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DISTRIBUTION OF THE WAITING AND SERVICE TIME IN AN M/M/m PREEMPTIVE-RESUME LCFS QUEUE WITH IMPATIENT CUSTOMERS

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Abstract: We study an M/M/m preemptive-resume last-come, first-served (PR-LCFS) queue with impatient customers without priority classes. We focus on the time interval from arrival to either service completion or abandonment, whichever occurs first, of an arbitrary customer in the steady state. The problem is formulated as a combination of two one-dimensional birth-and-death processes, each with two absorbing states. We provide explicit expressions in terms of Laplace-Stieltjes transform of the distribution function for the first passage time to service completion and abandonment, which is decomposed into the waiting and service time. As two special cases, an M/M/m preemptive-loss LCFS system with impatient customers and an M/M/m preemptive-resume LCFS queue with patient customers only are treated separately. Some numerical example is presented for computation of theoretical formulas.

AMS Subject Classification: 60K25, 90B22
Key Words: multiserver queue, preemptive-resume, last-come first-served, impatient customers, waiting time distribution, birth-and-death process, first passage time, absorbing state

1. Introduction

We consider an M/M/m queueing system with impatient customers without exogenous priority classes. Customers arrive according to a Poisson process at rate λ. The service time of each customer is exponentially distributed with mean $1/\mu$. There are m servers and a waiting room of infinite capacity. We

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define a parameter $\rho := \lambda/(m\mu)$, which represents the *traffic intensity* per server. At any time, each customer present in the system is either being served or staying in the waiting room. Each customer in the waiting room leaves the system (abandons the waiting process) with probability $\theta\Delta t$ within a short time interval $(t, t+\Delta t)$. That is to say, the patience time for each customer is exponentially distributed with mean $1/\theta$. We employ another parameter $\tau := \theta/\mu$, which is the ratio of the mean service time to the mean patience time. Customers never leave the system while being served before the service is completed.

When a tagged customer arrives to find that not all servers are busy, the service of this customer is initiated immediately. There are several service disciplines with respect to the handling of a tagged customer who arrives when all servers are busy. Some of them are described as follows:

- *First-come, first-served* (FCFS): The customer is placed at the tail of the queue in the waiting room.

- *Nonpreemptive last-come, first-served* (NP-LCFS): The customer is placed at the head of the queue in the waiting room. This discipline was assumed in the study of priority queues by Durr [1]. In both the FCFS and NP-LCFS disciplines, when one of the servers becomes available, a customer at the head of the queue, if any, is called in for service. In these disciplines, the serving of customers once started is never preempted.

- *Preemptive-resume last-come, first-served* (PR-LCFS): The arriving customer preempts the ongoing service of the customer who arrived first among those currently being served. The customer whose service is preempted is placed at the head of the queue in the waiting room. When one of the servers becomes available, a customer at the head of the queue, if any, is called in for his service to be resumed. This discipline is equivalent to the one called "preemptive *last-in, first-out*" for an M/G/1 queue by Wolff [19, p. 456].

- *Preemptive-loss last-come, first-served* (PL-LCFS): Similar to PR-LCFS, the arriving customer preempts the ongoing service of the customer who arrived first among those currently being served. However, the customer whose service is preempted is immediately lost from the system as studied by Klimov [7, p. 66].

We are concerned with an M/M/m PR-LCFS queue with impatient customers, and interested in the waiting and service time for an arbitrary customer during the interval from his arrival to departure (by either service completion or abandonment of waiting, whichever occurs first) in steady state. An M/M/m PL-LCFS queue can be treated as a special case with $\theta = \infty$ of a PR-LCFS queue. In our previous work on an M/M/m PR-LCFS queue with impatient customers [14, 15], we formulated the problem as a one-dimensional birth-and-death process with two absorbing states and considered the first passage times in this process. We provided explicit expressions for the probability of service completion and abandonment. Moreover, we obtained sets of computational formulas for calculating the mean and the second moment of the times until service completion and abandonment.

In the present paper, we turn our attention to the distribution of the time to service completion and abandonment We formulate the problem as a combination of two one-dimensional birth-and-death processes, each with two absorbing states, and consider the first passage times in the combined process. We provide explicit expressions in terms of Laplace-Stieltjes transform (LST) of the distribution function (DF) for the waiting time and service time of an arbitrary customer until departure either by service completion or abandonment of waiting. We also consider the initial waiting time until the service begins for the first time and the subsequent waiting time. Furthermore, the number of service preemptions and resumptions that an arbitrary customer experiences in the system is studied. Two special cases with $\theta = \infty$ (a preemptive-loss system) and $\theta = 0$ (a system with patient customers only), both with PR-LCFS discipline, are analyzed separately. We present some numerical example for the mean, second moment, and covariance of conditional waiting and service time.

Queues with impatient (or reneging) customers have been studied extensively. Recently, this system has attracted widespread attention as a basic model for evaluating the performance of handling inbound calls in telephone call centers [9, 18] where queues without priority structure are employed. The M/M/m FCFS nonpreemptive priority queue with impatient customers is studied by Jouini and Roubos [6] and Takagi [12]. The M/M/m LCFS queue is considered as well. Riordan [10, p. 112] refers to this model without priority structure, providing a difference-differential equation for the density function of the waiting time for a customer who arrives to find k other customers already waiting upon arrival. Jagerman [4, p. 226] derives a solution to this equation. Jouini [5] uses the distribution of the busy period duration from Iravani and Balcıoǧlu [3] to analyze the waiting time. Takagi [13] considers the first passage time until service completion and abandonment, and calculates the mean and second moment of the waiting times in the M/M/m LCFS nonpreemptive (priority and non-priority) queues. The same model is studied by Jouini and

Roubos [6]. To the best of our knowledge, no studies but ours [14, 15, 16] currently exist regarding the PR-LCFS queues in the open literature.

The analytic technique presented in this paper should be interesting in its own right. However, our framework is common to M/M/m queues with FCFS, NP-LCFS, and PR-LCFS queues. Moreover, it can be applied to the analysis of M/M/m preemptive-resume *priority* queues with impatient customers in which customers of the same class are served in either FCFS or LCFS fashion.

2. First Passage Time to Service Preemption and Completion from State k of Being Served, $0 \le k \le m-1$

We focus on a tagged customer in state k, signifying that there are k other customers who compete with him for service at any time in the steady state, where $k = 0, 1, 2, \ldots$. They are the customers who arrived after the tagged one and have been staying in the system until that time. According to the preemptive LCFS discipline, an arriving customer always joins the system at state $k = 0$ and his service is started immediately.

We first consider a finite-state birth-and-death process of state transitions for a tagged customer in state k, $0 \le k \le m-1$, in which he is being served. The service to this customer, with probability one, is eventually either preempted by a customer who arrives after him or completed without preemption.

2.1. Customer Behavior until Service Preemption and Completion

The state transition diagram for the discrete-time one-dimensional birth-and-death process modeling the behavior of a tagged customer in service is shown in Fig. 1. This process has m transient states $\{0, 1, 2, \ldots, m-1\}$ and two absorbing states denoted by “Pr” (state m) and “Sr”, representing the service preemption and completion, respectively. State transitions occur when another customer arrives and the service to one of customers is completed. Therefore, the state transition probabilities and the LST of the DF for the time spent by a tagged customer in state k are given by

$$\alpha_k = \frac{k\mu}{\lambda + (k+1)\mu} \quad ; \quad \beta_k = \frac{\mu}{\lambda + (k+1)\mu},$$
$$B_k^*(s) = \frac{\lambda + (k+1)\mu}{s + \lambda + (k+1)\mu}.$$

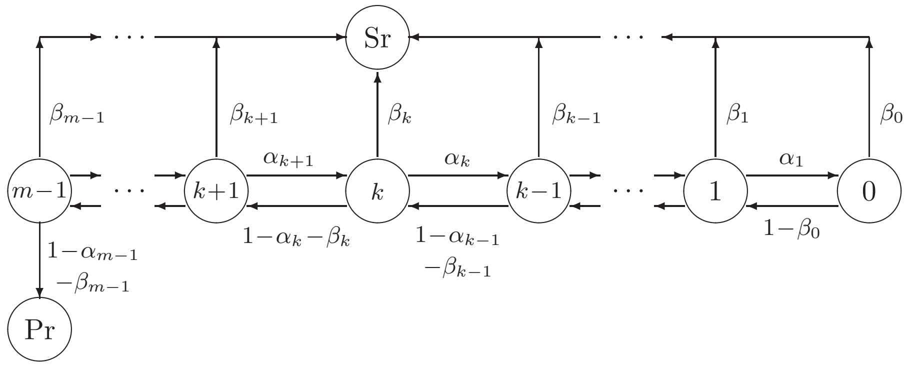

Figure 1: State transition of a tagged customer until service preemption and completion.

2.2. LST of the DF for the First Passage Time to Service Preemption and Completion

By $H_k^*(s,\mathrm{Pr})$, we denote the joint probability of service preemption and the LST of the DF for the first passage time from state k to state "Pr" without reaching state "Sr". In addition, we denote by $H_k^*(s,\mathrm{Sr})$ the joint probability of service completion and the LST of the DF for the first passage time from state k to state "Sr" without reaching state "Pr". Furthermore, let

$$H_k^*(s) := H_k^*(s,\mathrm{Pr}) + H_k^*(s,\mathrm{Sr})$$

be the unconditional LST of the DF for the first passage time from state k to state either "Pr" or "Sr", whichever occurs first.

Applying the *first step analysis* for the discrete-time Markov chain [8, p. 162], [17, p. 116], we have the following finite sets of equations for $\{H_k^*(s,\mathrm{Pr}); 0 \le k \le m-1\}$, $\{H_k^*(s,\mathrm{Sr}); 0 \le k \le m-1\}$, and $\{H_k^*(s); 0 \le k \le m-1\}$ respectively:

$$\begin{aligned}
(s+\lambda+\mu)H_0^*(s,\mathrm{Pr}) &= \lambda H_1^*(s,\mathrm{Pr}), \\
[s+\lambda+(k+1)\mu]H_k^*(s,\mathrm{Pr}) &= k\mu H_{k-1}^*(s,\mathrm{Pr}) + \lambda H_{k+1}^*(s,\mathrm{Pr}) \\
&\qquad\qquad 1 \le k \le m-2, \\
(s+\lambda+m\mu)H_{m-1}^*(s,\mathrm{Pr}) &= (m-1)\mu H_{m-2}^*(s,\mathrm{Pr}) + \lambda,
\end{aligned}$$

$$\begin{aligned}
(s+\lambda+\mu)H_0^*(s,\mathrm{Sr}) &= \mu + \lambda H_1^*(s,\mathrm{Sr}), \\
[s+\lambda+(k+1)\mu]H_k^*(s,\mathrm{Sr}) &= k\mu H_{k-1}^*(s,\mathrm{Sr}) + \mu + \lambda H_{k+1}^*(s,\mathrm{Sr})
\end{aligned}$$

$$
\begin{aligned}
& & 1 \le k \le m-2, \\
(s+\lambda+m\mu)H^*_{m-1}(s,\mathrm{Sr}) &= (m-1)\mu H^*_{m-2}(s,\mathrm{Sr}) + \mu, \\
(s+\lambda+\mu)H^*_0(s) &= \mu + \lambda H^*_1(s), \\
[s+\lambda+(k+1)\mu]H^*_k(s) &= k\mu H^*_{k-1}(s) + \mu + \lambda H^*_{k+1}(s) \\
& \qquad 1 \le k \le m-2, \\
(s+\lambda+m\mu)H^*_{m-1}(s) &= (m-1)\mu H^*_{m-2}(s) + \lambda + \mu.
\end{aligned}
$$

In addition, we let $H^*_m(s,\mathrm{Pr}) \equiv 1$, $H^*_m(s,\mathrm{Sr}) \equiv 0$, and $H^*_m(s) \equiv 1$. The solution can be obtained, in terms of functions $\{h^*_k(s); 0 \le k \le m\}$, in the form

$$
H^*_k(s,\mathrm{Pr}) = \frac{h^*_k(s)}{h^*_m(s)} \quad ; \quad H^*_k(s,\mathrm{Sr}) = \frac{\mu}{s+\mu}\left[1 - \frac{h^*_k(s)}{h^*_m(s)}\right],
$$

$$
H^*_k(s) = \frac{\mu}{s+\mu} + \frac{s}{s+\mu}\cdot\frac{h^*_k(s)}{h^*_m(s)} \qquad 0 \le k \le m.
$$

2.3. Solution for $\{h^*_k(s); 0 \le k \le m\}$

A finite set of equations for $\{h^*_k(s); 0 \le k \le m\}$ is given by

$$
h^*_0(s) = 1 \quad ; \quad s+\lambda+\mu = \lambda h^*_1(s),
$$

$$
[s+\lambda+(k+1)\mu]h^*_k(s) = k\mu h^*_{k-1}(s) + \lambda h^*_{k+1}(s) \qquad 1 \le k \le m-1.
$$

This can be written as the following set of recurrence relations:

$$
h^*_k(s) = \frac{s+\lambda+k\mu}{\lambda}h^*_{k-1}(s) - \frac{(k-1)\mu}{\lambda}h^*_{k-2}(s) \qquad 2 \le k \le m.
$$

The solution is given by *Cramer's rule* as the determinant

$$
h^*_k(s) = (-1)^k \left|\boldsymbol{H}^{(k)}\right| \qquad 2 \le k \le m,
$$

where $\boldsymbol{H}^{(k)}$ is a $k \times k$ tridiagonal matrix with nonzero elements

$$
\begin{aligned}
\boldsymbol{H}^{(k)}_{i,i} &= -\frac{s+\lambda+i\mu}{\lambda} \qquad 1 \le i \le k, \\
\boldsymbol{H}^{(k)}_{i,i+1} &= 1 \qquad 1 \le i \le k-1, \\
\boldsymbol{H}^{(k)}_{i+1,i} &= \frac{i\mu}{\lambda} \qquad 0 \le i \le k-1.
\end{aligned}
$$

Note that $h_k^*(s)$ is a kth degree polynomial in s, the coefficient of s^k being $(1/\lambda)^k$, $1 \le k \le m$.

At present, we do not have a simple expression for $h_k^*(s)$. However, for $s = 0$, we get

$$h_k^*(0) = \sum_{j=0}^{k} \frac{(m\rho)^j}{j!} \Big/ \frac{(m\rho)^k}{k!} = \frac{1}{B(k, m\rho)} \qquad 0 \le k \le m$$

using *Erlang's B formula*:

$$B(0,a) = 1 \quad ; \quad B(k,a) = \frac{a^k}{k!} \Big/ \sum_{j=0}^{k} \frac{a^j}{j!} \qquad k = 1, 2, \ldots .$$

Thus, we obtain the probability of service preemption and completion

$$\begin{aligned} p_k\{\mathrm{Pr}\} &= H_k^*(0, \mathrm{Pr}) = \frac{h_k^*(0)}{h_m^*(0)}, \\ p_k\{\mathrm{Sr}\} &= H_k^*(0, \mathrm{Sr}) = 1 - p_k\{\mathrm{Pr}\} \qquad 0 \le k \le m. \end{aligned}$$

In particular, we have

$$\begin{aligned} &p_0\{\mathrm{Pr}\} = B(m, m\rho) \quad ; \quad p_m\{\mathrm{Pr}\} = 1, \\ &p_{m-1}\{\mathrm{Pr}\} = \rho[1 - B(m, m\rho)]. \end{aligned}$$

The set of ℓth derivatives $\{h_k^{(\ell)}(s); \ell \le k \le m\}$ satisfies the following set of recurrence relations:

$$\begin{aligned} &h_0^{(\ell)}(s) = h_1^{(\ell)}(s) = h_2^{(\ell)}(s) = \cdots = h_{\ell-1}^{(\ell)}(s) = 0 \quad ; \quad h_\ell^{(\ell)}(s) = \frac{\ell!}{\lambda^\ell}, \\ &h_k^{(\ell)}(s) = \frac{s + \lambda + k\mu}{\lambda} h_{k-1}^{(\ell)}(s) - \frac{(k-1)\mu}{\lambda} h_{k-2}^{(\ell)}(s) + \frac{\ell}{\lambda} h_{k-1}^{(\ell-1)}(s) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ell + 1 \le k \le m. \end{aligned}$$

The ℓth derivatives of $\{h_k^*(s); 0 \le k \le m\}$ with respect to s at $s = 0$ can be calculated recursively by

$$\begin{aligned} &h_0^{(\ell)}(0) = h_1^{(\ell)}(0) = \cdots = h_{\ell-1}^{(\ell)}(0) = 0, \\ &h_k^{(\ell)}(0) = \frac{\ell}{\lambda} \sum_{j=1}^{k} \frac{(m\rho)^j}{j!} \sum_{l=0}^{j-1} h_l^{(\ell-1)}(0) \Big/ \frac{(m\rho)^k}{k!} \qquad \ell \le k \le m. \end{aligned}$$

3. First Passage Time to Service Resumption and Abandonment from State k of Waiting, $k \geq m$

We next consider another infinite-state birth-and-death process of state transitions for a tagged customer in state k, $k \geq m$, in which he is staying in the waiting room. With probability one, this customer, eventually, either is called in to resume his service or abandons waiting.

