International Journal of Pure and Applied Mathematics
Volume 116 No. 2 2017, 547-557
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v116i2.22

ijpam.eu

ON SOME MULTIPLIERS OF VECTOR-VALUED AMALGAM SPACES

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Abstract: We recall the vector-valued classical amalgam spaces $(L^p(G,A), \ell^q)$ and give several basic properties of this spaces. We also discuss some multipliers spaces from $L^1(G,A) \cap (L^p(G,A), \ell^q)$ to $L^1(G,A)$ and $L^1(G,A) \cap (L^p(G,A), \ell^q)$.

AMS Subject Classification: 43A15, 46E30, 43A22
Key Words: vector-valued amalgam spaces, convolution, multipliers

1. Introduction

Let A be a commutative Banach algebra with identity of norm 1 and G be a locally compact Abelian group with Haar measure μ. Amalgam spaces (L^p, ℓ^q) are based on the work of Wiener [19], and these spaces are considerable used in harmonic and time-frequency analysis. Extensive information about (L^p, ℓ^q) can be found in [6]. A scalar-valued amalgam space $(L^p, \ell^q)(G)$ $(1 \leq p, q \leq \infty)$ were studied by Bertrandis, Darty and Dupuis [2], Stewart [16], Busby and Smith [3] and Feichtinger [5]. In 2009, Lakshmi and Ray [10] defined firstly Banach-valued classical amalgam spaces $(L^p(\mathbb{R}, E), \ell^q)$ on the real line. They proved some fundamental properties of these spaces, such as embedding and separability. In their following paper [11], they investigated convolution product

Received: August 15, 2017
Revised: September 5, 2017
Published: October 7, 2017

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and obtained a similar result to Young's convolution theorem on $(L^p(\mathbb{R}, E), \ell^q)$. They also demonstrated classical result on Fourier transform of convolution product for $(L^p(\mathbb{R}, E), \ell^q)$.

In 1981, Tewari, Dutta and Vaidya (Theorem 4, [17]) proved the following multipliers result:

$$Hom_{L^1(G,A)}\left(L^1(G,A), L^1(G,A)\right) = M(G,A).$$

Moreover, in 1985, H. C. Lai [9] characterized the following multiplier of module homomorphism obtained more general result under some suitable conditions:

$$\begin{aligned} Hom_{L^1(G,A)}\left(L^1(G,A), L^p(G,X)\right) &= L^p(G,X),\ 1 < p < \infty; \\ Hom_{L^1(G,A)}\left(L^1(G,A), L^1(G,X)\right) &= M(G,X), \end{aligned}$$

where X is a Banach A-module. In 1999, Sağır [14] also proved the following multiplier

$$Hom_{L^1(G,A)}\left(L^1(G,A) \cap L^p(G,A), L^1(G,A)\right) = M(G,A),\ 1 < p < \infty$$

In this paper, we obtain some the following results of multipliers similar to [17], [9] and [14];

$$Hom_{L^1(G,A)}\left(L^1(G,A) \cap (L^p(G,A), \ell^q), L^1(G,A)\right) = M(G,A)$$

$$Hom_{L^1(G,A)}\left(L^1(G,A) \cap (L^p(G,A), \ell^q)\right) = M(G,A)$$

under some appropriate conditions, where $M(G,A)$ denotes the space of A-valued bounded regular Borel measures of bounded variation on G.

2. Preliminaries

Definition 1. The space $L^1(G,A)$, which consist of all A-valued Bochner integrable functions on G, is a commutative Banach algebra with respect to convolution given by

$$f * g(t) = \int_G f(ts^{-1})g(s)ds = \int_G f(s)g(ts^{-1})ds$$

and the norm

$$\|f\|_{L^1(G,A)} = \int_G \|f(t)\|_A \, dt$$

for $f, g \in L^1(G, A)$. The space $L^p(G, A)$ is the set of all strong measurable functions $f : G \to A$ such that $\|f(t)\|_A^p$ is integrable for $1 \le p < \infty$, that is, $\|f(t)\|_A^p \in L^1(G)$. The norm of $f \in L^p(G, A)$ is defined by

$$\|f\|_{L^p(G,A)} = \left(\int_G \|f(t)\|_A^p \, dt \right)^{\frac{1}{p}}$$

[4], [9].

