International Journal of Pure and Applied Mathematics
Volume 116 No. 2 2017, 489-499
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v116i2.20

REVERSE ALGORITHM FOR PRACTICAL STABILIZATION OF DISCRETE INCLUSIONS

Mariia S. Tairova[1], Yevhen M. Strakhov[2 §]

[1]Department of Computational Mathematics
Institute of Mathematics, Economics and Mechanics
Odessa I. I. Mechnikov National University
Dvoryanska st., 2, 65082, Odessa, UKRAINE

[2]Department of Optimal Control and Economic Cybernetics
Institute of Mathematics, Economics and Mechanics
Odessa I. I. Mechnikov National University
Dvoryanska st., 2, 65082, Odessa, UKRAINE

Abstract: This article discusses the issue of practical stabilization of linear discrete control inclusions. We introduce a concept of potential weakly stable solutions for discrete inclusions and propose a reverse stabilization method for linear case.

AMS Subject Classification: 34A60, 93D05
Key Words: discrete inclusions, practical stabilization, weakly stable solutions

1. Introduction

Discrete systems describe a large amount of real models, e. g. economic, mechanic, biological, social etc. If we deal with uncertainty in such kind of processes, then we obtain dicrete inclusions. In the research of discrete control systems, one of the most important issues is a practical stability and practical stabilization problem [2, 3, 4, 5]. This paper deals with practical stabilization

Received: June 25, 2017
Revised: August 20, 2017
Published: October 7, 2017

§Correspondence author

of dicrete control inclusions. We construct a reverse stabilization alogithm for practical stabilizing control in linear case.

Let us consider a linear discrete inclusion

$$x(k+1) \in A_k x(k) + u(k) + V(k), \quad k = 0, 1, \dots, N-1, \tag{1}$$

where $x \in \mathbb{R}^n$; A_k is invertible $n \times n$ matrix; $u(k) \in U(k)$ is a control, $0 \in U(k)$, $U(k) \in conv\,(\mathbb{R}^n)$; $0 \in V(k)$, $V(k) \in conv\,(\mathbb{R}^n)$, $k = 0, 1, \dots, N-1$; $\Phi(k) \in conv\,(\mathbb{R}^n)$ are sets of phase constraints, $k = 0, 1, \dots, N$.

Definition 1. [4] A function $x : \{0, 1, ..., N\} \mapsto \mathbb{R}^n$ is said to be a solution of discrete inclusion (1) if each pair of points $x(k)$, $x(k+1)$ $(k = 0, \dots, N-1)$ satisfies the following $x(k+1) \in A_k x(k) + u(k) + V(k)$.

2. Controllability Set and Potential Weak Practical Stability Sets

Definition 2. A set $Y\,(i, I, M_I) \subseteq \mathbb{R}^n$ $(i \leq I,\ M_I \subseteq \mathbb{R}^n)$ of points $y \in \mathbb{R}^n$ is called a controllability set of inclusion (1) if there exists a solution $x\,(k, x_0)$ $(k = \overline{0, N})$ such that $x\,(i, x_0) = y$ and $x\,(I, x_0) \in M_I$.

Further on, we consider controllability sets such that $i \in \{0, 1, \dots, N-1\}$, $I = i+1$, i.e. controllability for consecutive moments. By $Y_i\,(M_{i+1})$ we denote sets $Y\,(i, i+1, M_{i+1})$, where $i \in \{0, 1, \dots, N-1\}$.

In the following subsection, we obtain the formula for controllability set of inclusion (1).

Let x_{i+1} be an arbitrary point of the set M_{i+1}. Then

$$x_{i+1} \in A_i x_i + u(i) + V(i), \quad u_i \in U(i).$$

By definition of a controllability set we have: $x_i \in Y_i\,(M_{i+1})$. Therefore, the set M_{i+1} at the i-th step $(i = 0, 1, \dots, N-1)$ will have the form

$$M(i+1) = A_i Y_i\,(M_{i+1}) + U(i) + V(i).$$

Thus, we obtain a controllability set of inclusion (1)

$$Y_k\,(M_{k+1}) = A_k^{-1} \left(M_{k+1} + (-1) U(k) \overset{*}{-} V(k) \right), \tag{2}$$

where by $\overset{*}{-}$ we denote Minkowski difference of two sets. The usage of inverse matrix A_k^{-1} is correct, because A_k is invertible by the statement of the problem.