3.1. Customer Behavior until Service Resumption and Abandonment

The state transition diagram for the discrete-time one-dimensional birth-and-death process modeling the behavior of a tagged customer in the waiting room is shown in Fig. 2. The process has an infinite number of transient states $\{m, m+1, \ldots\}$ and two absorbing states denoted by "Rs" (state $m-1$) and "Ab", representing the service resumption and abandonment, respectively. State transitions occur when another customer arrives, the service to one of customers is completed, and a waiting customer before the tagged one abandons waiting. Therefore, the state transition probabilities and the LST of the DF for the time spent by a tagged customer in state k are given by

$$\alpha'_k = \frac{m\mu + (k-m)\theta}{\lambda + m\mu + (k+1-m)\theta} \quad ; \quad \beta'_k = \frac{\theta}{\lambda + m\mu + (k+1-m)\theta},$$

$$B'_k(s) = \frac{\lambda + m\mu + (k+1-m)\theta}{s + \lambda + m\mu + (k+1-m)\theta}.$$

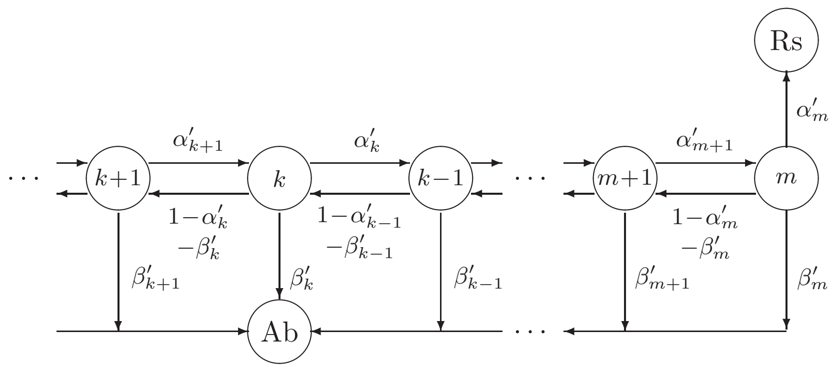

Figure 2: State transition of a tagged customer until service resumption and abandonment.

3.2. LST of the DF for the First Passage Time to Service Resumption and Abandonment

By $W_k^*(s,\mathrm{Rs})$, we denote the joint probability of service resumption and the LST of the DF for the first passage time from state k to state "Rs" without reaching state "Ab". In addition, we denote by $W_k^*(s,\mathrm{Ab})$ the joint probability of abandonment and the LST of the DF for the first passage time from state k to state "Ab" without reaching state "Rs". Furthermore, let

$$W_k^*(s) := W_k^*(s,\mathrm{Rs}) + W_k^*(s,\mathrm{Ab})$$

be the unconditional LST of the DF for the first passage time from state k to state either "Rs" or "Ab", whichever occurs first.

Three infinite sets of equations for $\{W_k^*(s,\mathrm{Rs}); k \geq m\}$, $\{W_k^*(s,\mathrm{Ab}); k \geq m\}$, and $\{W_k^*(s); k \geq m\}$ are respectively given by

$$\begin{aligned}
&(s+\lambda+m\mu+\theta)W_m^*(s,\mathrm{Rs}) = m\mu + \lambda W_{m+1}^*(s,\mathrm{Rs}),\\
&[s+\lambda+m\mu+(k+1-m)\theta]W_k^*(s,\mathrm{Rs})\\
&= [m\mu+(k-m)\theta]W_{k-1}^*(s,\mathrm{Rs}) + \lambda W_{k+1}^*(s,\mathrm{Rs}) \quad k \geq m+1,
\end{aligned}$$

$$\begin{aligned}
&(s+\lambda+m\mu+\theta)W_m^*(s,\mathrm{Ab}) = \theta + \lambda W_{m+1}^*(s,\mathrm{Ab}),\\
&[s+\lambda+m\mu+(k+1-m)\theta]W_k^*(s,\mathrm{Ab})\\
&= [m\mu+(k-m)\theta]W_{k-1}^*(s,\mathrm{Ab}) + \theta + \lambda W_{k+1}^*(s,\mathrm{Ab}) \quad k \geq m+1,
\end{aligned}$$

$$\begin{aligned}
&(s+\lambda+m\mu+\theta)W_m^*(s) = m\mu + \theta + \lambda W_{m+1}^*(s),\\
&[s+\lambda+m\mu+(k+1-m)\theta]W_k^*(s)\\
&= [m\mu+(k-m)\theta]W_{k-1}^*(s) + \theta + \lambda W_{k+1}^*(s) \quad k \geq m+1.
\end{aligned}$$

The solution is expressed, in terms of the set of functions $\{G_k^*(s); k \geq m\}$, in the form

$$\begin{aligned}
&W_k^*(s,\mathrm{Rs}) = G_k^*(s+\theta) \quad ; \quad W_k^*(s,\mathrm{Ab}) = \frac{\theta}{s+\theta}[1-G_k^*(s+\theta)],\\
&W_k^*(s) = \frac{\theta + sG_k^*(s+\theta)}{s+\theta} \qquad k \geq m.
\end{aligned}$$

Thus we obtain the probability of service resumption and abandonment

$$\begin{aligned}
p_k\{\mathrm{Rs}\} &= W_k^*(0,\mathrm{Rs}) = G_k^*(\theta),\\
p_k\{\mathrm{Ab}\} &= W_k^*(0,\mathrm{Ab}) = 1 - G_k^*(\theta) \qquad k \geq m
\end{aligned}$$

and the conditional and unconditional mean time to resumption and abandonment

$$E[W_k;\mathrm{Rs}] = -G_k'(\theta) \quad ; \quad E[W_k;\mathrm{Ab}] = \frac{1-G_k(\theta)}{\theta} + G_k'(\theta),$$
$$E[W_k] = E[W_k;\mathrm{Rs}] + E[W_k;\mathrm{Ab}] = \frac{1-G_k(\theta)}{\theta} \qquad k \geq m.$$

3.3. Busy Period

A *busy period* started with k ($\geq m$) customers in an M/M/m queue is the time interval, denoted by $\mathcal{G}_k$, from the instant at which there are k customers in the system (all servers are busy and $k-m$ customers are waiting) to the first instant at which any one of the servers becomes available. Let us denote by $f_{W_k}(t,\mathrm{Rs})$ and $f_{W_k}(t,\mathrm{Ab})$ the density functions of the time until service resumption and the time until abandonment, respectively, for a customer in state k, $k \geq m$. They are related with the density function $f_{\mathcal{G}_k}(t)$ for $\mathcal{G}_k$ and the probability $P\{\mathcal{G}_k > t\}$ as follows:

$$f_{W_k}(t,\mathrm{Rs}) = e^{-\theta t} f_{\mathcal{G}_k}(t) \quad ; \quad f_{W_k}(t,\mathrm{Ab}) = \theta e^{-\theta t} P\{\mathcal{G}_k > t\}.$$

The function $G_k^*(s)$ introduced in Section 3.2 is the LST of the DF for $\mathcal{G}_k$, $k \geq m$. The set of equations for $\{G_k^*(s), k \geq m\}$ is given by

$$\begin{aligned}
&(s+\lambda+m\mu)G_m^*(s) = \lambda G_{m+1}^*(s) + m\mu,\\
&[s+\lambda+m\mu+(k-m)\theta]G_k^*(s)\\
&= [m\mu+(k-m)\theta]G_{k-1}^*(s) + \lambda G_{k+1}^*(s) \qquad k \geq m+1.
\end{aligned}$$

Subba Rao [11] derives the LST of the DF for the duration of a busy period in an M/G/1 queue with impatient customers. Iravani and Balcıoğlu [3] provides the LST of the DF for the duration of a busy period in an M/M/m queue with exponentially distributed service times of mean $1/(m\mu)$ by modifying the result of Subba Rao as follows:

$$G_k^*(s) = \frac{\dfrac{m\mu}{s+m\mu} + \displaystyle\sum_{i=1}^{\infty}(-1)^i \psi_{i,k-m}(\lambda/\theta) \times \left[\prod_{j=0}^{i-1}\left(1-\frac{m\mu}{s+m\mu+j\theta}\right)\right]\frac{m\mu}{s+m\mu+i\theta}}{1+\displaystyle\sum_{i=1}^{\infty}\frac{(\lambda/\theta)^i}{i!}\prod_{j=0}^{i-1}\left(1-\frac{m\mu}{s+m\mu+j\theta}\right)} \qquad k \geq m,$$

where we define

$$\psi_{i,k}(x) := \sum_{j=\max\{0,i-k\}}^{i} \frac{(-x)^j}{j!}\binom{k}{i-j} \qquad i \geq 1,\ k \geq 0$$

with $\psi_{0,k}(x) \equiv 1$ for $k \geq 0$. Thus we get

$$G_k^*(\theta) = \sum_{i=0}^{\infty} \frac{i!(-\tau/m)^i \psi_{i,k-m}(\lambda/\theta)}{\prod_{j=1}^{i+1}(1+j\tau/m)} \Bigg/ \sum_{i=0}^{\infty} \frac{\rho^i}{\prod_{j=0}^{i}(1+j\tau/m)}\,.$$

In particular, for $k = m$, since $\psi_{i,0}(x) = (-x)^i/i!$, we obtain

$$G_m^*(s) = \frac{\dfrac{m\mu}{s+m\mu} + \displaystyle\sum_{i=1}^{\infty} \frac{(\lambda/\theta)^i}{i!} \times \left[\prod_{j=0}^{i-1}\left(1 - \frac{m\mu}{s+m\mu+j\theta}\right)\right] \frac{m\mu}{s+m\mu+i\theta}}{1 + \displaystyle\sum_{i=1}^{\infty} \frac{(\lambda/\theta)^i}{i!} \prod_{j=0}^{i-1}\left(1 - \frac{m\mu}{s+m\mu+j\theta}\right)},$$

$$1 - G_m^*(s+\theta) = \frac{\dfrac{s+\theta}{s+m\mu+\theta} + \displaystyle\sum_{i=1}^{\infty} \frac{(\lambda/\theta)^i}{i!} \times \left[\prod_{j=1}^{i}\left(1 - \frac{m\mu}{s+m\mu+j\theta}\right)\right] \frac{s+(i+1)\theta}{s+m\mu+(i+1)\theta}}{1 + \displaystyle\sum_{i=1}^{\infty} \frac{(\lambda/\theta)^i}{i!} \prod_{j=1}^{i}\left(1 - \frac{m\mu}{s+m\mu+j\theta}\right)}.$$

Thus we have

$$\begin{aligned} p_m\{\mathrm{Ab}\} &= 1 - G_m^*(\theta) \\ &= \frac{\theta}{\lambda} \sum_{i=1}^{\infty} \frac{i\rho^i}{\prod_{j=1}^{i}(1+j\tau/m)} \Bigg/ \sum_{i=0}^{\infty} \frac{\rho^i}{\prod_{j=0}^{i}(1+j\tau/m)}\,. \end{aligned}$$

The first and second derivatives of $G_k^*(s)$ with respect to s at $s = \theta$, $k \geq m$, are given in Appendix.

Alternatively, Jagerman [4, p. 231] shows that

$$G_k^*(s) = \frac{\left(\frac{m\mu}{\theta}\right)_{k-m+1} F\left(\frac{s}{\theta}, k-m+1+\frac{s+m\mu}{\theta}; \frac{\lambda}{\theta}\right)}{\left(\frac{s+m\mu}{\theta}\right)_{k-m+1} F\left(\frac{s}{\theta}, \frac{s+m\mu}{\theta}; \frac{\lambda}{\theta}\right)} \qquad k \geq m$$

and

$$G_m^*(s) = m\mu F\left(\frac{s}{\theta}, 1+\frac{s+m\mu}{\theta}; \frac{\lambda}{\theta}\right) \Big/ \left[(s+m\mu) F\left(\frac{s}{\theta}, \frac{s+m\mu}{\theta}; \frac{\lambda}{\theta}\right)\right],$$

where the *Kummer's series of the confluent hypergeometric function* [2, p. 248] and the ascending factorial are defined by

$$F(a,b;x) := 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!},$$

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+n-1) \qquad n \geq 1.$$

We can algebraically confirm that

$$1 + \sum_{i=1}^{\infty} \frac{(\lambda/\theta)^i}{i!} \prod_{j=0}^{i-1} \left(1 - \frac{m\mu}{s+m\mu+j\theta}\right) = F\left(\frac{s}{\theta}, \frac{s+m\mu}{\theta}; \frac{\lambda}{\theta}\right),$$

$$\frac{m\mu}{s+m\mu} + \sum_{i=1}^{\infty} \frac{(\lambda/\theta)^i}{i!} \left[\prod_{j=0}^{i-1} \left(1 - \frac{m\mu}{s+m\mu+j\theta}\right)\right] \frac{m\mu}{s+m\mu+i\theta}$$
$$= \frac{m\mu}{s+m\mu} F\left(\frac{s}{\theta}, 1+\frac{s+m\mu}{\theta}; \frac{\lambda}{\theta}\right),$$

and numerically confirm that

$$\frac{m\mu}{s+m\mu}$$
$$+ \sum_{i=1}^{\infty} (-1)^i \psi_{i,k-m}(\lambda/\theta) \left[\prod_{j=0}^{i-1} \left(1 - \frac{m\mu}{s+m\mu+j\theta}\right)\right] \frac{m\mu}{s+m\mu+i\theta}$$
$$= \left[\left(\frac{m\mu}{\theta}\right)_{k-m+1} \Big/ \left(\frac{s+m\mu}{\theta}\right)_{k-m+1}\right]$$
$$\times F\left(\frac{s}{\theta}, k-m+1+\frac{s+m\mu}{\theta}; \frac{\lambda}{\theta}\right) \qquad k \geq m.$$

4. Joint Distribution of the Waiting and Service Time until Departure

We are now in a position to consider the distribution of the time until departure (either abandonment or service completion) for the tagged customer in a combination of two birth-and-death processes whose state transitions are depicted in Figs. 1 and 2. We note that state "Pr" in Fig. 1 and state m in Fig. 2 are actually the common single state, so are state "Rs" in Fig. 2 and state $m-1$ in Fig. 1.