Definition 2. Let A be a Banach algebra. A Banach space B is said to be a Banach $A-$module if there exists a bilinear operation $\cdot : A \times B \to B$ such that

(i) $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for all $f, g \in A$, $h \in B$.

(ii) For some constant $C \ge 1$, $\|f \cdot h\|_B \le C \|f\|_A \|h\|_B$ for all $f \in A$, $h \in B$.

Since A is a Banach algebra under convolution, then the convolution $f * g(t)$ defines an element of A. It is known that $L^p(G, A)$,which has a bounded approximate identity, is an essential $L^1(G, A)$ -module according to convolution such that

$$\|f * g\|_{L^p(G,A)} \le \|f\|_{L^p(G,A)} \|g\|_{L^1(G,A)}$$

for $f \in L^p(G, A)$ and $g \in L^1(G, A)$, $1 \le p < \infty$ [9].

By the Structure Theorem (Theorem 24.30, [8]), $G = \mathbb{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H. Let $I = [0, 1)^a \times H$ and $J = \mathbb{Z}^a \times T$, where T is a transversal of H in G_1, i.e. $G_1 = \bigcup_{t \in T} (t + H)$ is a coset decomposition of G_1. If we define $I_\alpha = \alpha + I$ for $\alpha \in J$, then G is equal to the disjoint union of relatively compact sets I_α. We normalize μ so that $\mu(I) = \mu(I_\alpha) = 1$ for all α [15], [6].

We denote by $L^p_{loc}(G, A)$ $(1 \le p \le \infty)$ the space of (equivalence classes of) A-valued functions on G such that f restricted to any compact subset E of G belongs to $L^p(G, A)$.

Definition 3. Let $1 \le p, q < \infty$. The vector-valued classical amalgam spaces $(L^p(G, A), \ell^q)$ are the normed space

$$(L^p(G, A), \ell^q) = \left\{ f \in L^p_{loc}(G, A) : \|f\|_{(L^p(G,A),\ell^q)} < \infty \right\},$$

where

$$\|f\|_{(L^p(G,A),\ell^q)} = \left(\sum_{\alpha \in J} \|f \chi_{I_\alpha}\|^q_{L^p(G,A)} \right)^{1/q}, 1 \le p, q < \infty.$$

If $p = q$, then $(L^p(G,A), \ell^p) = L^p(G,A)$ [1].

Definition 4. A set $\{e_\alpha\}$ in a commutative, normed algebra B is an approximate identity, abbreviated a.i., if for all $a \in B$, $\lim_\alpha e_\alpha a = a$ in B.

Theorem 5. *Let $1 \le p, q < \infty$. If $\{e_\alpha\}$ is an a.i. in $L^1(G,A)$, then $\{e_\alpha\}$ is also an a.i. in $(L^p(G,A), \ell^q)$, i.e.*

$$\lim_\alpha \|e_\alpha * f - f\|_{(L^p(G,A),\ell^q)} = 0$$

for all $f \in (L^p(G,A), \ell^q)$.

Proof. The proof of this Theorem can be obtained easily similar to scalar-valued amalgam spaces [15]. □

Corollary 6. *Let $1 \le p, q < \infty$. Then the spaces $(L^p(G,A), \ell^q)$ is an essential $L^1(G,A)$-module.*

Now we give the Young's Theorem for scalar-valued $(L^p(G), \ell^q)$ amalgam spaces, $1 \le p, q < \infty$ [6].

Theorem 7. *Let $1 \le p_1, q_1, p_2, q_2 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} \ge 1$, $\frac{1}{q_1} + \frac{1}{q_2} \ge 1$, $\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and $\frac{1}{r_2} = \frac{1}{q_1} + \frac{1}{q_2} - 1$. If $f \in (L^{p_1}(G), \ell^{q_1})$ and $g \in (L^{p_2}(G), \ell^{q_2})$, then there exists a $C > 0$ (depending on the dimension of the factor $\mathbb{R}^a$ and on which subgroup H is chosen in G_1) such that $f * g \in (L^{r_1}(G), \ell^{r_2})$,*

$$\|f * g\|_{(L^{r_1}(G),\ell^{r_2})} \le C \|f\|_{(L^{p_1}(G),\ell^{q_1})} \|g\|_{(L^{p_2}(G),\ell^{q_2})}.$$