Definition 3. A set $\hat{Y}_i$ of points y_i such that $y_i \in \Phi(i)$ and there exists a solution $x\,(k, y_i)$ $(k = \overline{i, N})$ of the inclusion

$$x(k+1) \in A_k x(k) + U(k) + V(k), \quad k = i, i+1, \dots, N-1 \tag{3}$$

that satisfies the constraints $x\,(k, y_i) \in \Phi(k)$ for all $k = i, i+1, \dots, N$, is called a potential weak practical stability set of inclusion (1) at the i-th step, where $i \in \{0, \dots, N-1\}$.

We now prove the following statements.

Theorem 4. *Let $\hat{Y}_i$, $\hat{Y}_{i+1}$ be potentially stable sets at i-th and $(i+1)$-th steps of inclusion (1) respectively, $i \in \{0, \dots, N-1\}$. Then*

$$\hat{Y}_i \subseteq \Phi(i) \cap Y_i\left(\hat{Y}_{i+1}\right), \tag{4}$$

where $\Phi(i)$ is the set of phase constraints and Y_i is the controllability set of inclusion (1) at the step i.

Proof. We must prove that for an arbitrary point $y_i \in \hat{Y}_i$ the following statements hold:

$$y_i \in \Phi(i) \quad \text{and} \quad y_i \in Y_i\left(\hat{Y}_{i+1}\right). \tag{5}$$

The definition of a set $\hat{Y}_i$ implies the first part of (5).

Let us prove the second part. By definition of a set $\hat{Y}_i$, for $y_i \in \hat{Y}_i$ there exists a solution $x\,(k, y_i)$, $k = \overline{i, N}$ of inclusion (3) that satisfies the constraints $x\,(k, y_i) \in \Phi(k)$ for all $k = i, i+1, \dots, N$. Let $y_{i+1} = x\,(i+1, y_i)$ be the value of this solution at the $(i+1)$-th step. Consider all points of the solution $x\,(k, y_i)$ $(k = \overline{i, N})$ except $x\,(i, y_i)$, i. e. a subset

$$x\,(k, y_{i+1}) = x\,(k, y_i)\,, \quad k = \overline{i+1, N}.$$

For this subset of the solution the following statements hold:

$$x\,(k, y_{i+1}) \in \Phi(k) \quad \text{for all} \quad k = i+1, \dots, N,$$

and

$$x\,(k+1, y_{i+1}) \in A_k x\,(k, y_{i+1}) + U(k) + V(k), \quad k = i+1, \dots, N-1.$$

Hence, $x\,(k, y_{i+1})$ $(k = \overline{i+1, N})$ is a weakly stable solution of the inclusion

$$x(k+1) \in A_k x(k) + U(k) + V(k), \quad k = i+1, \dots, N-1. \tag{6}$$

From inclusion (6) and definition 3 we get $y_{i+1} \in \hat{Y}_{i+1}$.

Since $y_i = x(i, y_i)$ and $y_{i+1} = x(i+1, y_i)$, it follows that y_i and y_{i+1} belong to the same solution of inclusion (3). If we combine this with definition 2, we obtain $y_i \in Y_i(\{y_{i+1}\})$. Since $y_{i+1} \in \hat{Y}_{i+1}$, we have $y_i \in Y_i(\hat{Y}_{i+1})$.

