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SOME TYPES OF IDEALS IN DISTRIBUTIVE IMPLICATION GROUPOIDS

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Abstract: In this paper, the notions of $\mathcal{I}-$ideal, $\mathcal{N}-$ideal and $\mathcal{F}$-ideal in a distributive implication groupoid are introduced and proved that these notions are equivalent. Also, we introduced the notion of $\mathcal{O}-$ideal and maximal ideal in a distributive implication groupiod and studied the relation between $\mathcal{O}-$ideal and $\mathcal{N}-$ideal. Finally, we shown that $\mathcal{O}-$ideals and maximal ideals are equivalent.

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1. Introduction

Hilbert algebras play vital role for investigations in intuitionistic logic and other non-classical logics. Hilbert algebras represent the algebraic counterpart of the implicative fragment of intuitionistic propositional logic. These algebras were

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extensively studied by A.Diego [10], D. Buşneag [1], S.A.Celani [9], S. M. Hong [11], Y. B. Jun [12] and I.Chajda et al [7]. Later, I. Chajda and R. Halas [8] introduced the concept of implication groupoid as a generalization of the implication reduct of intuitionistic logic, i.e., Hilbert algebra and studied some connections among ideals, deductive systems and congruence kernels whenever the implication groupoid is distributive. Further, R.K.Bandaru et al. studied the properties of ideals of distributive implication groupoid [2] and introduced the concepts of fuzzy implication groupoids [6], fuzzy ideals [5], fuzzy implicative ideals [4] and vague ideals [3] in a distributive implication groupoid.

In this paper, we introduce the concept of $\mathcal{I}-$ideal, $\mathcal{N}-$ideal and $\mathcal{F}$-ideal in a distributive implication groupoid and prove that these notions are equivalent. Also, we introduce the notions of $\mathcal{O}-$ideal and maximal ideal in a distributive implication groupiod and give the relation between $\mathcal{O}-$ideal and $\mathcal{N}-$ideal. Finally, we show that $\mathcal{O}-$ideals and maximal ideals are equivalent.

2. Preliminaries

First, we recall certain definitions and properties of distributive implication groupoid which we use in later sections

Definition 2.1. [8] We call an algebra $(A, *, 1)$ of type $(2, 0)$ a distributive implication groupoid if A satisfies the following identities:

(1) $x * x = 1$

(2) $1 * x = x$

(3) $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in A$.

Lemma 2.2. *[8] Let $(A, *, 1)$ be a distributive implication groupoid. Then A satisfies the identities $x * 1 = 1$ and $x * (y * x) = 1$. Moreover, the induced relation $\leq$ is a quasi order on A and the following relationships are satisfied*

(i) $x \leq 1$

(ii) $x \leq y * x$

(iii) $x * ((x * y) * y) = 1$

(iv) $1 \leq x$ *implies* $x = 1$

(v) $y * z \leq (x * y) * (x * z)$

(vi) $x \leq y$ *implies* $y * z \leq x * z$

(vii) $x * (y * z) \leq y * (x * z)$

(viii) $x * y \leq (y * z) * (x * z)$

Definition 2.3. [8] Let $(A, *, 1)$ be a distributive implication groupoid. A non-empty subset I of A is called an ideal of A if, (i). $1 \in I$ (ii). $x \in I$ and $x * y \in I$ implies $y \in I$.

Denote $\mathcal{I}(A)$ the set of all ideals of A. For every subset $X \subseteq A$, the smallest ideal of A which contains X, that is the intersection of all ideals $I \supseteq X$, is said to be the ideal generated by X, and will be denoted by $(X]$. Obviously, $(\emptyset] = \{1\}$. Define $\prod_{i=1}^{n} a_i * x = a_n * (\cdots * (a_1 * x) \cdots)$

Theorem 2.4. *[2] Let $X(\neq \emptyset) \subseteq A$. Then*

$$(X] = \{x \in A \mid x = 1 \textit{ or } \prod_{i=1}^{n} a_i * x = 1, \textit{ for some } a_1, a_2, \cdots, a_n \in X\}.$$

Definition 2.5. [4] A non-empty subset I of A is an implicative ideal($\mathcal{I}$-ideal) of A if it satisfies the following conditions:

(I-1) $1 \in I$;

(I-2) $z * ((x * y) * x) \in I$,

and $z \in I$ implies that $x \in I$, for all $x, y, z \in A$.