Moreover, we have $f * g \in (L^{r_1}(G), \ell^{r_2})$ and

$$(L^{p_1}(G), \ell^{q_1}) * (L^{p_2}(G), \ell^{q_2}) \subset (L^{r_1}(G), \ell^{r_2}).$$

Theorem 8. *Let $1 \le p_1, q_1, p_2, q_2 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} \ge 1$, $\frac{1}{q_1} + \frac{1}{q_2} \ge 1$, $\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and $\frac{1}{r_2} = \frac{1}{q_1} + \frac{1}{q_2} - 1$. If $f \in (L^{p_1}(G,A), \ell^{q_1})$ and $g \in (L^{p_2}(G,A), \ell^{q_2})$, then there exists a $C > 0$ such that $f * g \in (L^{r_1}(G,A), \ell^{r_2})$,*

$$\|f * g\|_{(L^{r_1}(G,A),\ell^{r_2})} \le C \|f\|_{(L^{p_1}(G,A),\ell^{q_1})} \|g\|_{(L^{p_2}(G,A),\ell^{q_2})}.$$

Proof. Let $\widetilde{f}(x) = \|f(x)\|_A$ and $\widetilde{g}(x) = \|g(x)\|_A$ be given for any $x \in G$. If $f \in (L^{p_1}(G,A), \ell^{q_1})$ and $g \in (L^{p_2}(G,A), \ell^{q_2})$, then we have $\widetilde{f} \in (L^{p_1}(G), \ell^{q_1})$, $\widetilde{g} \in (L^{p_2}(G), \ell^{q_2})$ and

$$\left\|\widetilde{f}\right\|_{(L^{p_1}(G),\ell^{q_1})} = \|f\|_{(L^{p_1}(G,A),\ell^{q_1})},$$

$$\|\widetilde{g}\|_{(L^{p_2}(G),\ell^{q_2})} = \|g\|_{(L^{p_2}(G,A),\ell^{q_2})}.$$

Therefore, we get

$$\begin{aligned} \|f*g(x)\|_A &\leq \int_G \|f(x-y)g(y)\|_A \, dy \\ &\leq \int_G \|f(x-y)\|_A \|g(y)\|_A \, dy \\ &= \int_G \widetilde{f}(x-y)\widetilde{g}(y)dy \\ &= \left(\widetilde{f}*\widetilde{g}\right)(x). \end{aligned}$$

If we use definition of the norm of $\|.\|_{(L^{r_1}(G),\ell^{r_2})}$ and Theorem 7, then we have

$$\|f*g\|_{(L^{r_1}(G,A),\ell^{r_2})} \leq \left\|\widetilde{f}*\widetilde{g}\right\|_{(L^{r_1}(G),\ell^{r_2})} < \infty.$$

Therefore, $f*g(x)$ exists for almost all $x \in G$. Using Theorem 7 we get

$$\begin{aligned} \|f*g\|_{(L^{r_1}(G,A),\ell^{r_2})} &\leq \left\|\widetilde{f}*\widetilde{g}\right\|_{(L^{r_1}(G),\ell^{r_2})} \\ &\leq C\left\|\widetilde{f}\right\|_{(L^{p_1}(G),\ell^{q_1})} \|\widetilde{g}\|_{(L^{p_2}(G),\ell^{q_2})} \\ &= C\|f\|_{(L^{p_1}(G,A),\ell^{q_1})} \|g\|_{(L^{p_2}(G,A),\ell^{q_2})}. \end{aligned}$$