Thus, the second part of (5) is proved, and since a point $y_i \in \hat{Y}_i$ was arbitrary, we obtain (4). □

Theorem 5. *Let $\hat{Y}_i$, $\hat{Y}_{i+1}$ be potentially stable sets for i-th and $(i+1)$-th steps of inclusion* (1) *respectively, $i \in \{0, \ldots, N-1\}$. Then*

$$\hat{Y}_i \supseteq \Phi(i) \cap Y_i\left(\hat{Y}_{i+1}\right),$$

where $\Phi(i)$ is the set of phase constraints and Y_i is the controllability set of inclusion (1) *at the i-th step.*

Proof. We now prove that $y_i \in \hat{Y}_i$ for arbitrary point $y_i \in \Phi(i) \cap Y_i(\hat{Y}_{i+1})$. We clearly have

$$y_i \in \Phi(i) \quad \text{and} \quad y_i \in Y_i\left(\hat{Y}_{i+1}\right), \tag{7}$$

hence, there exists $y_{i+1} \in \hat{Y}_{i+1}$ such that

$$y_{i+1} \in A_i y_i + U(i) + V(i). \tag{8}$$

Since $y_{i+1} \in \hat{Y}_{i+1}$, by definition 3 it follows that there exists a solution $x(k, y_{i+1})$ $(k = \overline{i+1, N})$ of inclusion (6), i. e.

$$x(k+1, y_{i+1}) \in A_k x(k, y_{i+1}) + U(k) + V(k), \qquad k = i+1, \ldots, N-1 \tag{9}$$

and

$$x(k, y_{i+1}) \in \Phi(k), \quad k = i+1, \ldots, N. \tag{10}$$

Summing (7)–(10), we obtain that the solution

$$x(k, y_i) = \begin{cases} y_i, & k = i, \\ x(k, y_{i+1}), & k = i+1, \ldots, N \end{cases}$$

satisfies all the conditions of definition 3, so that $y_i \in \hat{Y}_i$. Since a point y_i was arbitrary, this completes the proof of Theorem 5. □

Theorems 4 and 5 imply the equality

$$\hat{Y}_k = \Phi(k) \cap Y_k\left(\hat{Y}_{k+1}\right),$$

where $k \in \{0, 1, \ldots, N-1\}$.

Let us obtain the form of the set $\hat{Y}_N$. From definition of a set $\hat{Y}_i$, it follows that for the last, N-th, step the set $\hat{Y}_N$ will coincide with the phase set $\Phi(N)$. Combining the results, we get a recurrence formula for potential weak stability sets of inclusion (1):

$$\hat{Y}_k = \begin{cases} \Phi(N), & k = N, \\ \Phi(k) \cap Y_k\left(\hat{Y}_{k+1}\right), & k = 0, \ldots, N-1. \end{cases} \tag{11}$$

Now it is easy to prove that potential weak stability sets $\hat{Y}_k$ of inclusion (1) have the following properties:

1. potential weak stability set $\hat{Y}_i$ ($i \in \{0, \ldots, N\}$) is convex;
2. $0 \in \hat{Y}_i$, where $i \in \{0, \ldots, N\}$;
3. if $\Phi(k)$ ($k = 0, \ldots, N$) and $U(k)$ ($k = 0, \ldots, N-1$) are symmetric with respect to 0, then $\hat{Y}_i$ ($i = 0, \ldots, N$) are symmetric with respect to 0.

Lemma 6. *The support function of a set $\hat{Y}_i$ has the recurrence form*

$$c\left(\hat{Y}_k, \psi\right) = \begin{cases} c\left(\Phi(N), \psi\right), & k = N, \\ co(\min\{c(\Phi(k), \psi), c(Y_k(\hat{Y}_{k+1}), \psi)\}), & k = \overline{0, N-1}, \end{cases} \tag{12}$$

where $\psi \in \mathbb{R}^n$.

Proof. This statement follows from formula (11) and a representation for the support function of intersection of sets. □

Note that a set $\hat{Y}_k$ for $k \in \{0, \ldots, N-1\}$ in the general case can be empty. Formula (12) can be simplified. Using (2), we get

$$c(Y_k(\hat{Y}_{k+1}), \psi) = co(c(A_k^{-1}\hat{Y}_{k+1}, \cdot) + \\ + c(A_k^{-1}(-1)U(k), \cdot) - c(A_k^{-1}V(k), \cdot))(\psi).$$

Denote

$$G(k, \psi) = co\big(\min\{c(\Phi(k), \cdot), c(Y_k(\hat{Y}_{k+1}), \cdot)\}\big)(\psi) =$$

$$= co\Big(\min\{c(\Phi(k),\cdot), co\big(c(A_k^{-1}\hat{Y}_{k+1},\cdot) + c(A_k^{-1}(-1)U(k),\cdot) - \\ - c(A_k^{-1}V(k),\cdot)\big)(\cdot)\}\Big)(\psi).$$

We will use the following definition.