Theorem 2.6. *[4] Every implicative ideal($\mathcal{I}$-ideal) is an ideal, but converse need not be true.*

Theorem 2.7. *[4] Let I be an ideal of A. Then the following are equivalent:*

(i) I is an $\mathcal{I}$-ideal.

(ii) $(x * y) * x \in I$ implies $x \in I$.

3. $\mathcal{F}$-Ideals and $\mathcal{N}$-Ideals

In this section, we introduce the concept of $\mathcal{N}$-ideal and $\mathcal{F}$-ideal in a distributive implication groupoid and we show that $\mathcal{I}$-ideals, $\mathcal{N}$-ideals and $\mathcal{F}$-ideals are equivalent.

Definition 3.1. A non-empty subset I of A is said to be fantastic ideal($\mathcal{F}-$ideal) of A if it is satisfies the following conditions:

(1) $1 \in I$

(2) $z * (y * x) \in I$ and $z \in I$ implies $((x * y) * y) * x \in I$, for all $x, y, z \in A$.

We can observe that every $\mathcal{F}-$ideal of A is an ideal of A but converse need not be true as shown in the following example.

Example 3.2. Let $A = \{1, a, b, c, d\}$ be a set with the following table:

∗	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(A, *, 1)$ is a distributive implication groupoid. Clearly $I = \{1, b\}$ is an ideal of A but not $\mathcal{F}-$ideal of A, since $b * (d * a) \in I$ and $b \in I$ but $((a * d) * d) * a = (d * d) * a = 1 * a = a \notin I$.

Theorem 3.3. *Let I be an ideal of A. Then I is an $\mathcal{F}-$ideal of A if and only if $y * x \in I$ implies $((x * y) * y) * x \in I$, for all $x, y \in A$.*

Proof. Let I be an $\mathcal{F}-$ideal of A and $y * x \in I$. Then $1 * (y * x) = y * x$ and hence $((x * y) * y) * x \in I$ and $1 \in I$. Conversely assume that I satisfies the condition. Let $x, y, z \in A$ such that $z * (y * x) \in I$ and $z \in I$. Then $y * x \in I$ and hence $((x * y) * y) * x \in I$. Therefore I is an $\mathcal{F}-$ideal of A. □

Theorem 3.4. *Let I be an ideal of A. Then the following are equivalent:*

(1) I is an $\mathcal{I}-$ideal

(2) I is an $\mathcal{F}-$ideal.

Proof. (1) ⇒ (2) : Assume (1) and $y * x \in I$. We know that $y * x \leq (x*y)*(y*x) = (x*y)*(((x*y)*y)*x) \leq [(((x*y)*y)*x)*y]*(((x*y)*y)*x)$, since $x \leq ((x * y) * y) * x$ implies that $(((x * y) * y) * x) * y \leq x * y$. Therefore $((((x * y) * y) * x) * y) * (((x * y) * y) * x) \in I$. That implies $((x * y) * y) * x \in I$, since I is $\mathcal{I}-$ideal of A.

(2) ⇒ (1) : Assume (2) and $(x*y)*x \in I$. Hence $((x*(x*y))*(x*y))*x \in I$, since I is an $\mathcal{F}-$ideal.That implies $((x*y)*(x*y))*x \in I$. That implies $1*x \in I$. Therefore $x \in I$. Hence, by Theorem 2.7, I is $\mathcal{I}-$ideal of A. □

Definition 3.5. An ideal I of A is called normal ideal($\mathcal{N}$-ideal) of A if, for any $x, y \in A$, $(x * y) * y \in I$ implies $(y * x) * x \in I$.