□

By Theorem 8, we have the following inequality

$$\begin{aligned} \|f*g\|_{(L^p(G,A),\ell^q)} &\leq C\|f\|_{(L^p(G,A),\ell^q)} \|g\|_{(L^1(G,A),\ell^1)} \\ &= C\|f\|_{(L^p(G,A),\ell^q)} \|g\|_{L^1(G,A)} \end{aligned}$$

for all $f \in (L^p(G,A),\ell^q)$ and $g \in L^1(G,A)$, where $C \geq 1$, i.e. the amalgam space $(L^p(G,A),\ell^q)$ is a Banach $L^1(G,A)$-module with respect to convolution. Moreover, it is easy to see that the amalgam space $\left(L^p(G,A),\ell^1\right)$ is a Banach algebra under convolution $p \geq 1$, if we define the norm $|||f|||_{(L^p(G,A),\ell^1)} = C\|f\|_{(L^p(G,A),\ell^1)}$ for $\left(L^p(G,A),\ell^1\right)$. Recall that $\left(L^p(G,A),\ell^1\right) \subset L^1(G,A)$.

3. Multipliers of $L^1(G,A)\cap(L^p(G,A),\ell^q)$ to $L^1(G,A)$, $1<p,q<\infty$

Definition 9. Let V and W be two Banach modules over a Banach algebra A. Then a multiplier from V into W is a bounded linear operator T from V into W, which commutes with module multiplication, i.e. $T(av)=aT(v)$ for $a\in A$ and $v\in V$. We denote by $Hom_A(V,W)$ the space of all multipliers from V into W. Also we write $Hom_A(V,V)=Hom_A(V)$. It is known that

$$Hom_A(V,W^*)\cong(V\otimes_A W)^*,$$

where W^* is dual of W, and $V\otimes_A W$ is the A-module tensor product of V and W (Corollary 2.13, [12]).

Let $1\le p,q<\infty$. We define the space $L^1(G,A)\cap(L^p(G,A),\ell^q)$ is a Banach space with the sum norm

$$\|f\|^1_{pq,A}=\|f\|_{L^1(G,A)}+\|f\|_{(L^p(G,A),\ell^q)}$$

where $f\in L^1(G,A)\cap(L^p(G,A),\ell^q)$.

Let A^* has RNP (Radon-Nikodym property) and $1<p,q,s,t<\infty$. Then, it is known that the dual space of $(L^p(G,A),\ell^s)$ is isometrically isomorphic to $\left(L^q(G,A^*),\ell^t\right)$ for $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{s}+\frac{1}{t}=1$ [1].

Corollary 10. *Using the Corollary 2.13 in [12], we have*

$$\begin{aligned}&Hom_{L^1(G,A)}\left(L^1(G,A),\left(L^q(G,A^*),\ell^t\right)\right)\\ \cong\ &\left(L^1(G,A)\otimes_{L^1(G,A)}(L^p(G,A),\ell^s)\right)^*=\left(L^q(G,A^*),\ell^t\right)\end{aligned}$$

for $\frac{1}{p}+\frac{1}{q}=1$ *and* $\frac{1}{s}+\frac{1}{t}=1$.

Consider the mapping Φ from $L^1(G,A)\cap(L^p(G,A),\ell^q)$ into $L^1(G,A)\times(L^p(G,A),\ell^q)$ defined by $\Phi(f)=(f,f)$. This is a linear isometry of $L^1(G,A)\cap(L^p(G,A),\ell^q)$ into $L^1(G,A)\times(L^p(G,A),\ell^q)$ with the norm

$$|||(f,f)|||=\|f\|_{L^1(G,A)}+\|f\|_{(L^p(G,A),\ell^q)},\ \left(f\in L^1(G,A)\cap(L^p(G,A),\ell^q)\right).$$

Hence it is easy to see that $L^1(G,A)\cap(L^p(G,A),\ell^q)$ is a closed subspace of the Banach space $L^1(G,A)\times(L^p(G,A),\ell^q)$. Let

$$H=\left\{(f,f):f\in L^1(G,A)\times(L^p(G,A),\ell^q)\right\}$$

and

$$K=\left\{\begin{array}{c}(\varphi,\psi):(\varphi,\psi)\in L^{\infty}(G,A^{*})\times\left(L^{q}\left(G,A^{*}\right),\ell^{t}\right),\\ \int\limits_{G}f(x)\varphi(x)dx+\int\limits_{G}f(y)\psi(y)dy=0,\text{ for all }\ (f,f)\in H\end{array}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{s}+\frac{1}{t}=1$.

The following proposition is easily proved by Duality Theorem 1.7 in [12].