The time until departure consists of the waiting time (the time that the customer spends staying in the waiting room) and the service time (the time during which the customer is being served). The waiting time and the service time are not independent of each other. Therefore, we will derive the joint LST of the DF for the waiting and service time for a customer who abandons waiting, denoted by $\mathcal{T}_k^*(s,s',\mathrm{Ab})$, and for a customer who gets served until completion, denoted by $\mathcal{T}_k^*(s,s',\mathrm{Sr})$. Then, we obtain the probability of abandonment and service completion, the marginal LST of the DF for the waiting time, the service time, and the total time spent in the system as follows:

$$\begin{aligned}
\mathcal{P}_k\{\mathrm{Ab}\} &:= \mathcal{T}_k^*(0,0,\mathrm{Ab}) \quad ; \quad \mathcal{P}_k\{\mathrm{Sr}\} := \mathcal{T}_k^*(0,0,\mathrm{Sr}),\\
\mathcal{W}_k^*(s,\mathrm{Ab}) &:= \mathcal{T}_k^*(s,0,\mathrm{Ab}) \quad ; \quad \mathcal{H}_k^*(s,\mathrm{Ab}) := \mathcal{T}_k^*(0,s,\mathrm{Ab}),\\
\mathcal{W}_k^*(s,\mathrm{Sr}) &:= \mathcal{T}_k^*(s,0,\mathrm{Sr}) \quad ; \quad \mathcal{H}_k^*(s,\mathrm{Sr}) := \mathcal{T}_k^*(0,s,\mathrm{Sr}),\\
\mathcal{T}_k^*(s,\mathrm{Ab}) &:= \mathcal{T}_k^*(s,s,\mathrm{Ab}) \quad ; \quad \mathcal{T}_k^*(s,\mathrm{Sr}) := \mathcal{T}_k^*(s,s,\mathrm{Sr}).
\end{aligned}$$

4.1. Waiting and Service Time until Abandonment

We first consider the waiting and service time until abandonment for an arbitrary customer who abandons waiting.

(1) For a customer being served in state k, $0 \le k \le m-1$, the first passage to abandonment ("Ab") during his waiting time consists of the following passages:

   (i) the initial passage from state k to state "Pr" in Fig. 1 (which is state m in Fig. 2),

   (ii) no or several cycles of transitions from state m to state "Rs" in Fig. 2 and from state $m-1$ back to state "Pr" in Fig. 1, followed by

   (iii) the final passage from state m to state "Ab" in Fig. 2.

Owing to the Markovian property of state transitions, the times to take these passages in succession are independent of each other. Therefore, we get

$$\begin{aligned}
&\mathcal{T}_k^*(s,s',\mathrm{Ab}) = H_k^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Ab}) \\
&+H_k^*(s',\mathrm{Pr})[W_m^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Pr})]W_m^*(s,\mathrm{Ab}) \\
&+H_k^*(s',\mathrm{Pr})[W_m^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Pr})]^2W_m^*(s,\mathrm{Ab}) + \cdots \\
&= H_k^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Ab})\sum_{n=0}^{\infty}[W_m^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Pr})]^n \\
&= \frac{H_k^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Ab})}{1-W_m^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Pr})} \\
&= \frac{\theta}{s+\theta}\cdot\frac{h_k^*(s')[1-G_m^*(s+\theta)]}{h_m^*(s')-h_{m-1}^*(s')G_m^*(s+\theta)}.
\end{aligned}$$

This joint distribution leads to the marginal distribution

$$\begin{aligned}
\mathcal{W}_k^*(s,\mathrm{Ab}) &= \frac{p_k\{\mathrm{Pr}\}W_m^*(s,\mathrm{Ab})}{1-p_{m-1}\{\mathrm{Pr}\}W_m^*(s,\mathrm{Rs})} \\
&= \frac{\theta}{s+\theta}\cdot\frac{h_k^*(0)[1-G_m^*(s+\theta)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(s+\theta)}, \\
\mathcal{H}_k^*(s,\mathrm{Ab}) &= \frac{p_m\{\mathrm{Ab}\}H_k^*(s,\mathrm{Pr})}{1-p_m\{\mathrm{Rs}\}H_{m-1}^*(s,\mathrm{Pr})} \\
&= \frac{h_k^*(s)[1-G_m^*(\theta)]}{h_m^*(s)-h_{m-1}^*(s)G_m^*(\theta)}, \\
\mathcal{T}_k^*(s,\mathrm{Ab}) &= \frac{H_k^*(s,\mathrm{Pr})W_m^*(s,\mathrm{Ab})}{1-W_m^*(s,\mathrm{Rs})H_{m-1}^*(s,\mathrm{Pr})} \\
&= \frac{\theta}{s+\theta}\cdot\frac{h_k^*(s)[1-G_m^*(s+\theta)]}{h_m^*(s)-h_{m-1}^*(s)G_m^*(s+\theta)}.
\end{aligned}$$

Then we obtain the probability of abandonment

$$\mathcal{P}_k\{\mathrm{Ab}\} = \frac{p_k\{\mathrm{Pr}\}p_m\{\mathrm{Ab}\}}{1-p_m\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}} = \frac{h_k^*(0)[1-G_m^*(\theta)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)},$$

the mean waiting and service time

$$E[\mathcal{W}_k,\mathrm{Ab}] = \frac{1}{\theta}\mathcal{P}_k\{\mathrm{Ab}\} + \frac{h_k^*(0)[h_m^*(0)-h_{m-1}^*(0)]G_m'(\theta)}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2},$$

$$\begin{aligned} E[\mathcal{H}_k, \mathrm{Ab}] &= -[1-G_m^*(\theta)]\left\{\frac{h_k'(0)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)}\right. \\ &\quad \left. -\frac{h_k^*(0)[h_m'(0)-h_{m-1}'(0)G_m^*(\theta)]}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2}\right\}, \end{aligned}$$

$$\begin{aligned} E[\mathcal{T}_k, \mathrm{Ab}] &= \frac{1}{\theta}\mathcal{P}_k\{\mathrm{Ab}\} - \frac{h_k'(0)[1-G_m^*(\theta)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)} \\ &\quad + \frac{h_k^*(0)\left\{\begin{array}{l}[h_m^*(0)-h_{m-1}^*(0)]G_m'(\theta) \\ +\,[h_m'(0)-h_{m-1}'(0)G_m^*(\theta)][1-G_m^*(\theta)]\end{array}\right\}}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2}, \end{aligned}$$

and the second moment of the waiting and service time

$$\begin{aligned} &E[\mathcal{W}_k^2, \mathrm{Ab}] = \frac{2}{\theta}E[\mathcal{W}_k, \mathrm{Ab}] - h_k^*(0)[h_m^*(0)-h_{m-1}^*(0)] \\ &\times\left\{\frac{G_m''(\theta)}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2} + \frac{2h_{m-1}^*(0)[G_m'(\theta)]^2}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^3}\right\}, \\ &E[\mathcal{H}_k^2, \mathrm{Ab}] = [1-G_m^*(\theta)]\left\{\frac{h_k''(0)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)}\right. \\ &\qquad - \frac{2h_k'(0)[h_m'(0)-h_{m-1}'(0)G_m^*(\theta)]}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2} \\ &\qquad - \frac{h_k^*(0)[h_m''(0)-h_{m-1}''(0)G_m^*(\theta)]}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2} \\ &\qquad \left. + \frac{2h_k^*(0)[h_m'(0)-h_{m-1}'(0)G_m^*(\theta)]^2}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^3}\right\}, \\ &E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Ab}] = \frac{1}{\theta}E[\mathcal{H}_k, \mathrm{Ab}] \\ &+ G_m'(\theta)\left\{\frac{2h_k^*(0)[h_m^*(0)-h_{m-1}^*(0)][h_m'(0)-h_{m-1}'(0)G_m^*(\theta)]}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^3}\right. \\ &\left. - \frac{h_k^*(0)[h_m'(0)-h_{m-1}'(0)] + h_k'(0)[h_m^*(0)-h_{m-1}^*(0)]}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2}\right\}. \end{aligned}$$

For the total time spent in the system, we can write

$$\mathcal{T}_k^*(s, \mathrm{Ab}) = \frac{\theta}{s+\theta}\cdot h_k^*(s)U^*(s),$$

$$\text{where} \quad U^*(s) := \frac{1-G_m^*(s+\theta)}{h_m^*(s)-h_{m-1}^*(s)G_m^*(s+\theta)}.$$

Then, we have

$$\mathcal{P}_k\{\text{Ab}\} = h_k^*(0)U^*(0),$$

$$E[\mathcal{T}_k^\ell, \text{Ab}] = \frac{\ell!}{\theta^\ell}\sum_{l=0}^{\ell}\frac{(-\theta)^l}{l!}\sum_{n=0}^{l}\binom{l}{n}h_k^{(n)}(0)U^{(l-n)}(0)$$

$$\ell = 1, 2, \ldots,$$

where the derivatives of $U^*(s)$ with respect to s at $s = 0$ are given by

$$U^*(0) = \frac{1 - G_m^*(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)},$$

$$U'(0) = -\frac{\left\{\begin{array}{l}[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta) \\ +\ [h_m'(0) - h_{m-1}'(0)G_m^*(\theta)][1 - G_m^*(\theta)]\end{array}\right\}}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2},$$

$$U''(0) = -\frac{\left\{\begin{array}{l}[h_m^*(0) - h_{m-1}^*(0)]G_m''(\theta) \\ +[h_m''(0) - h_{m-1}''(0)G_m^*(\theta) - 2h_{m-1}'(0)G_m'(\theta)] \\ \times\ [1 - G_m^*(\theta)]\end{array}\right\}}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2} + \frac{2\left\{\begin{array}{l}[h_m'(0) - h_{m-1}'(0)G_m^*(\theta) - h_{m-1}^*(0)G_m'(\theta)] \\ \times\{[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta) \\ +\ [h_m'(0) - h_{m-1}'(0)G_m^*(\theta)][1 - G_m^*(\theta)]\}\end{array}\right\}}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^3}.$$

(2) For a customer waiting in state k, $k \geq m$, the first passage to abandonment ("Ab") is either

  (a) a direct passage from state k to state "Ab" without reaching state "Rs" in Fig. 2, or

  (b) a sequence of the following passages:

    (i) the initial passage from state k to state "Rs" in Fig. 2 (which is state $m - 1$ in Fig. 1),

    (ii) no or several cycles of transitions from state $m-1$ to state "Pr" in Fig. 1 and from state m back to state "Rs" in Fig. 2,

    (iii) the passage from state $m-1$ to state "Pr" in Fig. 1, followed by

(iv) the final passage from state m to state "Ab" in Fig. 2.

Therefore, from the Markovian property of state transitions, we get

$$\begin{aligned}
&\mathcal{T}_k^*(s,s',\mathrm{Ab}) = W_k^*(s,\mathrm{Ab})\\
&+W_k^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Ab})\\
&+W_k^*(s,\mathrm{Rs})[H_{m-1}^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Rs})]H_{m-1}^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Ab})\\
&+W_k^*(s,\mathrm{Rs})[H_{m-1}^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Rs})]^2H_{m-1}^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Ab})\\
&+\cdots\\
&= W_k^*(s,\mathrm{Ab}) + W_k^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Ab})\\
&\qquad\qquad\times\sum_{n=0}^{\infty}[H_{m-1}^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Rs})]^n\\
&= W_k^*(s,\mathrm{Ab}) + \frac{W_k^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Ab})}{1-W_m^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Pr})}\\
&= \frac{\theta}{s+\theta}\left\{1-\frac{[h_m^*(s')-h_{m-1}^*(s')]G_k^*(s+\theta)}{h_m^*(s')-h_{m-1}^*(s')G_m^*(s+\theta)}\right\}.
\end{aligned}$$

This joint distribution leads to the marginal distribution

$$\begin{aligned}
\mathcal{W}_k^*(s,\mathrm{Ab}) &= W_k^*(s,\mathrm{Ab}) + \frac{p_{m-1}\{\mathrm{Pr}\}W_k^*(s,\mathrm{Rs})W_m^*(s,\mathrm{Ab})}{1-p_{m-1}\{\mathrm{Pr}\}W_m^*(s,\mathrm{Rs})}\\
&= \frac{\theta}{s+\theta}\left\{1-\frac{[h_m^*(0)-h_{m-1}^*(0)]G_k^*(s+\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(s+\theta)}\right\},\\
\mathcal{H}_k^*(s,\mathrm{Ab}) &= p_k\{\mathrm{Ab}\} + \frac{p_k\{\mathrm{Rs}\}p_m\{\mathrm{Ab}\}H_{m-1}^*(s,\mathrm{Pr})}{1-p_m\{\mathrm{Rs}\}H_{m-1}^*(s,\mathrm{Pr})}\\
&= 1-\frac{[h_m^*(s)-h_{m-1}^*(s)]G_k^*(\theta)}{h_m^*(s)-h_{m-1}^*(s)G_m^*(\theta)},\\
\mathcal{T}_k^*(s,\mathrm{Ab}) &= W_k^*(s,\mathrm{Ab}) + \frac{W_k^*(s,\mathrm{Rs})H_{m-1}^*(s,\mathrm{Pr})W_m^*(s,\mathrm{Ab})}{1-W_m^*(s,\mathrm{Rs})H_{m-1}^*(s,\mathrm{Pr})}\\
&= \frac{\theta}{s+\theta}\left\{1-\frac{[h_m^*(s)-h_{m-1}^*(s)]G_k^*(s+\theta)}{h_m^*(s)-h_{m-1}^*(s)G_m^*(s+\theta)}\right\}.
\end{aligned}$$

Therefore, we obtain the probability of abandonment

$$\begin{aligned}
\mathcal{P}_k\{\mathrm{Ab}\} &= p_k\{\mathrm{Ab}\} + \frac{p_k\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}p_m\{\mathrm{Ab}\}}{1-p_{m-1}\{\mathrm{Pr}\}p_m\{\mathrm{Rs}\}}\\
&= 1-\frac{[h_m^*(0)-h_{m-1}^*(0)]G_k^*(\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)},
\end{aligned}$$

the mean waiting and service time

$$
\begin{aligned}
&E[\mathcal{W}_k,\mathrm{Ab}] = \frac{1}{\theta}\mathcal{P}_k\{\mathrm{Ab}\} + [h_m^*(0) - h_{m-1}^*(0)] \\
&\times \left\{ \frac{G_k'(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)} + \frac{G_k^*(\theta)h_{m-1}^*(0)G_m'(\theta)}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2} \right\}, \\
&E[\mathcal{H}_k,\mathrm{Ab}] = \frac{G_k^*(\theta)[h_m'(0)h_{m-1}^*(0) - h_m^*(0)h_{m-1}'(0)][1 - G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2}, \\
&E[\mathcal{T}_k,\mathrm{Ab}] = \frac{1}{\theta}\mathcal{P}_k\{\mathrm{Ab}\} + \frac{G_k'(\theta)[h_m^*(0) - h_{m-1}^*(0)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)} \\
&+ \frac{G_k^*(\theta) \left\{ \begin{array}{l} [h_m^*(0) - h_{m-1}^*(0)]h_{m-1}^*(0)G_m'(\theta) \\ + [h_m'(0)h_{m-1}^*(0) - h_m^*(0)h_{m-1}'(0)][1 - G_m^*(\theta)] \end{array} \right\}}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2},
\end{aligned}
$$

and the second moment of the waiting and service time

$$
\begin{aligned}
&E[\mathcal{W}_k^2,\mathrm{Ab}] = \frac{2}{\theta}E[\mathcal{W}_k,\mathrm{Ab}] - [h_m^*(0) - h_{m-1}^*(0)] \\
&\times \left\{ \frac{G_k''(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)} + \frac{2G_k^*(\theta)[h_{m-1}^*(0)G_m'(\theta)]^2}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^3} \right. \\
&\quad \left. + \frac{h_{m-1}^*(0)[G_k^*(\theta)G_m''(\theta) + 2G_k'(\theta)G_m'(\theta)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2} \right\}, \\
&E[\mathcal{H}_k^2,\mathrm{Ab}] = G_k^*(\theta)[1 - G_m^*(\theta)] \\
&\times \left\{ \frac{h_m^*(0)h_{m-1}''(0) - h_m''(0)h_{m-1}^*(0)}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2} \right. \\
&\left. - \frac{2[h_m^*(0)h_{m-1}'(0) - h_m'(0)h_{m-1}^*(0)][h_m'(0) - h_{m-1}'(0)G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^3} \right\}, \\
&E[\mathcal{W}_k\mathcal{H}_k,\mathrm{Ab}] = \frac{1}{\theta}E[\mathcal{H}_k,\mathrm{Ab}] \\
&+ [h_m'(0)h_{m-1}^*(0) - h_{m-1}'(0)h_m^*(0)] \\
&\times \left\{ \frac{G_k^*(\theta)G_m'(\theta)[h_m^*(0) - 2h_{m-1}^*(0) + h_{m-1}^*(0)G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^3} \right. \\
&\quad \left. - \frac{G_k'(\theta)[1 - G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2} \right\}.
\end{aligned}
$$