Definition 7. [1] The convex hull $co\,(f(\cdot))\,(\psi)$ of a function $f(\psi)$ $(\psi \in \mathbb{R}^n)$ is the biggest convex function which is less or equal to $f(\psi)$ for all $\psi \in \mathbb{R}^n$.

Taking into account Definition 7, in (12) we get

$$G(k,\psi) = co\Big(\min\{c(\Phi(k),\cdot), \\ c(A_k^{-1}\hat{Y}_{k+1},\cdot) + c(A_k^{-1}(-1)U(k),\cdot) - c(A_k^{-1}V(k),\cdot)\}\Big)(\psi).$$

It can be easily shown that the following statements hold for the above function:

1. $G(k,\psi)$ is convex;
2. $G(k,\psi) \le c(\Phi(k),\psi)\ \forall \psi \in S$;
3. $G(k,\psi) \le c(A_k^{-1}\hat{Y}_{k+1},\psi) + c(A_k^{-1}(-1)U(k),\psi) - c(A_k^{-1}V(k),\psi)\ \forall \psi \in S$;
4. $G(k,\psi)$ is maximum.

Furthermore, statements 1), 3), 4) imply that

$$G(k,\psi) \le co\big(c(A_k^{-1}\hat{Y}_{k+1},\cdot) + c(A_k^{-1}(-1)U(k),\cdot) - c(A_k^{-1}V(k),\cdot)\big)(\psi)$$

for any $\psi \in S$, that is, $G(k,\psi)$ satisfies all the conditions in (12).

Thus, expression (12) can be written in simplified form

$$c(\hat{Y}_k,\psi) = \begin{cases} c(\Phi(N),\psi), & k = N, \\ co\big(\min\{c(\Phi(k),\psi), \mu_k(\psi)\}\big), & k = \overline{0, N-1}, \end{cases} \tag{13}$$

where $\mu_k(\psi) = c(A_k^{-1}\hat{Y}_{k+1},\psi) + c(A_k^{-1}(-1)U(k),\psi) - c(A_k^{-1}V(k),\psi)$.

3. A Reverse Algorithm for Practical Stabilization of Linear Discrete Inclusions

Let $\hat{G}_0$ be a set of initial values for inclusion (1), $\hat{G}_0 \subseteq \Phi(0)$.

Definition 8. [4] A control $u \in U$ is said to be practically stabilizing control for inclusion (1) from a point $x_0 \in \hat{G}_0$ in phase constraints $\Phi(k)$ ($k = 0, \dots, N$) if $x(k, x_0, u) \in \Phi(k)$ for all $k = 0, \dots, N$ for any solution $x(k, x_0, u)$ ($k = \overline{0, N}$) of inclusion (1).

Definition 9. [4] Discrete inclusion (1) is called $\left\{\hat{G}_0, \Phi(k), 0, N\right\}$-stabilized if for any point $x_0 \in \hat{G}_0$ in phase constraints $\Phi(k)$ ($k = 0, \dots, N$) there exists feasible stabilizing control $u \in U$ in those constraints.

We introduce an additional constraint $0 \in \hat{G}_0$ to guarantee nonemptiness of a stabilizing set of inclusion (1): $\hat{G}_0 \neq \emptyset$.

Definition 10. A stabilizing problem for inclusion (1) from initial point $x_0 \in \hat{G}_0$ is to find a control $u \in U$ that practically stabilizes this inclusion.

By $\hat{G}_*$ denote the maximum set of practical stability, i. e. maximum set of all initial conditions $\hat{G}_0$ such that for any point $x_0 \in \hat{G}_0$ there exists feasible control $u \in U$ such that $x(k, x_0, u) \in \Phi(k)$ for all $k = 0, \dots, N$ for an arbitrary solution of inclusion (1) starting from this point.