Proposition 11. *The dual space of* $L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)$ *is isomorphic to* $L^{\infty}(G,A^{*})\times\left(L^{q}\left(G,A^{*}\right),\ell^{t}\right)/K$.

Proposition 12. $L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)$ *is a essential Banach* $L^{1}(G,A)$*-module.*

Proof. Let $f\in L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)$ and $g\in L^{1}(G,A)$. Since $\left(L^{p}\left(G,A\right),\ell^{q}\right)$ is an essential Banach $L^{1}(G,A)$-module, then we have

$$\begin{aligned}\left\|f*g\right\|_{pq,A}^{1} &= \left\|f*g\right\|_{L^{1}(G,A)}+\left\|f*g\right\|_{(L^{p}(G,A),\ell^{q})}\\ &\leq \left\|f\right\|_{L^{1}(G,A)}\left\|g\right\|_{L^{1}(G,A)}+\left\|f\right\|_{(L^{p}(G,A),\ell^{q})}\left\|g\right\|_{L^{1}(G,A)}\\ &= \left\|f\right\|_{pq,A}^{1}\left\|g\right\|_{L^{1}(G,A)}.\end{aligned}$$

Also, using Theorem 5 , then $\left\|e_{\alpha}*f-f\right\|_{pq,A}^{1}\to 0$. Hence $L^{1}(G,A)*L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)=L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)$ by Module Factorization Theorem [18]. This completes the proof. □

Proposition 13. *The multiplier space* $Hom_{L^{1}(G,A)}\left(L^{1}(G,A),\left(L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)\right)^{*}\right)$ *is isomorphic to* $L^{\infty}(G,A^{*})\times\left(L^{q}\left(G,A^{*}\right),\ell^{t}\right)/K$.

Proof. By Proposition 12 we write $L^{1}(G,A)*L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)=L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)$. Hence, by Corollary 2.13 in [13] and Proposition 11 we have

$$\begin{aligned}&Hom_{L^{1}(G,A)}\left(L^{1}(G,A),\left(L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)\right)^{*}\right)\\ =&\left(L^{1}(G,A)*L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)\right)^{*}\\ =&\left(L^{1}(G,A)\cap\left(L^{p}\left(G,A\right),\ell^{q}\right)\right)^{*}=L^{\infty}(G,A^{*})\times\left(L^{q}\left(G,A^{*}\right),\ell^{t}\right)/K.\end{aligned}$$

□

For $t, s \in G$ and $f \in (L^p(G,A), \ell^q)$, $\tau_s f$ is the function on G defined by $\tau_s f(t) = f(ts^{-1})$. If $\tau_s f \in (L^p(G,A), \ell^q)$ for all $f \in (L^p(G,A), \ell^q)$, then the space is $(L^p(G,A), \ell^q)$ said to be translation invariant. It is well known that $(L^p(G,A), \ell^q)$ is translation invariant [10].

The following lemma is well known for scalar-valued amalgam spaces (Lemma 12.3, [15]) and vector-valued Lebesgue spaces (Lemma 1, [14]). Therefore, the proof of Lemma can be proved similarly for $(L^p(G,A), \ell^q)$.

Lemma 14. *Let G be a non-compact locally compact Abelian group and $1 \leq p, q < \infty$. If $f \in (L^p(G,A), \ell^q)$, then*

$$\lim_{s \to \infty} \|f + \tau_s f\|_{(L^p(G,A),\ell^q)} = 2^{\frac{1}{q}} \|f\|_{(L^p(G,A),\ell^q)}.$$

Theorem 15. *Let G be a noncompact locally compact Abelian group and $1 < p, q < \infty$. Then the following are equivalent:*

(**i**) $T \in Hom_{L^1(G,A)}\left(L^1(G,A) \cap (L^p(G,A), \ell^q), L^1(G,A)\right)$.

(**ii**) *There exists a unique A-valued vector measure $\mu \in M(G,A)$ such that $Tf = f * \mu$ for all $f \in L^1(G,A) \cap (L^p(G,A), \ell^q)$.*

Moreover, the correspondence between T and μ defines an isomorphism from $Hom_{L^1(G,A)}\left(L^1(G,A) \cap (L^p(G,A), \ell^q), L^1(G,A)\right)$ to $M(G,A)$.