For the total time spent in the system, we can write

$$
\mathcal{T}_k^*(s,\mathrm{Ab}) = \frac{\theta}{s+\theta}[1 - G_k^*(s+\theta)V^*(s)],
$$

$$\text{where} \quad V^*(s) := \frac{h_m^*(s) - h_{m-1}^*(s)}{h_m^*(s) - h_{m-1}^*(s)G_m^*(s+\theta)}.$$

Then, we have

$$\begin{aligned}
\mathcal{P}_k\{\text{Ab}\} &= 1 - G_k^*(\theta)V^*(0), \\
E[\mathcal{T}_k^\ell, \text{Ab}] &= \frac{\ell!}{\theta^\ell}\left[1 - \sum_{l=0}^{\ell}\frac{(-\theta)^l}{l!}\sum_{n=0}^{l}\binom{l}{n}G_k^{(n)}(\theta)V^{(l-n)}(0)\right] \\
&\qquad\qquad \ell = 1, 2, \dots,
\end{aligned}$$

where the derivatives of $V^*(s)$ with respect to s at $s = 0$ are given by

$$V^*(0) = \frac{h_m^*(0) - h_{m-1}^*(0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)},$$

$$V'(0) = \frac{\left\{\begin{array}{l} [h_m^*(0) - h_{m-1}^*(0)]h_{m-1}^*(0)G_m'(\theta) \\ + [h_m'(0)h_{m-1}^*(0) - h_m^*(0)h_{m-1}'(0)][1 - G_m^*(\theta)] \end{array}\right\}}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2},$$

$$\begin{aligned}
V''(0) = {} & \frac{\left\{\begin{array}{l} [h_m^*(0) - h_{m-1}^*(0)] \\ \times[h_{m-1}^*(0)G_m''(\theta) + 2h_{m-1}'(0)G_m'(\theta)] \\ + [h_m''(0)h_{m-1}^*(0) - h_m^*(0)h_{m-1}''(0)][1 - G_m^*(\theta)] \end{array}\right\}}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2} \\
& - \frac{2\left\{\begin{array}{l} [h_m'(0) - h_{m-1}'(0)G_m^*(\theta) - h_{m-1}^*(0)G_m'(\theta)] \\ \times \{[h_m^*(0) - h_{m-1}^*(0)]h_{m-1}^*(0)G_m'(\theta) \\ + [h_m'(0)h_{m-1}^*(0) - h_m^*(0)h_{m-1}'(0)][1 - G_m^*(\theta)]\} \end{array}\right\}}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^3}.
\end{aligned}$$

From the relation $h_m^*(s)U^*(s) = 1 - G_m^*(s+\theta)V^*(s)$, we have two equivalent expressions

$$\begin{aligned}
\mathcal{T}_m^*(s, \text{Ab}) &= \frac{\theta}{s+\theta} \cdot h_m^*(s)U^*(s) \\
&= \frac{\theta}{s+\theta}[1 - G_m^*(s+\theta)V^*(s)]
\end{aligned}$$

$$= \frac{\theta}{s+\theta} \cdot \frac{h_m^*(s)[1-G_m^*(s+\theta)]}{h_m^*(s)-h_{m-1}^*(s)G_m^*(s+\theta)}$$

with

$$\sum_{n=0}^{l} \binom{l}{n} h_m^{(n)}(0)U^{(l-n)}(0) + \sum_{n=0}^{l} \binom{l}{n} G_m^{(n)}(\theta)V^{(l-n)}(0) = 0 \qquad l = 1, 2, \ldots .$$

4.2. Waiting and Service Time until Service Completion

We next consider the waiting and service time until service completion for an arbitrary customer who gets served.

(1) For a customer being served in state k, $0 \le k \le m-1$, by an argument similar to the one in Section 4.1 (2), we get

$$\begin{aligned}
&\mathcal{T}_k^*(s, s', \mathrm{Sr}) = H_k^*(s', \mathrm{Sr}) \\
&+H_k^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})H_{m-1}^*(s', \mathrm{Sr}) \\
&+H_k^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})[H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})]H_{m-1}^*(s', \mathrm{Sr}) \\
&+H_k^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})[H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})]^2 H_{m-1}^*(s', \mathrm{Sr}) \\
&+\cdots \\
&= H_k^*(s', \mathrm{Sr}) + H_k^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})H_{m-1}^*(s', \mathrm{Sr}) \\
&\qquad \times \sum_{n=0}^{\infty} [H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})]^n \\
&= H_k^*(s', \mathrm{Sr}) + \frac{H_k^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})H_{m-1}^*(s', \mathrm{Sr})}{1 - W_m^*(s, \mathrm{Rs})H_{m-1}^*(s', \mathrm{Pr})} \\
&= \frac{\mu}{s'+\mu}\left\{1 - \frac{h_k^*(s')[1-G_m^*(s+\theta)]}{h_m^*(s')-h_{m-1}^*(s')G_m^*(s+\theta)}\right\}.
\end{aligned}$$

This joint distribution leads to the marginal distribution

$$\begin{aligned}
\mathcal{W}_k^*(s, \mathrm{Sr}) &= p_k\{\mathrm{Sr}\} + \frac{p_k\{\mathrm{Pr}\}p_{m-1}\{\mathrm{Sr}\}W_m^*(s, \mathrm{Rs})}{1 - p_{m-1}\{\mathrm{Pr}\}W_m^*(s, \mathrm{Rs})} \\
&= 1 - \frac{h_k^*(0)[1-G_m^*(s+\theta)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(s+\theta)}, \\
\mathcal{H}_k^*(s, \mathrm{Sr}) &= H_k^*(s, \mathrm{Sr}) + \frac{p_m\{\mathrm{Rs}\}H_k^*(s, \mathrm{Pr})H_{m-1}^*(s, \mathrm{Sr})}{1 - p_m\{\mathrm{Rs}\}H_{m-1}^*(s, \mathrm{Pr})}
\end{aligned}$$

$$= \frac{\mu}{s+\mu}\left\{1 - \frac{h_k^*(s)[1-G_m^*(\theta)]}{h_m^*(s) - h_{m-1}^*(s)G_m^*(\theta)}\right\},$$

$$\begin{aligned}\mathcal{T}_k^*(s,\mathrm{Sr}) &= H_k^*(s,\mathrm{Sr}) + \frac{H_k^*(s,\mathrm{Pr})W_m^*(s,\mathrm{Rs})H_{m-1}^*(s,\mathrm{Sr})}{1 - W_m^*(s,\mathrm{Rs})H_{m-1}^*(s,\mathrm{Pr})} \\ &= \frac{\mu}{s+\mu}\left\{1 - \frac{h_k^*(s)[1-G_m^*(s+\theta)]}{h_m^*(s) - h_{m-1}^*(s)G_m^*(s+\theta)}\right\}.\end{aligned}$$

Then we obtain the probability of service completion

$$\begin{aligned}\mathcal{P}_k\{\mathrm{Sr}\} &= p_k\{\mathrm{Sr}\} + \frac{p_k\{\mathrm{Pr}\}p_{m-1}\{\mathrm{Sr}\}p_m\{\mathrm{Rs}\}}{1 - p_{m-1}\{\mathrm{Pr}\}p_m\{\mathrm{Rs}\}} \\ &= 1 - \frac{h_k^*(0)[1-G_m^*(\theta)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)} = 1 - \mathcal{P}_k\{\mathrm{Ab}\},\end{aligned}$$

the mean waiting and service time

$$E[\mathcal{W}_k,\mathrm{Sr}] = -\frac{h_k^*(0)[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta)}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2},$$

$$E[\mathcal{H}_k,\mathrm{Sr}] = \frac{1}{\mu}\mathcal{P}_k\{\mathrm{Sr}\} + [1-G_m^*(\theta)]$$

$$\times\left\{\frac{h_k'(0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)} - \frac{h_k^*(0)[h_m'(0) - h_{m-1}'(0)G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2}\right\},$$

$$E[\mathcal{T}_k,\mathrm{Sr}] = \frac{1}{\mu}\mathcal{P}_k\{\mathrm{Sr}\} + \frac{h_k'(0)[1-G_m^*(\theta)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)}$$

$$-\frac{h_k^*(0)\left\{\begin{array}{l}[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta) \\ +\,[h_m'(0) - h_{m-1}'(0)G_m^*(\theta)][1-G_m^*(\theta)]\end{array}\right\}}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2},$$

and the second moment of the waiting and service time

$$E[\mathcal{W}_k^2,\mathrm{Sr}] = h_k^*(0)[h_m^*(0) - h_{m-1}^*(0)]$$

$$\times\left\{\frac{G_m''(\theta)}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2} + \frac{2h_{m-1}^*(0)[G_m'(\theta)]^2}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^3}\right\},$$

$$E[\mathcal{H}_k^2,\mathrm{Sr}] = \frac{2}{\mu}E[\mathcal{H}_k,\mathrm{Sr}] - [1-G_m^*(\theta)]$$

$$\times\left\{\frac{h_k''(0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)} - \frac{h_k^*(0)[h_m''(0) - h_{m-1}''(0)G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2}\right.$$

$$-\frac{2h'_k(0)[h'_m(0)-h''_{m-1}(0)G^*_m(\theta)]}{[h^*_m(0)-h^*_{m-1}(0)G^*_m(\theta)]^2}$$
$$\left.+\frac{2h^*_k(0)[h'_m(0)-h'_{m-1}(0)G^*_m(\theta)]^2}{[h^*_m(0)-h^*_{m-1}(0)G^*_m(\theta)]^3}\right\},$$

$$E[\mathcal{W}_k\mathcal{H}_k,\mathrm{Sr}]=\frac{1}{\mu}E[\mathcal{W}_k,\mathrm{Sr}]+G'_m(\theta)$$
$$\times\left\{\frac{h^*_k(0)[h'_m(0)-h'_{m-1}(0)]+h'_k(0)[h^*_m(0)-h^*_{m-1}(0)]}{[h^*_m(0)-h^*_{m-1}(0)G^*_m(\theta)]^2}\right.$$
$$\left.-\frac{2h^*_k(0)[h^*_m(0)-h^*_{m-1}(0)][h'_m(0)-h'_{m-1}(0)G^*_m(\theta)]}{[h^*_m(0)-h^*_{m-1}(0)G^*_m(\theta)]^3}\right\}.$$

For the total time spent in the system, we can write

$$\mathcal{T}^*_k(s,\mathrm{Sr})=\frac{\mu}{s+\mu}[1-h^*_k(s)U^*(s)],$$

where $U^*(s)$ is given in Section 4.1(1). Then, we get

$$\mathcal{P}_k\{\mathrm{Sr}\} = 1-h^*_k(0)U^*(0),$$
$$E[\mathcal{T}^\ell_k,\mathrm{Sr}] = \frac{\ell!}{\mu^\ell}\left[1-\sum_{l=0}^{\ell}\frac{(-\mu)^l}{l!}\sum_{n=0}^{l}\binom{l}{n}h_k^{(n)}(0)U^{(l-n)}(0)\right]$$
$$\ell=1,2,\ldots.$$

(2) For a customer waiting in state k, $k\geq m$, by an argument similar to the one in Section 4.1(1), we get

$$\mathcal{T}^*_k(s,s',\mathrm{Sr})=W^*_k(s,\mathrm{Rs})H^*_{m-1}(s',\mathrm{Sr})$$
$$+W^*_k(s,\mathrm{Rs})[H^*_{m-1}(s',\mathrm{Pr})W^*_m(s,\mathrm{Rs})]H^*_{m-1}(s,\mathrm{Sr})$$
$$+W^*_k(s,\mathrm{Rs})[H^*_{m-1}(s',\mathrm{Pr})W^*_m(s,\mathrm{Rs})]^2H^*_{m-1}(s',\mathrm{Sr})+\cdots$$
$$=W^*_k(s,\mathrm{Rs})H^*_{m-1}(s',\mathrm{Sr})\sum_{n=0}^{\infty}[H^*_{m-1}(s',\mathrm{Pr})W^*_m(s,\mathrm{Rs})]^n$$
$$=\frac{W^*_k(s,\mathrm{Rs})H^*_{m-1}(s',\mathrm{Sr})}{1-W^*_m(s,\mathrm{Rs})H^*_{m-1}(s',\mathrm{Pr})}$$
$$=\frac{\mu}{s'+\mu}\cdot\frac{[h^*_m(s')-h^*_{m-1}(s')]G^*_k(s+\theta)}{h^*_m(s')-h^*_{m-1}(s')G^*_m(s+\theta)}.$$

This joint distribution leads to the marginal distribution

$$\begin{aligned}\mathcal{W}_k^*(s,\mathrm{Sr}) &= \frac{p_{m-1}\{\mathrm{Sr}\}W_k^*(s,\mathrm{Rs})}{1-p_{m-1}\{\mathrm{Pr}\}W_m^*(s,\mathrm{Rs})}\\ &= \frac{[h_m^*(0)-h_{m-1}^*(0)]G_k^*(s+\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(s+\theta)},\end{aligned}$$

$$\begin{aligned}\mathcal{H}_k^*(s,\mathrm{Sr}) &= \frac{p_k\{\mathrm{Rs}\}H_{m-1}^*(s,\mathrm{Sr})}{1-p_m\{\mathrm{Rs}\}H_{m-1}^*(s,\mathrm{Pr})}\\ &= \frac{\mu}{s+\mu}\cdot\frac{[h_m^*(s)-h_{m-1}^*(s)]G_k^*(\theta)}{h_m^*(s)-h_{m-1}^*(s)G_m^*(\theta)},\\ \mathcal{T}_k^*(s,\mathrm{Sr}) &= \frac{W_k^*(s,\mathrm{Rs})H_{m-1}^*(s,\mathrm{Sr})}{1-W_m^*(s,\mathrm{Rs})H_{m-1}^*(s,\mathrm{Pr})},\\ &= \frac{\mu}{s+\mu}\cdot\frac{[h_m^*(s)-h_{m-1}^*(s)]G_k^*(s+\theta)}{h_m^*(s)-h_{m-1}^*(s)G_m^*(s+\theta)}.\end{aligned}$$

