned} &= \Gamma[X,Y] - \lambda[X,Y] \\ &= (\Gamma - \lambda I)[X,Y] \\ &= 0. \end{aligned}$$

So $\Gamma[X,Y] = [X,\Gamma Y]$. Hence $X \in A_\Gamma$. □

6. Study of the nullity space of the curvature by invariante connection Γ

Let X, Y be two nullity fields of the curvature R associated to Γ.

Proposition 6.1. *We suppose that the connection Γ bi-invariante (invariante) for any polynomial vector fields of TM. Then the nullity space $\mathfrak{N}_R$ is involutive.*

Proof. Let $Z \in \chi(M)$ be. According to expression of R we have

$$\begin{aligned} R([X,Y],Z) &= [\Gamma[X,Y],Z] + [[X,Y],Z] - \Gamma[\Gamma[X,Y],Z] - \Gamma[[X,Y],\Gamma Z] \\ &= [[\Gamma X,Y],\Gamma Z] + [[X,Y],Z] - [[\Gamma X,Y],\Gamma Z] - [\Gamma[X,Y],\Gamma Z] \\ &= [[\Gamma X,Y],\Gamma Z] + [[X,Y],Z] - [[\Gamma X,Y],\Gamma Z] - \Gamma^2[[X,Y],Z] \\ &= 0 \end{aligned}$$

Thus $[X,Y] \in \mathfrak{N}_R$. Hence $\mathfrak{N}_R$ is involutive. □

Remark 6.2. If the connection Γ is bi-invariante (or invariante) uniformly on nullity vector fields of the curvature R then the nullity space $\mathfrak{N}_R$ is not necessarily involutive on $\chi(M)$.

Theorem 6.3. *If Γ is a bi-invariante connection of A_Γ on M. The following assertions are equivalent:*

1. *For any $X, Y \in A_\Gamma$,*
2. *All derivation of $A_\Gamma \cap \mathfrak{N}_R$ is inner,*
3. *The curvature corresponding to the connection Γ is null ($R(X,Y) = 0$), that is, Γ is flat.*

Proof. (1. $\Longrightarrow$ 2.): We consider two elements X, Y of A_Γ. Let D be a derivation of $A_\Gamma \cap \mathfrak{N}_R$. Using the Definition of two vector fields derivation and a proof in cf.[10], we have $D[X,Y] = D_{|_U}[X,Y] = [Z,[X,Y]]$ and $D(X) = D_{|_U}(X) = [Z,X]$ for all $Z \in \chi(M)$. But $[Z,[X,Y]] = [[Z,X],Y] + [X,[Z,Y]]$ so we obtain $[D(X),Y] + [X,D(Y)] = [[Z,X],Y] + [X,D(Y)]$. Thus $[D(X) - [Z,X],Y] = 0$ because compared with $+$ the bracket $[.,.]$ is distributive. As Γ is bi-invariante then the centralizer $\mathcal{C}$ of A_Γ is $\{0\}$ as well as $A_\Gamma \cap \mathfrak{N}_R$. In other words, $D(X) - [Z,X] \in \mathcal{C}$ for all $Z \in \chi(M)$. Consequently $D(X) = [Z,X]$ for all $Z \in \chi(M)$ and D is inner. Hence the result.

(2. $\Longrightarrow$ 3.): This is immediate by adaptation of the Theorem 4.4 cf.[10].
(3. $\Longrightarrow$ 1.): Suppose that Γ is flat and for any $X, Y \in \chi(M)$, $\quad R(X,Y) = 0$ we prove that $X, Y \in A_\Gamma$. By absurd, suppose that $X, Y \notin A_\Gamma$. According to computation, we have $[\Gamma X, Z] \neq \Gamma[X,Z]$ and $[\Gamma Y, Z] \neq \Gamma[Y,Z]$. According to the expression of R, we obtain $[X,Y] + [\Gamma X, \Gamma Y] - \Gamma[X, \Gamma Y] - \Gamma[\Gamma X, Y] = 0$. But Γ is bi-invariante on A_Γ, so we have a contradiction. Hence necessarily $X, Y \in A_\Gamma$. □

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