Proof. If $\mu \in M(G,A)$ and $f \in L^1(G,A) \cap (L^p(G,A), \ell^q)$, then the convolution

$$f * \mu(t) = \int_G f(ts^{-1}) d\mu(s) = \int_G \tau_s f(t) d\mu(s)$$

defines an element in $L^1(G,A)$ (Theorem 9, [7]). Hence, we have the mapping $T(f) = f * \mu$, which is a bounded linear map from $L^1(G,A) \cap (L^p(G,A), \ell^q)$ to $L^1(G,A)$, and

$$\begin{aligned} \|Tf\|_{L^1(G,A)} &= \|\mu * f\|_{L^1(G,A)} \\ &\leq \|\mu\| \|f\|_{L^1(G,A)} \\ &\leq \|\mu\| \|f\|^1_{pq,A}, \end{aligned}$$

$\|T\| \leq \|\mu\|$. Moreover, for $f \in L^1(G,A) \cap (L^p(G,A), \ell^q)$ and $g \in L^1(G,A)$ we get

$$T(g * f) = (g * f) * \mu = g * (f * \mu) = g * T(f)$$

and $T \in Hom_{L^1(G,A)}\left(L^1(G,A) \cap (L^p(G,A), \ell^q), L^1(G,A)\right)$ and $\|T\| \leq \|\mu\|$.

Conversely assume that
$T \in Hom_{L^1(G,A)}\left(L^1(G,A) \cap (L^p(G,A), \ell^q), L^1(G,A)\right)$. Then, for each $f \in L^1(G,A) \cap (L^p(G,A), \ell^q)$ we have

$$\|Tf\|_{L^1(G,A)} \leq \|T\| \|f\|^1_{pq,A} = \|T\| \left(\|f\|_{L^1(G,A)} + \|f\|_{(L^p(G,A),\ell^q)}\right).$$

By Lemma 14 we have

$$\begin{aligned} & 2\|Tf\|_{L^1(G,A)} \\ = & \lim_{s\to\infty} \|Tf + \tau_s Tf\|_{L^1(G,A)} = \lim_{s\to\infty} \|T(f+\tau_s f)\|_{L^1(G,A)} \\ \leq & \lim_{s\to\infty} \|T\| \left(\|f+\tau_s f\|_{L^1(G,A)} + \|f+\tau_s f\|_{(L^p(G,A),\ell^q)}\right) \\ \leq & \lim_{s\to\infty} \|T\| \left(2\|f\|_{L^1(G,A)} + \|f+\tau_s f\|_{(L^p(G,A),\ell^q)}\right) \\ = & \|T\| \left(2\|f\|_{L^1(G,A)} + 2^{\frac{1}{q}} \|f\|_{(L^p(G,A),\ell^q)}\right) \end{aligned}$$

for each $f \in L^1(G,A) \cap (L^p(G,A), \ell^q)$. Thus, we have

$$\|Tf\|_{L^1(G,A)} \leq \|T\| \left(\|f\|_{L^1(G,A)} + 2^{\frac{1}{q}-1} \|f\|_{(L^p(G,A),\ell^q)}\right),$$

and repeating this process n times we see that

$$\|Tf\|_{L^1(G,A)} \leq \|T\| \left(\|f\|_{L^1(G,A)} + 2^{n\left(\frac{1}{q}-1\right)} \|f\|_{(L^p(G,A),\ell^q)}\right).$$

Since $q > 1$, then we have $\lim_{n\to\infty} 2^{n\left(\frac{1}{q}-1\right)} = 0$, and

$$\|Tf\|_{L^1(G,A)} \leq \|T\| \|f\|_{L^1(G,A)}.$$

Thus, T, which commutes with translation operators, that is, $T\tau_s = \tau_s T$ for each $s \in G$, is a continuous linear transformation from $L^1(G,A) \cap (L^p(G,A), \ell^q)$ viewed as a subspace of $L^1(G,A)$. Therefore, since $L^1(G,A) \cap (L^p(G,A), \ell^q)$ is a dense subspace of $L^1(G,A)$, then T determines a unique element $T^\star$ of $Hom_{L^1(G,A)}\left(L^1(G,A)\right)$ and $\|T^\star\| \leq \|T\|$. By Theorem 4 in [17] and Theorem 9 in [9] there exists a unique $\mu \in M(G,A)$ such that $T^\star f = \mu * f$ for each $f \in L^1(G,A)$ and $\|\mu\| = \|T^\star\|$. Consequently, $Tf = \mu * f$ for $f \in L^1(G,A) \cap (L^p(G,A), \ell^q)$ and $\|\mu\| \leq \|T\|$. Therefore, (i) and (ii) are equivalent. □