Then we get the probability of service completion

$$\begin{aligned}\mathcal{P}_k\{\mathrm{Sr}\} &= \frac{p_k\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Sr}\}}{1-p_m\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}}\\ &= \frac{[h_m^*(0)-h_{m-1}^*(0)]G_k^*(\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)} = 1-\mathcal{P}_k\{\mathrm{Ab}\},\end{aligned}$$

the mean waiting and service time

$$\begin{aligned}&E[\mathcal{W}_k,\mathrm{Sr}] = -[h_m^*(0)-h_{m-1}^*(0)]\\ &\times\left\{\frac{G_k'(\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)}+\frac{G_k^*(\theta)h_{m-1}^*(0)G_m'(\theta)}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2}\right\},\\ &E[\mathcal{H}_k,\mathrm{Sr}] = \frac{1}{\mu}\mathcal{P}_k\{\mathrm{Sr}\}\\ &-\frac{G_k^*(\theta)[h_m'(0)h_{m-1}^*(0)-h_m^*(0)h_{m-1}'(0)][1-G_m^*(\theta)]}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2},\\ &E[\mathcal{T}_k,\mathrm{Sr}] = \frac{1}{\mu}\mathcal{P}_k\{\mathrm{Sr}\}-\frac{G_k'(\theta)[h_m^*(0)-h_{m-1}^*(0)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)}\\ &-\frac{G_k^*(\theta)\left\{\begin{array}{l}[h_m^*(0)-h_{m-1}^*(0)]h_{m-1}^*(0)G_m'(\theta)\\ +[h_m'(0)h_{m-1}^*(0)-h_m^*(0)h_{m-1}'(0)][1-G_m^*(\theta)]\end{array}\right\}}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2},\end{aligned}$$

and the second moment of the waiting and service time

$$E[\mathcal{W}_k^2, \mathrm{Sr}] = [h_m^*(0) - h_{m-1}^*(0)] \left\{ \frac{G_k''(\theta)}{h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)} \right.$$
$$+ \frac{h_{m-1}^*(0)[G_k^*(\theta) G_m''(\theta) + 2G_k'(\theta) G_m'(\theta)]}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)]^2}$$
$$\left. + \frac{2G_k^*(\theta)[h_{m-1}^*(0) G_m'(\theta)]^2}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)]^3} \right\},$$

$$E[\mathcal{H}_k^2, \mathrm{Sr}] = \frac{2}{\mu} E[\mathcal{H}_k, \mathrm{Sr}] - G_k^*(\theta)[1 - G_m^*(\theta)]$$
$$\times \left\{ \frac{h_m^*(0) h_{m-1}''(0) - h_m''(0) h_{m-1}^*(0)}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)]^2} \right.$$
$$\left. - \frac{2[h_m^*(0) h_{m-1}'(0) - h_m'(0) h_{m-1}^*(0)][h_m'(0) - h_{m-1}'(0) G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)]^3} \right\},$$

$$E[\mathcal{W}_k \mathcal{H}_k, \mathrm{Sr}] = \frac{1}{\mu} E[\mathcal{W}_k, \mathrm{Sr}] - [h_m'(0) h_{m-1}^*(0) - h_{m-1}'(0) h_m^*(0)]$$
$$\times \left\{ \frac{G_k^*(\theta) G_m'(\theta)[h_m^*(0) - 2h_{m-1}^*(0) + h_{m-1}^*(0) G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)]^3} \right.$$
$$\left. - \frac{G_k'(\theta)[1 - G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)]^2} \right\}.$$

For the total time spent in the system, we can write

$$\mathcal{T}_k^*(s, \mathrm{Sr}) = \frac{\mu}{s + \mu} G_k^*(s + \theta) V^*(s),$$

where $V^*(s)$ is given in Section 4.1(2). Then, we have

$$\mathcal{P}_k\{\mathrm{Sr}\} = G_k^*(\theta) V^*(0),$$
$$E[\mathcal{T}_k^\ell, \mathrm{Sr}] = \frac{\ell!}{\mu^\ell} \sum_{l=0}^{\ell} \frac{(-\mu)^l}{l!} \sum_{n=0}^{l} \binom{l}{n} G_k^{(n)}(\theta) V^{(l-n)}(0) \qquad \ell = 1, 2, \ldots.$$

We note that

$$\mathcal{T}_m^*(s, \mathrm{Sr}) = \frac{\mu}{s + \mu}[1 - h_m^*(s) U^*(s)] = \frac{\mu}{s + \mu} G_m^*(s + \theta) V^*(s)$$
$$= \frac{\mu}{s + \mu} \cdot \frac{[h_m^*(s) - h_{m-1}^*(s)] G_m^*(s + \theta)}{h_m^*(s) - h_{m-1}^*(s) G_m^*(s + \theta)}.$$

4.3. Waiting and Service Time until Departure

We finally consider the waiting and service time until departure (either abandonment or service completion) for an arbitrary customer in state k ($k \geq 0$). Let

$$\mathcal{T}_k^*(s,s') := \mathcal{T}_k^*(s,s',\mathrm{Ab}) + \mathcal{T}_k^*(s,s',\mathrm{Sr}) \qquad k \geq 0$$

be the unconditional joint LST of the DF for the waiting and service time for a customer in state k. Then, we obtain the marginal LST of the DF for the waiting time, the service time, and the total time spent in the system as follows for $k \geq 0$:

$$\mathcal{W}_k^*(s) := \mathcal{T}_k^*(s,0) \quad ; \quad \mathcal{H}_k^*(s) := \mathcal{T}_k^*(0,s) \quad ; \quad \mathcal{T}_k^*(s) := \mathcal{T}_k^*(s,s).$$

(1) For a customer being served in state k, $0 \leq k \leq m-1$, we have

$$\begin{aligned}\mathcal{T}_k^*(s,s') &= \frac{\mu}{s'+\mu} + \left(\frac{\theta}{s+\theta} - \frac{\mu}{s'+\mu}\right) \\ &\quad \times \frac{h_k^*(s')[1-G_m^*(s+\theta)]}{h_m^*(s') - h_{m-1}^*(s')G_m^*(s+\theta)}.\end{aligned}$$

This joint distribution leads to the marginal distribution

$$\begin{aligned}\mathcal{W}_k^*(s) &= \mathcal{W}_k^*(s,\mathrm{Ab}) + \mathcal{W}_k^*(s,\mathrm{Sr}) \\ &= 1 - \frac{s}{s+\theta} \cdot \frac{h_k^*(0)[1-G_m^*(s+\theta)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(s+\theta)} \\ &= 1 - \frac{s}{\theta}\mathcal{W}_k^*(s,\mathrm{Ab}) = \frac{\theta}{s+\theta} + \frac{s}{s+\theta}\mathcal{W}_k^*(s,\mathrm{Sr}), \\ \mathcal{H}_k^*(s) &= \mathcal{H}_k^*(s,\mathrm{Ab}) + \mathcal{H}_k^*(s,\mathrm{Sr}) \\ &= \frac{\mu}{s+\mu} + \frac{s}{s+\mu} \cdot \frac{h_k^*(s)[1-G_m^*(\theta)]}{h_m^*(s) - h_{m-1}^*(s)G_m^*(\theta)} \\ &= \frac{\mu}{s+\mu} + \frac{s}{s+\mu}\mathcal{H}_k^*(s,\mathrm{Ab}) = 1 - \frac{s}{\mu}\mathcal{H}_k^*(s,\mathrm{Sr}), \\ \mathcal{T}_k^*(s) &= \mathcal{T}_k^*(s,\mathrm{Ab}) + \mathcal{T}_k^*(s,\mathrm{Sr}) \\ &= 1 - \frac{s}{\theta}\mathcal{T}_k^*(s,\mathrm{Ab}) - \frac{s}{\mu}\mathcal{T}_k^*(s,\mathrm{Sr}) \\ &= \frac{\mu}{s+\mu} + \left(\frac{\theta}{s+\theta} - \frac{\mu}{s+\mu}\right) \\ &\quad \times \frac{h_k^*(s)[1-G_m^*(s+\theta)]}{h_m^*(s) - h_{m-1}^*(s)G_m^*(s+\theta)}\end{aligned}$$

$$= \frac{\mu}{s+\mu} + \left(\frac{\theta}{s+\theta} - \frac{\mu}{s+\mu}\right) h_k^*(s)U^*(s),$$

where $U^*(s)$ is given in Section 4.1(1). Then, we obtain the mean waiting and service time

$$\begin{aligned}
E[\mathcal{W}_k] &= \frac{h_k^*(0)[1-G_m^*(\theta)]}{\theta[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]},\\
E[\mathcal{H}_k] &= \frac{1}{\mu}\left\{1 - \frac{h_k^*(0)[1-G_m^*(\theta)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)}\right\},\\
E[\mathcal{T}_k] &= E[\mathcal{W}_k] + E[\mathcal{H}_k]\\
&= \frac{1}{\mu} + \left(\frac{1}{\theta} - \frac{1}{\mu}\right)\frac{h_k^*(0)[1-G_m^*(\theta)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)},
\end{aligned}$$

and the second moment of the waiting and service time

$$\begin{aligned}
E[\mathcal{W}_k^2] = &\frac{2h_k^*(0)[1-G_m^*(\theta)]}{\theta^2[h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta)]}\\
&+ \frac{2h_k^*(0)[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta)}{\theta[h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta)]^2},\\
E[\mathcal{H}_k^2] = &\frac{2}{\mu^2} - \frac{2[1-G_m^*(\theta)]}{\mu^2}\left\{\frac{h_k^*(0) - \mu h_k'(0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)}\right.\\
&\left. + \frac{h_k^*(0)[h_m'(0) - h_{m-1}'(0)G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2}\right\},\\
E[\mathcal{T}_k^2] = &\frac{2}{\mu^2} + 2\left(\frac{1}{\theta^2} - \frac{1}{\mu^2}\right)\frac{h_k^*(0)[1-G_m^*(\theta)]}{h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta)}\\
&+ 2\left(\frac{1}{\theta} - \frac{1}{\mu}\right)\frac{h_k^*(0)[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta)}{[h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta)]^2}\\
&- 2\left(\frac{1}{\theta} - \frac{1}{\mu}\right)[1-G_m^*(\theta)]\left\{\frac{h_k'(0)}{h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta)}\right.\\
&\left. - \frac{h_k^*(0)[h_m'(0) - h_{m-1}'(0)G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta)]^2}\right\},\\
E[\mathcal{W}_k\mathcal{H}_k] = &-\frac{h_k^*(0)[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta)}{\mu[h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta)]^2}\\
&- \frac{1-G_m^*(\theta)}{\theta}\left\{\frac{h_k'(0)}{h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta)}\right.\\
&\left. - \frac{h_k^*(0)[h_m'(0) - h_{m-1}'(0)G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta)]^2}\right\}.
\end{aligned}$$

We generally have

$$E[T_k^\ell] = \frac{\ell!}{\mu^\ell} + (-1)^\ell \sum_{l=1}^{\ell} (-1)^l \binom{\ell}{l} l! \left(\frac{1}{\theta^l} - \frac{1}{\mu^l}\right) \sum_{n=0}^{\ell-l} \binom{\ell-l}{n} h_k^{(n)}(0) U^{(\ell-l-n)}(0) \qquad \ell = 1, 2, \ldots .$$

(2) For a customer waiting in state k, $k \geq m$, we have

$$\mathcal{T}_k^*(s, s') = \frac{\theta}{s+\theta} + \left(\frac{\mu}{s'+\mu} - \frac{\theta}{s+\theta}\right) \times \frac{[h_m^*(s') - h_{m-1}^*(s')] G_k^*(s+\theta)}{h_m^*(s') - h_{m-1}^*(s') G_m^*(s+\theta)}.$$

This joint distribution leads to the marginal distribution

$$\begin{aligned}
\mathcal{W}_k^*(s) &= \frac{\theta}{s+\theta} + \frac{s}{s+\theta} \cdot \frac{[h_m^*(0) - h_{m-1}^*(0)] G_k^*(s+\theta)}{h_m^*(0) - h_{m-1}^*(0) G_m^*(s+\theta)} \\
&= 1 - \frac{s}{\theta} \mathcal{W}_k^*(s, \mathrm{Ab}) = \frac{\theta}{s+\theta} + \frac{s}{s+\theta} \mathcal{W}_k^*(s, \mathrm{Sr}), \\
\mathcal{H}_k^*(s) &= 1 - \frac{s}{s+\mu} \cdot \frac{[h_m^*(s) - h_{m-1}^*(s)] G_k^*(\theta)}{h_m^*(s) - h_{m-1}^*(s) G_m^*(\theta)} \\
&= \frac{\mu}{s+\mu} + \frac{s}{s+\mu} \mathcal{H}_k^*(s, \mathrm{Ab}) = 1 - \frac{s}{\mu} \mathcal{H}_k^*(s, \mathrm{Sr}), \\
\mathcal{T}_k^*(s) &= \mathcal{T}_k^*(s, \mathrm{Ab}) + \mathcal{T}_k^*(s, \mathrm{Sr}) \\
&= 1 - \frac{s}{\theta} \mathcal{T}_k^*(s, \mathrm{Ab}) - \frac{s}{\mu} \mathcal{T}_k^*(s, \mathrm{Sr}) \\
&= \frac{\theta}{s+\theta} + \left(\frac{\mu}{s+\mu} - \frac{\theta}{s+\theta}\right) \times \frac{[h_m^*(s) - h_{m-1}^*(s)] G_k^*(s+\theta)}{h_m^*(s) - h_{m-1}^*(s) G_m^*(s+\theta)} \\
&= \frac{\theta}{s+\theta} + \left(\frac{\mu}{s+\mu} - \frac{\theta}{s+\theta}\right) G_k^*(s+\theta) V^*(s),
\end{aligned}$$

where $V^*(s)$ is given in Section 4.1(2). Then, we obtain the mean waiting and service time

$$E[\mathcal{W}_k] = \frac{1}{\theta} \left\{1 - \frac{h_m^*(0) - h_{m-1}^*(0)] G_k^*(\theta)}{h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)}\right\},$$

$$\begin{aligned} E[\mathcal{H}_k] &= \frac{[h_m^*(0)-h_{m-1}^*(0)]G_k^*(\theta)}{\mu[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]}, \\ E[\mathcal{T}_k] &= E[\mathcal{W}_k]+E[\mathcal{H}_k] \\ &= \frac{1}{\theta}+\left(\frac{1}{\mu}-\frac{1}{\theta}\right)\frac{h_m^*(0)-h_{m-1}^*(0)]G_k^*(\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)}, \end{aligned}$$

and the second moment of the waiting and service time

$$E[\mathcal{W}_k^2] = \frac{2}{\theta^2}\left\{1-\frac{[h_m^*(0)-h_{m-1}^*(0)][G_k^*(\theta)-\theta G_k'(\theta)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)}\right\} + \frac{2G_k^*(\theta)[h_m^*(0)-h_{m-1}^*(0)]h_{m-1}^*(0)G_m'(\theta)}{\theta[h_m^*(0)-h_{m-1}^*(s)G_m^*(\theta)]^2},$$

$$E[\mathcal{H}_k^2] = 2G_k^*(\theta)\left\{\frac{h_m^*(0)-h_{m-1}^*(0)}{\mu^2[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]} - \frac{[h_m'(0)h_{m-1}^*(0)-h_m^*(0)h_{m-1}'(0)][1-G_m^*(\theta)]}{\mu[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2}\right\},$$

$$\begin{aligned} E[\mathcal{T}_k^2] &= \frac{2}{\theta^2}+2\left(\frac{1}{\mu^2}-\frac{1}{\theta^2}\right)\frac{[h_m^*(0)-h_{m-1}^*(0)]G_k^*(\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)} \\ &\quad + 2\left(\frac{1}{\theta}-\frac{1}{\mu}\right)\frac{[h_m^*(0)-h_{m-1}^*(0)]G_k'(\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)} \\ &\quad + 2\left(\frac{1}{\theta}-\frac{1}{\mu}\right)\frac{\left\{\begin{array}{l} G_k^*(\theta)[h_m^*(0)-h_{m-1}^*(0)]h_{m-1}^*(0)G_m'(\theta) \\ +[h_m'(0)h_{m-1}^*(0)-h_m^*(0)h_{m-1}'(0)][1-G_m^*(\theta)] \end{array}\right\}}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2}, \end{aligned}$$

$$\begin{aligned} E[\mathcal{W}_k\mathcal{H}_k] &= -\frac{[h_m^*(0)-h_{m-1}^*(0)]G_k'(\theta)}{\mu[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]} \\ &\quad - \frac{G_k^*(\theta)[h_m^*(0)-h_{m-1}^*(0)]h_{m-1}^*(0)G_m'(\theta)}{\mu[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2} \\ &\quad + \frac{G_k^*(\theta)[h_m'(0)h_{m-1}^*(0)-h_m^*(0)h_{m-1}'(0)][1-G_m^*(\theta)]}{\theta[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2}. \end{aligned}$$