Lemma 11. *The maximum set of practical stabilization $\hat{G}_*$ coincides with a potential weak stability set $\hat{Y}_0$, that is*

$$\hat{G}_* = \hat{Y}_0.$$

Proof. The proof follows from definitions of $\hat{G}_*$ and $\hat{Y}_0$. □

Having the formula for support function of a controllability set $\hat{Y}_k$, we can construct approximation algorithms for those sets as well as an algorithm of practical stabilization for inclusions (1).

A stabilization algorithm for linear discrete inclusion coincides with an algorithm for systems [5], $\hat{Y}_i$ $\hat{Y}_i$.

By $X(\cdot, u, x_0)$ denote a set of solutions of inclusion (1) that corresponds to a control $u \in U$ and initial condition $x(0) = x_0$, i. e.

$$X(k, X_0, u) = \cup_{x_0 \in X_0} \cup_{x \in X(\cdot, u, x_0)} x(k, x_0, u),$$

and by

$$X(k, X_0) = \cup_{u \in U} X(k, X_0, u)$$

denote the reachable set of (1).

We introduce some additional denotations.

$$P(i, j) = \begin{cases} E, & i \geq j, \\ A_j A_{j-1} \dots A_{i+1} A_i & \text{otherwise;} \end{cases}$$

$$\Lambda(k) = \sum_{i=0}^{k} P(i+1,k)V(i).$$

Then for some step $k \in \{0, 1, \ldots, N\}$ we have

$$X\left(k, x_0, u\right) = x_k + \Lambda(k),$$

where $k \in \{0, 1, \ldots, N\}$.

Since sets $\Lambda(k)$ do not depend on the selected stabilizing control, vector $x(k)$ can be considered as the central vector of solution in the algorithm.

Let us find the control successively, i. e. in order $u(0)$, $u(1)$, $\ldots$, $u(N-1)$.

$k = 0, 1, ..., N-1$:

$u(k) : x(k+1) = A_k x_k + u(k) \in \tilde{Y}_{k+1}$;

$x_{k+1} = A_k x_k + u(k)$.

Thus, for a fixed step number $i \in \{0, \ldots, N-1\}$ and a state $x_i \in \tilde{Y}_i$ at this step we must find a vector $u_i \in U(i)$ such that

$$x_{i+1} = A_i x_i + u_i \in \tilde{Y}_{i+1}.$$

By the property of a support function it follows that $x_{i+1} \in \tilde{Y}_{i+1}$ if

$$\langle x_{i+1}, \psi\rangle \leq c\left(\tilde{Y}_{i+1}, \psi\right) \quad \forall\psi \in S,$$

that is,

$$\langle A_i x_i + u_i, \psi\rangle \leq c\left(\tilde{Y}_{i+1}, \psi\right) \quad \forall\psi \in S.$$

We find a control vectors $u(i)$ in the form

$$u(i) = P_i \xi_i,$$

where $P_i \in \mathbb{R}$, $P_i \geq 0$ is a number; $\xi_i \in S$ is a vector of the unit sphere, $i = 0, \ldots, N-1$.

Since $u(i) = P_i \xi_i$ must belong to $U(i)$, an additional restriction needed:

$$P_i \in \left[0, P_{\max}\left(i, \xi_i\right)\right],$$
$$P_{\max}\left(i, \xi_i\right) = d_*\left(U(i), \xi_i\right),$$

where $d_*\left(U(i), \xi_i\right)$ is the value of a deformation function of a set $U(i)$ in the direction of the vector ξ_i.

In this terms we can find a stabilizing control $u(k)$ ($k = \overline{0, N-1}$) by the following formulas:

$$u(k) = P_k \xi_k, \tag{14}$$

where

$$\xi_k = \arg\max_{\xi \in S} \max_{P \in [0, P_{\max}(i,\xi)]} F_k(P, \xi); \tag{15}$$

$$P_k = \arg\max_{P \in [0, P_{\max}(i,\xi_k)]} F_k(P, \xi_k); \tag{16}$$

$$x(k+1) = A_k x(k) + u(k). \tag{17}$$

In addition,

$$F_k(P, \xi) = \min_{\psi \in S} \big(c(\tilde{Y}_{k+1}, \psi) - \langle A_k x(k), \psi \rangle - \langle P\xi, \psi \rangle \big); \tag{18}$$

$$P_{\max}(k, \xi) = d_*(U(k), \xi), \tag{19}$$

where $d_*\left(U(k), \xi\right)$ is the value of a deformation function of a set $U(i)$ in the direction of the vector ξ_i and S is the unit sphere.