Corollary 16. *Let G be a noncompact locally compact Abelian group and $1 < p, q < \infty$. Then the following are equivalent:*

(**i**) $T \in Hom_{L^1(G,A)}\left(L^1(G,A) \cap \left(L^p\left(G,A\right),\ell^q\right)\right)$.

(**ii**) *There exists a unique A-valued vector measure* $\mu \in M(G,A)$ *such that* $Tf = f * \mu$ *for all* $f \in L^1(G,A) \cap \left(L^p\left(G,A\right),\ell^q\right)$.

Moreover, the correspondence between T *and* μ *defines an isomorphism from* $Hom_{L^1(G,A)}\left(L^1(G,A) \cap \left(L^p\left(G,A\right),\ell^q\right)\right)$ *to* $M(G,A)$.

Proof. For $T \in Hom_{L^1(G,A)}\left(L^1(G,A) \cap \left(L^p\left(G,A\right),\ell^q\right)\right)$ and $1 < p,q < \infty$ we have

$$\|Tf\|_{L^1(G,A)} \leq \|Tf\|_{pq,A}^1 \leq |||T|||\,\|f\|_{pq,A}^1$$

where $|||T|||$ denotes the norm of the operator T consider as element of $Hom_{L^1(G,A)}$ $\left(L^1(G,A) \cap \left(L^p\left(G,A\right),\ell^q\right)\right)$. Moreover, we have also

$$T \in Hom_{L^1(G,A)}\left(L^1(G,A) \cap \left(L^p\left(G,A\right),\ell^q\right), L^1(G,A)\right)$$

and $\|T\| \leq |||T|||$. Applying Theorem 15 we deduce that there exists a unique A-valued vector measure $\mu \in M(G,A)$ such that $Tf = f * \mu$ for all $f \in L^1(G,A) \cap \left(L^p\left(G,A\right),\ell^q\right)$ and for which $\|\mu\| = \|T\| \leq |||T|||$.

Conversely, if $\mu \in M(G,A)$ is such that $Tf = f * \mu$ for each $f \in L^1(G,A) \cap \left(L^p\left(G,A\right),\ell^q\right)$, then using the method in Proposition 2.1 in [7] we obtain

$$\begin{aligned}
& \|Tf\|_{L^1(G,A)\cap(L^p(G,A),\ell^q)} \\
=\ & \|\mu * f\|_{L^1(G,A)} + \|\mu * f\|_{(L^p(G,A),\ell^q)} \\
=\ & \|\mu * f\|_{L^1(G,A)} + \left\|\int_G \tau_s f(t)\,d\mu(s)\right\|_{(L^p(G,A),\ell^q)} \\
\leq\ & \|\mu * f\|_{L^1(G,A)} + \int_G \|\tau_s f\|_{(L^p(G,A),\ell^q)}\,d|\mu|(s) \\
\leq\ & \|\mu\|\,\|f\|_{L^1(G,A)} + C\,\|\mu\|\,\|f\|_{(L^p(G,A),\ell^q)} \\
\leq\ & \|\mu\|\left(\|f\|_{L^1(G,A)} + C\,\|f\|_{(L^p(G,A),\ell^q)}\right) \\
\leq\ & (C+1)\,\|\mu\|\,\|f\|_{L^1(G,A)\cap(L^p(G,A),\ell^q)},
\end{aligned}$$

$T \in Hom_{L^1(G,A)}\left(L^1(G,A) \cap \left(L^p\left(G,A\right),\ell^q\right)\right)$ and $|||T||| \leq (C+1)\,\|\mu\|$. This completes the proof. $\square$

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