We generally have

$$\begin{aligned} E[T_k^\ell] &= \frac{\ell!}{\theta^\ell}+(-1)^\ell\sum_{l=1}^{\ell}(-1)^l\binom{\ell}{l}l!\left(\frac{1}{\mu^l}-\frac{1}{\theta^l}\right) \\ &\quad \times\sum_{n=0}^{\ell-l}\binom{\ell-l}{n}G_k^{(n)}(\theta)V^{(\ell-l-n)}(0) \quad \ell=1,2,\ldots. \end{aligned}$$

(3) Recursive relation among moments of distribution for the waiting and service time

From the explicit expressions for $\mathcal{T}_k^*(s,s',\text{Ab})$, $\mathcal{T}_k^*(s,s',\text{Sr})$, and $\mathcal{T}_k^*(s,s')$ given above, it can be shown that the unconditional and conditional joint LST of the DF for the waiting and service time until departure for a customer in state k satisfy the following same relation in both cases $0 \leq k \leq m-1$ and $k \geq m$:

$$\mathcal{T}_k^*(s,s') = 1 - \frac{s}{\theta}\mathcal{T}_k^*(s,s',\text{Ab}) - \frac{s'}{\mu}\mathcal{T}_k^*(s,s',\text{Sr}) \qquad k \geq 0.$$

This yields the recursive relation among unconditional and conditional moments as follows:

$$E[\mathcal{W}_k^{\ell}\mathcal{H}_k^{\ell'}] = \frac{\ell}{\theta}E[\mathcal{W}_k^{\ell-1}\mathcal{H}_k^{\ell'},\text{Ab}] + \frac{\ell'}{\mu}E[\mathcal{W}_k^{\ell}\mathcal{H}_k^{\ell'-1},\text{Sr}] \qquad \ell,\ell' = 0,1,2,\dots.$$

In particular, we get

$$\begin{aligned} E[\mathcal{W}_k] &= \frac{1}{\theta}\mathcal{P}_k\{\text{Ab}\} \quad ; \quad E[\mathcal{H}_k] = \frac{1}{\mu}\mathcal{P}_k\{\text{Sr}\}, \\ E[\mathcal{W}_k\mathcal{H}_k] &= \frac{1}{\theta}E[\mathcal{H}_k,\text{Ab}] + \frac{1}{\mu}E[\mathcal{W}_k,\text{Sr}], \\ E[\mathcal{W}_k^{\ell}] &= \frac{\ell}{\theta}E[\mathcal{W}_k^{\ell-1},\text{Ab}] \quad ; \quad E[\mathcal{H}_k^{\ell}] = \frac{\ell}{\mu}E[\mathcal{H}_k^{\ell-1},\text{Sr}] \\ &\qquad \ell = 2,3,\dots. \end{aligned}$$

Furthermore, it follows from the relation

$$\mathcal{T}_k^*(s) = 1 - \frac{s}{\theta}\mathcal{T}_k^*(s,\text{Ab}) - \frac{s}{\mu}\mathcal{T}_k^*(s,\text{Sr}) \qquad k \geq 0$$

(or from $\mathcal{T}_k = \mathcal{W}_k + \mathcal{H}_k$) that

$$\begin{aligned} E[\mathcal{T}_k] &= \frac{1}{\theta}\mathcal{P}_k\{\text{Ab}\} + \frac{1}{\mu}\mathcal{P}_k\{\text{Sr}\}, \\ E[\mathcal{T}_k^{\ell}] &= \frac{\ell}{\theta}E[\mathcal{T}_k^{\ell-1},\text{Ab}] + \frac{\ell}{\mu}E[\mathcal{T}_k^{\ell-1},\text{Sr}] \qquad \ell = 2,3,\dots. \end{aligned}$$

4.4. Waiting and Service Time of an Arriving Customer

According to the preemptive LCFS discipline, an arriving customer always joins the system at state $k = 0$. Therefore, the mean of his waiting and service time is given by

$$E[\mathcal{W}_0] = \frac{1}{\theta}\mathcal{P}_0\{\text{Ab}\} = \frac{E[\mathcal{L}]}{\lambda} \quad ; \quad E[\mathcal{H}_0] = \frac{1}{\mu}\mathcal{P}_0\{\text{Sr}\} = \frac{E[\mathcal{S}]}{\lambda}$$

with an instance of Little's theorem [19, p. 235]

$$E[\mathcal{T}_0] = \frac{1}{\theta}\mathcal{P}_0\{\text{Ab}\} + \frac{1}{\mu}\mathcal{P}_0\{\text{Sr}\} = \frac{E[\mathcal{N}]}{\lambda},$$

where $E[\mathcal{L}]$, $E[\mathcal{S}]$, and $E[\mathcal{N}]$ are the mean number of customers present in the waiting room, the service facility, and the entire system, respectively, in the steady state [15].

An arbitrary tagged customer is affected, regardless of whether he is being served ($0 \leq k \leq m-1$) or waiting ($k \geq m$), by only those customers who arrive after him. None of the customers present in the system at his arrival time competes for service with him at any time. Therefore, for an arbitrary arriving customer, we have

$$\begin{aligned}
\mathcal{T}^*(s, s', \text{Sr}) &= \mathcal{T}_0^*(s, s', \text{Sr}) \\
&= \frac{\mu}{s'+\mu}\left[1 - \frac{1 - G_m^*(s+\theta)}{h_m^*(s') - h_{m-1}^*(s')G_m^*(s+\theta)}\right], \\
\mathcal{T}^*(s, s', \text{Ab}) &= \mathcal{T}_0^*(s, s', \text{Ab}) \\
&= \frac{\theta}{s+\theta} \cdot \frac{1 - G_m^*(s+\theta)}{h_m^*(s') - h_{m-1}^*(s')G_m^*(s+\theta)},
\end{aligned}$$

which lead to

$$\begin{aligned}
\mathcal{P}\{\text{Sr}\} &= 1 - \frac{1 - G_m^*(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)}, \\
\mathcal{P}\{\text{Ab}\} &= \frac{1 - G_m^*(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)},
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}^*(s, \text{Sr}) &= 1 - \frac{1 - G_m^*(s+\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(s+\theta)}, \\
\mathcal{H}^*(s, \text{Sr}) &= \frac{\mu}{s+\mu}\left[1 - \frac{1 - G_m^*(\theta)}{h_m^*(s) - h_{m-1}^*(s)G_m^*(\theta)}\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^*(s,\mathrm{Sr}) &= \frac{\mu}{s+\mu}\left[1-\frac{1-G_m^*(s+\theta)}{h_m^*(s)-h_{m-1}^*(s)G_m^*(s+\theta)}\right], \\
\mathcal{W}^*(s,\mathrm{Ab}) &= \frac{\theta}{s+\theta}\cdot\frac{1-G_m^*(s+\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(s+\theta)}, \\
\mathcal{H}^*(s,\mathrm{Ab}) &= \frac{1-G_m^*(\theta)}{h_m^*(s)-h_{m-1}^*(s)G_m^*(\theta)}, \\
\mathcal{T}^*(s,\mathrm{Ab}) &= \frac{\theta}{s+\theta}\cdot\frac{1-G_m^*(s+\theta)}{h_m^*(s)-h_{m-1}^*(s)G_m^*(s+\theta)}.
\end{aligned}$$

4.5. Initial and Subsequent Waiting Time

In addition, for a customer waiting in state k, $k \geq m$, let us call a portion of the waiting time until the service begins for the first time the *initial waiting time*, denoted $\mathcal{W}_k^\circ$, and call the rest of the waiting time the *subsequent waiting time*, denoted by $\mathcal{W}_k^\bullet$. We can find the joint distribution of $\mathcal{W}_k^\circ$ and $\mathcal{W}_k^\bullet$, and observe that $\mathcal{W}_k^\circ$ and $\mathcal{W}_k^\bullet$ are not independent.

Let $\mathcal{W}_k^*(s,s',\mathrm{Ab})$ and $\mathcal{W}_k^*(s,s',\mathrm{Sr})$ be the joint LSTs of the DF for $\mathcal{W}_k^\circ$ and $\mathcal{W}_k^\bullet$ for a tagged customer in state k, $k \geq m$, who abandons waiting and who gets served, respectively.

(1) Waiting time until abandonment

By an analysis similar to the one in Section 4.1(2), we obtain

$$\begin{aligned}
\mathcal{W}_k^*(s,s',\mathrm{Ab}) &= W_k^*(s,\mathrm{Ab}) + \frac{W_k^*(s,\mathrm{Rs})p_{m-1}\{\mathrm{Pr}\}W_m^*(s',\mathrm{Ab})}{1-p_{m-1}\{\mathrm{Pr}\}W_m^*(s',\mathrm{Rs})} \\
&= \frac{\theta}{s+\theta}[1-G_k^*(s+\theta)] \\
&+ \frac{\theta}{s'+\theta}\cdot\frac{h_{m-1}^*(0)G_k^*(s+\theta)[1-G_m^*(s'+\theta)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(s'+\theta)}.
\end{aligned}$$

Then, the LST of the DF for the total waiting time of a customer who abandons waiting is given by $\mathcal{W}_k^*(s,\mathrm{Ab}) = \mathcal{W}_k^*(s,s,\mathrm{Ab})$, which agrees with the result in Section 4.1(2). The marginal distribution, mean, and second moment of $\mathcal{W}_k^\circ$ and $\mathcal{W}_k^\bullet$ are given as follows:

$$\begin{aligned}
\mathcal{W}_k^\circ(s,\mathrm{Ab}) = \mathcal{W}_k^*(s,0,\mathrm{Ab}) &= \frac{\theta}{s+\theta}[1-G_k^*(s+\theta)] \\
&+ \frac{h_{m-1}^*(0)G_k^*(s+\theta)[1-G_m^*(\theta)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)},
\end{aligned}$$

$$
\begin{aligned}
E[\mathcal{W}_k^\circ, \mathrm{Ab}] &= \frac{1 - G_k^*(\theta)}{\theta} + \frac{[h_m^*(0) - h_{m-1}^*(0)]G_k'(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)}, \\
E[(\mathcal{W}_k^\circ)^2, \mathrm{Ab}] &= \frac{2[1 - G_k^*(\theta)]}{\theta^2} + \frac{2G_k'(\theta)}{\theta} \\
&\quad - \frac{[h_m^*(0) - h_{m-1}^*(0)]G_k''(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)}, \\
\mathcal{W}_k^\bullet(s, \mathrm{Ab}) &= \mathcal{W}_k^*(0, s, \mathrm{Ab}) = 1 - G_k^*(\theta) \\
&\quad + \frac{\theta}{s+\theta} \cdot \frac{h_{m-1}^*(0)G_k^*(\theta)[1 - G_m^*(s+\theta)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(s+\theta)}, \\
E[\mathcal{W}_k^\bullet, \mathrm{Ab}] &= h_{m-1}^*(0)G_k^*(\theta)\left\{\frac{1 - G_m^*(\theta)}{\theta[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]}\right. \\
&\quad \left. + \frac{[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta)}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2}\right\}, \\
E[(\mathcal{W}_k^\bullet)^2, \mathrm{Ab}] &= h_{m-1}^*(0)G_k^*(\theta)\left\{\frac{2[1 - G_m^*(\theta)]}{\theta^2[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]}\right. \\
&\quad + \frac{[h_m^*(0) - h_{m-1}^*(0)][2G_m'(\theta) - \theta G_m''(\theta)]}{\theta[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2} \\
&\quad \left. - \frac{2[h_m^*(0) - h_{m-1}^*(0)]h_{m-1}(0)[G_m'(\theta)]^2}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^3}\right\}, \\
E[\mathcal{W}_k^\circ\mathcal{W}_k^\bullet, \mathrm{Ab}] &= -h_{m-1}^*(0)G_k'(\theta)\left\{\frac{1 - G_m^*(\theta)}{\theta[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]}\right. \\
&\quad \left. + \frac{[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta)}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2}\right\}.
\end{aligned}
$$

Since $\mathcal{W}_k^*(s, s' \,|\, \mathrm{Ab}) \neq \mathcal{W}_k^\circ(s \,|\, \mathrm{Ab})\mathcal{W}_k^\bullet(s' \,|\, \mathrm{Ab})$, $\mathcal{W}_k^\circ$ and $\mathcal{W}_k^\bullet$ are not independent for a customer who abandons waiting.

(2) Waiting time until service completion

By an analysis similar to the one in Section 4.2(2), we obtain

$$
\begin{aligned}
\mathcal{W}_k^*(s, s', \mathrm{Sr}) &= \frac{W_k^*(s, \mathrm{Rs})p_{m-1}\{\mathrm{Sr}\}}{1 - p_{m-1}\{\mathrm{Pr}\}W_m^*(s', \mathrm{Rs})} \\
&= \frac{[h_m^*(0) - h_{m-1}^*(0)]G_k^*(s+\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(s'+\theta)}.
\end{aligned}
$$

Then, the LST of the DF for the total waiting time of a customer who gets served is given by $\mathcal{W}_k^*(s, \mathrm{Sr}) = \mathcal{W}_k^*(s, s, \mathrm{Sr})$, which agrees with the result in Section 4.2(2). The marginal distribution, mean, and second moment of $\mathcal{W}_k^\circ$ and $\mathcal{W}_k^\bullet$ are given as follows:

$$\mathcal{W}_k^\circ(s,\mathrm{Sr}) = \mathcal{W}_k^*(s,0,\mathrm{Sr}) = \frac{[h_m^*(0)-h_{m-1}^*(0)]G_k^*(s+\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)},$$
$$E[\mathcal{W}_k^\circ,\mathrm{Sr}] = -\frac{[h_m^*(0)-h_{m-1}^*(0)]G_k'(\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)},$$
$$E[(\mathcal{W}_k^\circ)^2,\mathrm{Sr}] = \frac{[h_m^*(0)-h_{m-1}^*(0)]G_k''(\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)},$$
$$\mathcal{W}_k^\bullet(s,\mathrm{Sr}) = \mathcal{W}_k^*(0,s,\mathrm{Sr}) = \frac{[h_m^*(0)-h_{m-1}^*(0)]G_k^*(\theta)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(s+\theta)},$$
$$E[\mathcal{W}_k^\bullet,\mathrm{Sr}] = -\frac{[h_m^*(0)-h_{m-1}^*(0)]h_{m-1}^*(0)G_k^*(\theta)G_m'(\theta)}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2},$$
$$E[(\mathcal{W}_k^\bullet)^2,\mathrm{Sr}] = [h_m^*(0)-h_{m-1}^*(0)]G_k^*(\theta)$$
$$\times\left\{\frac{h_{m-1}^*(0)G_m''(\theta)}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2} + \frac{2[h_{m-1}^*(0)G_m'(\theta)]^2}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^3}\right\},$$
$$E[\mathcal{W}_k^\circ\mathcal{W}_k^\bullet,\mathrm{Sr}] = \frac{[h_m^*(0)-h_{m-1}^*(0)]h_{m-1}^*(0)G_k'(\theta)G_m'(\theta)}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta)]^2}.$$

Since $\mathcal{W}_k^*(s,s'\,|\,\mathrm{Sr}) \neq \mathcal{W}_k^\circ(s\,|\,\mathrm{Sr})\mathcal{W}_k^\bullet(s'\,|\,\mathrm{Sr})$, $\mathcal{W}_k^\circ$ and $\mathcal{W}_k^\bullet$ are not independent for a customer who gets served.