As we have shown, if a point x_0 belongs to a set I_*, then

$$F_k(P_k, \xi_k) = \min_{\psi \in S} \big(c(\tilde{Y}_{k+1}, \psi) - \langle A_k x(k), \psi \rangle - \langle P_k \xi_k, \psi \rangle \big) \geq 0$$

for a control $u(k)$ that satisfies conditions (14)–(19), that is, the obtained control exists, is feasible and stabilizing.

We obtain the stabilization algorithm for linear discrete inclusions in two-dimensional case.

Input: $x(0) = X\,(0, x_0) = x_0 \in \hat{G}_0$.
for $k = 0, \ldots, N-1$ **do**
1. Specify a partition $\hat{S} = \{\xi_i\}_{i=0}^{m-1} = \big\{\big(\cos(\frac{2\pi i}{m}), \sin(\frac{2\pi i}{m})\big)\big\}_{i=0}^{m-1}$.
2. Specify a minimum value for the length of a control $P_{\min} = INF$ and corresponding vector $\xi_{\min} = (1, 0)$.
3. **foreach** $\xi_i \in \hat{S}$, $i = 0, \ldots, m-1$ **do**
   3.1. Find value $P_{\max}(k, \xi_i) = d_*(U(k), \xi_i)$.
   3.2. Find by a maximization method of zero order
   $P^* = \underset{P \in [0, P_{\max}(k,\xi_i)]}{\arg\max} F_k(P, \xi_i)$,
   where $F_k(P, \xi) = \min\limits_{\psi_j \in \hat{S}} \big(c(\hat{Y}_{k+1}, \psi_j) - \langle A_k x(k), \psi_j \rangle - \langle P\xi, \psi_j \rangle\big)$,
   $c(\hat{Y}_i, \psi)$ is calculated by (13).
   3.3. **if** $F_k(P^*, \xi_i) < 0$ **then**
      continue
   end
   3.4. **if** $P_{\min} > P^*$ **then**
      $P_{\min} = P^*, \xi_{\min} = \xi_i$.
   end
end
4. **if** $P_{\min} = INF$ **then**
   return *"No solution found"*
end
else
   $u(k) = P_{\min}\xi_{\min}$,
   $x(k+1) = A_k x(k) + u(k)$,
   $X(k+1, x_0) = x(k+1) + \Lambda(k+1)$
end
end
return $u(k), k = \overline{0, N-1}$; $x(k), X(k, x_0), k = \overline{0, N}$.

4. Conclusion

In this paper we have introduced a new reverse practical stabilization algorithm for discrete linear inclusions. Thus, we can obtain the control that guarantees a practical stability of the system.

References

[1] N.L. Grigorenko, D.V. Kamzolkin, D.G.Pivovarchuk, *Linear Differential Games. Numerical Solutions*, Moscow, 2007, In Russian.

[2] V.V. Pichkur, M.S. Sasonkina, Approximation of maximal set of initial conditions in a practical stabilization problem for systems with multi-valued right-hand side, *Visnyk Odes'kogo natsional'nogo universitetu. Matematyka i mekhanika*, **17** (2012), 121–128, In Ukrainian.

[3] V.V. Pichkur, M.S. Sasonkina, Optimal sets of initial conditions in a practical stability problem for discrete inclusions, In: *Mathematical Analysis, Differential Equations and their Applications*, September 15-20, 2010, Sunny Beach, Bulgaria (2011), 291–300.

[4] V.V. Pichkur, M.S. Sasonkina, Practical stabilization of discrete control systems, *International Journal of Pure and Applied Mathematics*, **81** (2012), 877–884.

[5] T.O. Romanova, M.S. Tairova, Reverse practical stabilization method for discrete linear systems, *Visnyk Odes'kogo natsional'nogo universitetu. Matematyka i mekhanika*, **20** (2015), 59–70, In Ukrainian.