4.6. Number of Service Preemptions and Resumptions

We are interested in the number of service preemptions and resumptions that an arbitrary customer experiences before he departs from the system. Let $\mathcal{J}_k$ and $\mathcal{K}_k$ be the number of preemptions and resumptions, respectively, until departure (either by service completion or abandonment) for a customer in state k (≥ 0). According to the Markovian property of transitions, these numbers are geometrically distributed.

(1) Preemptions and resumptions until abandonment

(i) For a customer in state k, $0 \leq k \leq m-1$, we have

$$P\{\mathcal{J}_k = n, \mathrm{Ab}\} = p_k\{\mathrm{Pr}\}p_m\{\mathrm{Ab}\}[p_m\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}]^{n-1}$$
$$n = 1,2,\ldots.$$

The probability distribution and the mean of $\mathcal{J}_k$ for a customer who abandons are given by

$$P\{\mathcal{J}_k = n \mid \mathrm{Ab}\} = \frac{P\{\mathcal{J}_k = n, \mathrm{Ab}\}}{\mathcal{P}_k\{\mathrm{Ab}\}}$$
$$= (1 - p_m\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\})[p_m\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}]^{n-1},$$
$$E[\mathcal{J}_k \mid \mathrm{Ab}] = \sum_{n=1}^{\infty} nP\{\mathcal{J}_k = n \mid \mathrm{Ab}\}$$
$$= \frac{1}{1 - p_m\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}} = \frac{h_m^*(0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)}.$$

Since $\mathcal{J}_k = \mathcal{K}_k + 1$, we have

$$P\{\mathcal{K}_k = n, \mathrm{Ab}\} = P\{\mathcal{J}_k = n+1, \mathrm{Ab}\}$$
$$= p_k\{\mathrm{Pr}\}p_m\{\mathrm{Ab}\}[p_m\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}]^n \qquad n = 0, 1, 2, \dots,$$
$$E[\mathcal{K}_k \mid \mathrm{Ab}] = \frac{p_m\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}}{[1 - p_m\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}]^2}$$
$$= \frac{h_{m-1}^*(0)G_m^*(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)} = E[\mathcal{J}_k \mid \mathrm{Ab}] - 1.$$

The distribution of $\mathcal{J}_k$ and $\mathcal{K}_k$ for a customer who is being served and eventually abandons does not depend on k, since it is only related to the number of preemptions and resumptions before abandonment.

(ii) For a customer in state k, $k \geq m$, we have $\mathcal{J}_k = \mathcal{K}_k$ and

$$P\{\mathcal{J}_k = n, \mathrm{Ab}\} = P\{\mathcal{K}_k = n, \mathrm{Ab}\}$$
$$= \begin{cases} p_k\{\mathrm{Ab}\} & n = 0 \\ p_k\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}p_m\{\mathrm{Ab}\}[p_{m-1}\{\mathrm{Pr}\}p_m\{\mathrm{Rs}\}]^{n-1} & \\ & n = 1, 2, \dots, \end{cases}$$
$$E[\mathcal{J}_k, \mathrm{Ab}] = E[\mathcal{K}_k, \mathrm{Ab}] = \frac{p_k\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}p_m\{\mathrm{Ab}\}}{[1 - p_{m-1}\{\mathrm{Pr}\}p_m\{\mathrm{Rs}\}]^2}$$
$$= \frac{h_{m-1}^*(0)h_m^*(0)G_k^*(\theta)[1 - G_m^*(\theta)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)]^2}.$$

In this case, the distribution of $\mathcal{J}_k$ and $\mathcal{K}_k$ for a customrer who is waiting and abandons, $P\{\mathcal{J}_k = n \mid \mathrm{Ab}\} = P\{\mathcal{K}_k = n \mid \mathrm{Ab}\} = P\{\mathcal{J}_k = n, \mathrm{Ab}\}/\mathcal{P}_k\{\mathrm{Ab}\}$, *does* depend on k, since the possibilty of reaching "Ab" before reaching "Rs" depends on k in Fig. 2.

(2) Preemptions and resumptions until service completion

(i) For a customer in state k, $0 \le k \le m-1$, we have $\mathcal{J}_k = \mathcal{K}_k$ and

$$\begin{aligned} P\{\mathcal{J}_k = n, \mathrm{Sr}\} &= P\{\mathcal{K}_k = n, \mathrm{Sr}\} \\ &= \begin{cases} p_k\{\mathrm{Sr}\} & n = 0 \\ p_k\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\} p_{m-1}\{\mathrm{Sr}\}[p_{m-1}\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\}]^{n-1} & \\ & n = 1, 2, \ldots, \end{cases} \\ E[\mathcal{J}_k, \mathrm{Sr}] &= E[\mathcal{K}_k, \mathrm{Sr}] = \frac{p_k\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\} p_{m-1}\{\mathrm{Sr}\}}{[1 - p_m\{\mathrm{Rs}\} p_{m-1}\{\mathrm{Pr}\}]^2} \\ &= \frac{h_k^*(0)[h_m^*(0) - h_{m-1}^*(0)] G_m^*(\theta)}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)]^2}. \end{aligned}$$

(ii) For a customer in state k, $k \ge m$, we have

$$P\{\mathcal{K}_k = n, \mathrm{Sr}\} = p_k\{\mathrm{Rs}\} p_{m-1}\{\mathrm{Sr}\}[p_{m-1}\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\}]^{n-1} \qquad n = 1, 2, \ldots$$

The distribution and the mean of $\mathcal{K}_k$ for a customer who gets served are given by

$$\begin{aligned} P\{\mathcal{K}_k = n \mid \mathrm{Sr}\} &= \frac{P\{\mathcal{K}_k = n, \mathrm{Sr}\}}{\mathcal{P}_k\{\mathrm{Sr}\}} \\ &= (1 - p_{m-1}\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\})[p_{m-1}\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\}]^{n-1} \\ E[\mathcal{K}_k \mid \mathrm{Sr}] &= \sum_{n=1}^{\infty} n P\{\mathcal{K}_k = n \mid \mathrm{Sr}\} \\ &= \frac{1}{1 - p_{m-1}\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\}} = \frac{h_m^*(0)}{h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)}. \end{aligned}$$

Since $\mathcal{K}_k = \mathcal{J}_k + 1$, we have

$$\begin{aligned} P\{\mathcal{J}_k = n, \mathrm{Sr}\} &= P\{\mathcal{K}_k = n+1, \mathrm{Sr}\} \\ &= p_k\{\mathrm{Rs}\} p_{m-1}\{\mathrm{Sr}\}[p_{m-1}\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\}]^{n} \\ & \qquad n = 0, 1, 2, \ldots, \end{aligned}$$

$$\begin{aligned} E[\mathcal{J}_k \mid \mathrm{Sr}] &= \frac{p_{m-1}\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\}}{1 - p_{m-1}\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\}} \\ &= \frac{h_{m-1}^*(0) G_m^*(\theta)}{h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta)} = E[\mathcal{K}_k \mid \mathrm{Sr}] - 1. \end{aligned}$$

The distribution of $\mathcal{J}_k$ and $\mathcal{K}_k$ for a customer who is waiting and eventually gets served does not depend on k for the reason similar to the one in (1)(i).

(3) Preemptions and resumptions until departure

The unconditional mean number of preemptions and resumptions until departure is given as follows.

(i) For a customer in state k, $0 \le k \le m-1$, we have

$$E[\mathcal{J}_k] = \frac{h_k^*(0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)},$$
$$E[\mathcal{K}_k] = E[\mathcal{J}_k] - \mathcal{P}_k\{\text{Ab}\} = \frac{h_k^*(0)G_m^*(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)}.$$

(ii) For a customer in state k, $k \ge m$, we have

$$E[\mathcal{J}_k] = \frac{h_{m-1}^*(0)G_k^*(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)},$$
$$E[\mathcal{K}_k] = E[\mathcal{J}_k] + \mathcal{P}_k\{\text{Sr}\} = \frac{h_m^*(0)G_k^*(\theta)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta)}.$$

5. Numerical Example

In Tables 1–5, we present some numerical example of our theoretical formulas. We assume that $m = 5$, $\mu = 1$, $\theta = 2$, and $\lambda = 10$ ($\rho = 2, \tau = 2$). Our experience reveals that the numerical computation using the distribution formulas in this paper is much faster and more stable than that using recursive relations of moments given in our previous work [14, 15].

6. Special Cases

We consider two special cases of our model with respect to θ. Recursive computational formulas for calculating the moments of the time until service completion and abandonment in these special cases, as well as numerical results, are provided in [15]. Direct numerical calculation, using the distribution formulas below results in the same values. Therefore, in this section, we only show the explicit LST of the DF for the time until service completion and abandonment.

Table 1: Numerical example.

k	$\mathcal{P}_k\{\text{Ab}\}$	$\mathcal{P}_k\{\text{Sr}\}$	$E[\mathcal{W}_k, \text{Ab}]$	$E[\mathcal{H}_k, \text{Ab}]$	$E[\mathcal{T}_k, \text{Ab}]$	$E[\mathcal{W}_k, \text{Sr}]$
0	0.51270	0.48730	0.24299	0.30992	0.55291	0.01336
1	0.56396	0.43604	0.26729	0.28965	0.55693	0.01470
2	0.62549	0.37451	0.29645	0.26019	0.55663	0.01630
3	0.70034	0.29966	0.33192	0.21777	0.54969	0.01825
4	0.79283	0.20717	0.37576	0.15678	0.53254	0.02066
5	0.90911	0.09089	0.43087	0.06878	0.49965	0.02369
6	0.94907	0.05093	0.45368	0.03854	0.49222	0.02085
7	0.96686	0.03314	0.46548	0.02508	0.49056	0.01795
8	0.97624	0.02376	0.47252	0.01798	0.49050	0.01560
9	0.98181	0.01819	0.47713	0.01377	0.49090	0.01378
10	0.98541	0.01459	0.48038	0.01104	0.49142	0.01233
15	0.99293	0.00707	0.48827	0.00535	0.49362	0.00819
20	0.99541	0.00459	0.49146	0.00347	0.49493	0.00624
30	0.99732	0.00268	0.49432	0.00203	0.49635	0.00434

k	$E[\mathcal{H}_k, \text{Sr}]$	$E[\mathcal{T}_k, \text{Sr}]$	$E[\mathcal{W}_k]$	$E[\mathcal{H}_k]$	$E[\mathcal{T}_k]$	$E[\mathcal{W}_k\mathcal{H}_k; \text{Ab}]$
0	0.17738	0.19074	0.25635	0.48730	0.74365	0.15700
1	0.14639	0.16108	0.28198	0.43604	0.71802	0.14840
2	0.11432	0.13062	0.31274	0.37451	0.68726	0.13565
3	0.08189	0.10014	0.35017	0.29966	0.64983	0.11702
4	0.05039	0.07105	0.39642	0.20717	0.60358	0.08994
5	0.02211	0.04579	0.45456	0.09089	0.54544	0.05053
6	0.01239	0.03324	0.47454	0.05093	0.52546	0.03405
7	0.00806	0.02601	0.48343	0.03314	0.51567	0.02547
8	0.00578	0.02138	0.48812	0.02376	0.51188	0.02033
9	0.00442	0.01820	0.49090	0.01819	0.50910	0.01695
10	0.00355	0.01588	0.49270	0.01459	0.50730	0.01456
15	0.00172	0.00991	0.49647	0.00707	0.50353	0.00874
20	0.00112	0.00736	0.49771	0.00459	0.50229	0.00637
30	0.00065	0.00499	0.49866	0.00268	0.50134	0.00424

Table 2: Numerical example – continued.

k	$E[\mathcal{W}_k\mathcal{H}_k;\mathrm{Sr}]$	$E[\mathcal{W}_k\mathcal{H}_k]$	$E[\mathcal{W}_k^2,\mathrm{Ab}]$	$E[\mathcal{H}_k^2,\mathrm{Ab}]$	$E[\mathcal{T}_k^2,\mathrm{Ab}]$
0	0.01132	0.16832	0.23301	0.24230	0.78930
1	0.01112	0.15952	0.25631	0.20454	0.75765
2	0.01074	0.14639	0.28427	0.16329	0.71886
3	0.01011	0.12713	0.31829	0.11933	0.67166
4	0.00911	0.09905	0.36033	0.07452	0.61473
5	0.00755	0.05808	0.41317	0.03269	0.54692
6	0.00608	0.04012	0.43606	0.01832	0.52247
7	0.00502	0.03049	0.44855	0.01192	0.51141
8	0.00426	0.02459	0.45641	0.00855	0.50562
9	0.00371	0.02066	0.46184	0.00654	0.50227
10	0.00329	0.01785	0.46583	0.00524	0.50020
15	0.00213	0.01087	0.47650	0.00254	0.48652
20	0.00161	0.00798	0.48413	0.00165	0.49583
30	0.00111	0.00535	0.48638	0.00096	0.49583

k	$E[\mathcal{W}_k^2,\mathrm{Sr}]$	$E[\mathcal{H}_k^2,\mathrm{Sr}]$	$E[\mathcal{T}_k^2,\mathrm{Sr}]$	$E[\mathcal{T}_k^2]$	$E[\mathcal{W}_k^2]$	$E[\mathcal{H}_k^2]$
0	0.00998	0.11247	0.14509	0.93439	0.22499	0.35476
1	0.01098	0.08824	0.12145	0.87910	0.26729	0.29278
2	0.01217	0.06536	0.09902	0.81788	0.29645	0.22865
3	0.01363	0.04445	0.07831	0.74997	0.33192	0.16378
4	0.01543	0.02625	0.05990	0.67463	0.37576	0.10077
5	0.01769	0.01152	0.04432	0.59124	0.43087	0.04421
6	0.01762	0.00645	0.03623	0.55870	0.45368	0.02477
7	0.01693	0.00420	0.03117	0.54257	0.46548	0.01612
8	0.01611	0.00301	0.02764	0.53326	0.47252	0.01156
9	0.01530	0.00230	0.02502	0.52729	0.47713	0.00885
10	0.01455	0.00185	0.02297	0.52317	0.48037	0.00710
15	0.01177	0.00090	0.01693	0.51345	0.48827	0.00344
20	0.01003	0.00058	0.01383	0.50965	0.49146	0.00223
30	0.00794	0.00034	0.01050	0.50633	0.49432	0.00130

Table 3: Numerical example – continued.

k	$E[\mathcal{W}_k\mathcal{H}_k]$	$E[\mathcal{W}_k^2\mathcal{H}_k]$	$E[\mathcal{W}_k\mathcal{H}_k^2]$	$E[\mathcal{W}_k^3]$	$E[\mathcal{H}_k^3]$	$E[\mathcal{T}_k^3]$
0	0.16832	0.16697	0.14830	0.34952	0.33740	1.61923
1	0.15952	0.15937	0.12451	0.38447	0.26471	1.50083
2	0.14639	0.14782	0.10313	0.42641	0.19608	1.37535
3	0.12713	0.13065	0.07989	0.47744	0.13336	1.24242
4	0.09905	0.10537	0.05548	0.54059	0.07875	1.10179
5	0.05808	0.06822	0.03145	0.61976	0.03545	0.95333
6	0.04012	0.05167	0.02131	0.65409	0.01936	0.89240
7	0.03049	0.04240	0.01600	0.67283	0.01260	0.86061
8	0.02459	0.03644	0.01280	0.68462	0.00903	0.84136
9	0.02066	0.03224	0.01069	0.69275	0.00691	0.82847
10	0.01785	0.02911	0.00920	0.69874	0.00554	0.81921
15	0.01087	0.02050	0.00554	0.71476	0.00269	0.79556
20	0.00798	0.01640	0.00404	0.72215	0.00174	0.78522
30	0.00535	0.01218	0.00270	0.72957	0.00102	0.77524

6.1. Special Case $\theta = \infty$, a Preemptive-Loss System

We first consider an extreme case $\theta = \infty$, in which customers who are pushed out of service leave the system immediately. This service discipline can be referred to as *preemptive-loss* [7, p. 66]. We still assume the LCFS discipline.

In this special case, there are no customers in the waiting room. Therefore, the time $\mathcal{T}_k$ spent by an arbitrary customer in state k, $0 \le k \le m-1$, is equivalent to the first passage time from state k until service preemption and completion, which is considered in Section 2. From the results in Section 2.2, we simply have

$$\mathcal{T}_k^*(s, \mathrm{Ab}) = \frac{h_k^*(s)}{h_m^*(s)} \quad ; \quad \mathcal{T}_k^*(s, \mathrm{Sr}) = \frac{\mu}{s+\mu}\left[1 - \frac{h_k^*(s)}{h_m^*(s)}\right],$$
$$\mathcal{T}_k^*(s) = \mathcal{T}_k^*(s, \mathrm{Ab}) + \mathcal{T}_k^*(s, \mathrm{Sr}) = \frac{\mu}{s+\mu} + \frac{s}{s+\mu} \cdot \frac{h_k^*(s)}{h_m^*(s)}$$
$$0 \le k \le m,$$

where the set of functions $\{h_k^*(s); 0 \le k \le m\}$ is given in Section 2.3. The probability that a customer in state k is lost is given by

$$\mathcal{P}_k\{\mathrm{Ab}\} = \mathcal{T}_k^*(0, \mathrm{Ab}) = \frac{h_k^*(0)}{h_m^*(0)} = \frac{B(m, m\rho)}{B(k, m\rho)}.$$

Table 4: Numerical example for the initial and subsequent waiting time.

k	$E[\mathcal{W}_k^\circ, \text{Ab}]$	$E[\mathcal{W}_k^\bullet, \text{Ab}]$	$E[(\mathcal{W}_k^\circ)^2, \text{Ab}]$	$E[(\mathcal{W}_k^\bullet)^2, \text{Ab}]$	$E[\mathcal{W}_k^\circ \mathcal{W}_k^\bullet, \text{Ab}]$
5	0.26601	0.16485	0.20204	0.15808	0.02653
6	0.36131	0.09237	0.29026	0.08858	0.02861
7	0.40538	0.06011	0.33780	0.05764	0.02656
8	0.42943	0.04309	0.36708	0.04232	0.02400
9	0.44414	0.03299	0.38681	0.03164	0.02169
10	0.45391	0.02646	0.40100	0.02538	0.01972
15	0.47546	0.01282	0.43704	0.01229	0.01358
20	0.48314	0.00832	0.45246	0.00798	0.01049
30	0.48974	0.00486	0.46696	0.00466	0.00738

k	$E[\mathcal{W}_k^\circ, \text{Sr}]$	$E[\mathcal{W}_k^\bullet, \text{Sr}]$	$E[(\mathcal{W}_k^\circ)^2, \text{Sr}]$	$E[(\mathcal{W}_k^\bullet)^2, \text{Sr}]$	$E[\mathcal{W}_k^\circ \mathcal{W}_k^\bullet, \text{Sr}]$
5	0.01463	0.00906	0.00801	0.00677	0.00146
6	0.01577	0.00508	0.01069	0.00379	0.00157
7	0.01464	0.00330	0.01154	0.00247	0.00146
8	0.01323	0.00237	0.01170	0.00177	0.00131
9	0.01196	0.00181	0.01156	0.00135	0.00119
10	0.01087	0.00145	0.01129	0.00109	0.00108
15	0.00749	0.00070	0.00975	0.00053	0.00075
20	0.00579	0.00046	0.00853	0.00034	0.00058
30	0.00407	0.00027	0.00693	0.00020	0.00041

6.2. Special Case $\theta = 0$, in which All Customers are Patient

We next consider another extreme case $\theta = 0$, in which all customers are very patient in an M/M/m PR-LCFS queue. No customers abandon waiting. That is, all customers are served until completion. Therefore, $\mathcal{P}_k\{\text{Sr}\} = 1$ for $k \geq 0$. The same queue with FCFS discipline is an ordinary M/M/m queue.

The LST $G_k^*(s)$, $k \geq m$, of the DF for the busy period $\mathcal{G}_k$, considered in Section 3.3, now satisfies the following set of recursive equations:

$$\lambda G_{k+1}^*(s) - (s + \lambda + \mu) G_k^*(s) + m\mu G_{k-1}^*(s) = 0 \qquad k \geq m.$$

We note that $\mathcal{G}_k$ is equivalent to the busy period started with k customers in an M/M/1 queue with mean service time $1/(m\mu)$. The solution is given in the form

$$G_k^*(s) = [G^*(s)]^{k-m+1} \qquad k \geq m-1,$$

Table 5: Numerical example for the number of service preemptions and resumptions.

k	$E[\mathcal{J}_k \mid \mathrm{Ab}]$	$E[\mathcal{K}_k \mid \mathrm{Ab}]$	$E[\mathcal{J}_k \mid \mathrm{Sr}]$	$E[\mathcal{K}_k \mid \mathrm{Sr}]$	$E[\mathcal{J}_k]$	$E[\mathcal{K}_k]$
0	1.61971	0.61971	0.17037	0.17037	0.91344	0.40074
1	1.61971	0.61971	0.20944	0.20944	1.00478	0.44082
2	1.61971	0.61971	0.27045	0.27045	1.11440	0.48891
3	1.61971	0.61971	0.37846	0.37846	1.24776	0.54742
4	1.61971	0.61971	0.61971	0.61971	1.41254	0.61971
5	0.61971	0.61971	0.61971	1.61971	1.41254	0.61971
6	0.33262	0.33262	0.61971	1.61971	0.61971	0.71060
7	0.21246	0.21246	0.61971	1.61971	0.22595	0.25909
8	0.15085	0.15085	0.61971	1.61971	0.16199	0.18575
9	0.11484	0.11484	0.61971	1.61971	0.12403	0.14222
10	0.09178	0.09178	0.61971	1.61971	0.09948	0.11407
15	0.04411	0.04411	0.61971	1.61971	0.04818	0.05525
20	0.02858	0.02858	0.61971	1.61971	0.03129	0.03588
30	0.01665	0.01665	0.61971	1.61971	0.01826	0.02094

where

$$G^*(s) := \frac{s+\lambda+m\mu-\sqrt{(s+\lambda+m\mu)^2-4m\lambda\mu}}{2\lambda}$$

with $G^*(0)=1$ and

$$G^*(0)=1, \quad G'(0)=-\frac{1}{m\mu(1-\rho)}, \quad G''(0)=\frac{2}{(m\mu)^2(1-\rho)^3},$$
$$G'''(0)=-\frac{6(1+\rho)}{(m\mu)^3(1-\rho)^5}.$$

For a customer being served in state k, $0\le k\le m-1$, from the result in Section 4.2(1), we have

$$\mathcal{T}_k^*(s,s')=\frac{\mu}{s'+\mu}\left\{1-\frac{h_k^*(s')[1-G^*(s)]}{h_m^*(s')-h_{m-1}^*(s')G^*(s)}\right\},$$
$$\mathcal{W}_k^*(s)=1-\frac{h_k^*(0)[1-G^*(s)]}{h_m^*(0)-h_{m-1}^*(0)G^*(s)} \quad ; \quad \mathcal{H}_k^*(s)=\frac{\mu}{s+\mu},$$
$$\mathcal{T}_k^*(s)=\frac{\mu}{s+\mu}\left\{1-\frac{h_k^*(s)[1-G^*(s)]}{h_m^*(s)-h_{m-1}^*(s)G^*(s)}\right\},$$

where the set of functions $\{h_k^*(s); 0\le k\le m\}$ is given in Section 2.3.

For a customer waiting in state k, $k \geq m$, from the result in Section 4.2(2), we have

$$\mathcal{T}_k^*(s,s') = \frac{\mu}{s'+\mu} \cdot \frac{[h_m^*(s') - h_{m-1}^*(s')][G^*(s)]^{k-m+1}}{h_m^*(s') - h_{m-1}^*(s')G^*(s)},$$

$$\mathcal{W}_k^*(s) = \frac{[h_m^*(0) - h_{m-1}^*(0)][G^*(s)]^{k-m+1}}{h_m^*(0) - h_{m-1}^*(0)G^*(s)} \quad ; \quad \mathcal{H}_k^*(s) = \frac{\mu}{s+\mu},$$

$$\mathcal{T}_k^*(s) = \frac{\mu}{s+\mu} \cdot \frac{[h_m^*(s) - h_{m-1}^*(s)][G^*(s)]^{k-m+1}}{h_m^*(s) - h_{m-1}^*(s)G^*(s)}.$$

We note that the time interval in which a customer is being served has the same exponential distribution as the original service time requirement. However it is not independent of the waiting time, because $\mathcal{T}_k^*(s,s') \neq \mathcal{W}_k^*(s)\mathcal{H}_k^*(s')$ for $k \geq 0$.

Appendix: Derivatives of function $G_k^*(s)$ at $s = \theta$

The function $G_k^*(s)$ is given in Section 3.3 for $k \geq m$. We show the first and second derivatives of $G_k^*(s)$ with respect to s at $s = \theta$ for $k \geq m$ and $k = m$ particularly. For the brevity of notation, let us use the convention

$$A := A(m,\mu,\theta,\lambda) = \sum_{i=0}^{\infty} \frac{\rho^i}{\prod_{j=0}^{i}(1+j\tau/m)} \qquad \rho := \frac{\lambda}{m\mu},$$

$$I(i) := I(i,m,\mu,\theta) = \sum_{j=1}^{i} \frac{1}{(1+j\tau/m)j\tau/m} \qquad \tau := \frac{\theta}{\mu}, \quad i \geq 0.$$

The function $\psi_{i,k}(x), i \geq 0, k \geq 0$, is defined in Section 3.3.

(1) $G_k'(\theta) := [dG_k^*(s)/ds]_{s=\theta},\ k \geq m$

$$\begin{aligned} m\mu G_k'(\theta) &= \frac{1}{A}\left[\sum_{i=1}^{\infty} \frac{i!(-\tau/m)^i \psi_{i,k-m}(\lambda/\theta) I(i)}{\prod_{j=1}^{i+1}(1+j\tau/m)} \right. \\ &\qquad \left. - \sum_{i=0}^{\infty} \frac{i!(-\tau/m)^i \psi_{i,k-m}(\lambda/\theta)}{[1+(i+1)\tau/m]\prod_{j=1}^{i+1}(1+j\tau/m)}\right] \\ &\quad - \frac{1}{A^2}\left[\sum_{i=0}^{\infty} \frac{i!(-\tau/m)^i \psi_{i,k-m}(\lambda/\theta)}{\prod_{j=1}^{i+1}(1+j\tau/m)}\right] \sum_{i=1}^{\infty} \frac{\rho^i I(i)}{\prod_{j=1}^{i}(1+j\tau/m)}. \end{aligned}$$

(2) $G'_m(\theta) := [dG^*_m(s)/ds]_{s=\theta}$

$$\begin{aligned}
&m\mu G'_m(\theta)\\
&= \frac{1}{A}\left[\sum_{i=1}^{\infty}\frac{\rho^i I(i)}{\prod_{j=1}^{i+1}(1+j\tau/m)} - \sum_{i=0}^{\infty}\frac{\rho^i}{[1+(i+1)\tau/m]\prod_{j=1}^{i+1}(1+j\tau/m)}\right]\\
&-\frac{1}{A^2}\left[\sum_{i=0}^{\infty}\frac{\rho^i}{\prod_{j=1}^{i+1}(1+j\tau/m)}\right]\sum_{i=1}^{\infty}\frac{\rho^i I(i)}{\prod_{j=1}^{i}(1+j\tau/m)}.
\end{aligned}$$

(3) $G''_k(\theta) := [d^2G^*_k(s)/ds^2]_{s=\theta},\ k \geq m$

$$\begin{aligned}
(m\mu)^2 G''_k(\theta) &= \frac{1}{A}\sum_{i=0}^{\infty}\frac{i!(-\tau/m)^i\psi_{i,k-m}(\lambda/\theta)}{\prod_{j=1}^{i+1}(1+j\tau/m)}\\
&\times\left\{[I(i)]^2 - 2\sum_{j=1}^{i}\frac{1}{(1+j\tau/m)^2 j\tau/m} - \sum_{j=1}^{i}\frac{1}{[(1+j\tau/m)j\tau/m]^2}\right.\\
&\qquad\left. - \frac{2I(i)}{1+(i+1)\tau/m} + \frac{2}{[1+(i+1)\tau/m]^2}\right\}\\
&-\frac{1}{A^2}\left\langle 2\left\{\sum_{i=0}^{\infty}\frac{i!(-\tau/m)^i\psi_{i,k-m}(\lambda/\theta)}{\prod_{j=1}^{i+1}(1+j\tau/m)}\left[I(i) - \frac{1}{1+(i+1)\tau/m}\right]\right\}\right.\\
&\times\left[\sum_{i=1}^{\infty}\frac{\rho^i I(i)]}{\prod_{j=1}^{i}(1+j\tau/m)}\right] + \left[\sum_{i=0}^{\infty}\frac{i!(-\tau/m)^i\psi_{i,k-m}(\lambda/\theta)}{\prod_{j=1}^{i+1}(1+j\tau/m)}\right]\\
&\quad\left.\times\sum_{i=1}^{\infty}\frac{\rho^i}{\prod_{j=1}^{i}(1+j\tau/m)}\left\{[I(i)]^2 - \sum_{j=1}^{i}\frac{1+2j\tau/m}{[(1+j\tau/m)j\tau/m]^2}\right\}\right\rangle\\
&+\frac{2}{A^3}\left[\sum_{i=0}^{\infty}\frac{i!(-\tau/m)^i\psi_{i,k-m}(\lambda/\theta)}{\prod_{j=1}^{i+1}(1+j\tau/m)}\right]\left[\sum_{i=1}^{\infty}\frac{\rho^i I(i)}{\prod_{j=1}^{i}(1+j\tau/m)}\right]^2.
\end{aligned}$$

(4) $G''_m(\theta) := [d^2G^*_m(s)/ds^2]_{s=\theta}$

$$\begin{aligned}
(m\mu)^2 G''_m(\theta) &= \frac{1}{A}\sum_{i=0}^{\infty}\frac{\rho^i}{\prod_{j=1}^{i+1}(1+j\tau/m)}\\
&\times\left\{[I(i)]^2 - 2\sum_{j=1}^{i}\frac{1}{(1+j\tau/m)^2 j\tau/m} - \sum_{j=1}^{i}\frac{1}{[(1+j\tau/m)j\tau/m]^2}\right.\\
&\left. - \frac{2I(i)}{1+(i+1)\tau/m} + \frac{2}{[1+(i+1)\tau/m]^2}\right\}\\
&-\frac{1}{A^2}\left\langle 2\left\{\sum_{i=0}^{\infty}\frac{\rho^i}{\prod_{j=1}^{i+1}(1+j\tau/m)}\left[I(i) - \frac{1}{1+(i+1)\tau/m}\right]\right\}\right.
\end{aligned}$$

$$\times \left[\sum_{i=1}^{\infty} \frac{\rho^i I(i)}{\prod_{j=1}^{i}(1+j\tau/m)}\right] + \left[\sum_{i=0}^{\infty} \frac{\rho^i}{\prod_{j=1}^{i+1}(1+j\tau/m)}\right]$$

$$\times \sum_{i=1}^{\infty} \frac{\rho^i}{\prod_{j=1}^{i}(1+j\tau/m)} \left\{[I(i)]^2 - \sum_{j=1}^{i} \frac{1+2j\tau/m}{[(1+j\tau/m)j\tau/m]^2}\right\}\Bigg\rangle$$

$$+ \frac{2}{A^3}\left[\sum_{i=0}^{\infty} \frac{\rho^i}{\prod_{j=1}^{i+1}(1+j\tau/m)}\right]\left[\sum_{i=1}^{\infty} \frac{\rho^i I(i)}{\prod_{j=1}^{i}(1+j\tau/m)}\right]^2.$$

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