Example 3.6. Let $A = \{1, a, b, c\}$ be a set with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	c	1	c
c	1	1	b	1

Then $(A, *, 1)$ is a distributive implication groupoid and $I = \{1, b\}$ is an $\mathcal{N}$-ideal of A.

Theorem 3.7. *Let I be an ideal of A. Then the following are equivalent:*

(1) I is an $\mathcal{I}-$ ideal

(2) $a \in I,\ (x * y) * (a * x) \in I$ implies $x \in I$.

Proof. Suppose that I is an $\mathcal{I}-$ ideal of A and $a \in I,\ (x*y)*(a*x) \in I$. We know that $a \leq (x*y)*a$ which implies that $[(x*y)*a]*[(x*y)*x] \leq a*[(x*y)*x]$. So that $(x * y) * (a * x) \leq a * ((x * y) * x)$ and hence $a * ((x * y) * x) \in I$, since I is an ideal of A. Therefore $(x * y) * x \in I$ and hence $x \in I$ since I is $\mathcal{I}-$ ideal. Conversely, assume that the condition holds. Let $(x * y) * x \in I$. Then $(x * y) * (((x * y) * x) * x) = (((x * y) * (x * y) * (x * y) * x)) * (x * y) * x = ((x * y) * x) * ((x * y) * x) = 1 \in I$. Hence, by assumption, $x \in I$. Therefore I is $\mathcal{I}-$ ideal. □

Theorem 3.8. *Let I be an ideal of A. Then the following are equivalent:*

(1) I is an $\mathcal{I}-$ideal

(2) I is an $\mathcal{N}-$ideal.

Proof. (1) $\Rightarrow$ (2) : Assume (1) and $(x * y) * y \in I$. By Lemma 2.2(vi), $x \leq (y * x) * x$ implies that $[(y * x) * x] * y \leq x * y \longrightarrow (*)$. Now

$$\begin{aligned} (x * y) * y &\leq (y * x) * [(x * y) * x] && (\text{ by Lemma 2.2(viii)}) \\ &\leq (x * y) * ((y * x) * x) && (\text{ by Lemma 2.2(vii) }) \\ &\leq [((y * x) * x) * y] * ((y * x) * x) && (\text{ by } (*) \text{ and Lemma 2.2(vi) }) \end{aligned}$$

Since I is an ideal of A, we have $(((y * x) * x) * y) * ((y * x) * x) \in I$. Hence, by Theorem 2.7, $(y * x) * x \in I$. Thus (2) follows.

(2) $\Rightarrow$ (1) : Assume (2) and $(x*y)*x \in I$. We know that $x \leq (x*y)*y$. Then, by Lemma 2.2(v), $(x*y)*x \leq (x*y)*((x*y)*y)$ and hence $(x*y)*((x*y)*y) \in I$, since I is an ideal. That implies $(x * y) * y \in I$. Therefore $(y * x) * x \in I$. Also, $y \leq x*y$ which implies that $(x*y)*x \leq y*x$. Hence $y*x \in I$. Now $(y*x)*x \in I$ and $y * x \in I$ implies that $x \in I$. Thus (1) follows. □

We conclude this section with the following theorem.

Theorem 3.9. *Let I be an ideal of A. Then the following are equivalent.*

(1) I is an $\mathcal{I}-$ideal

(2) I is an $\mathcal{N}-$ideal

(3) I is an $\mathcal{F}-$ideal.

4. $\mathcal{O}$-Ideals and Maximal Ideals

In this section, we introduce the notions of $\mathcal{O}$-ideals and maximal ideals in a distributive implication groupoid and prove that these two are equivalent. Also, we give relation between $\mathcal{O}$-ideals and $\mathcal{I}-$ideals.

Let $I \in \mathcal{I}(A)$ and $a \in A$, put $I_a = \{x \in A \mid a*x \in I\}$. Then I_a is non-empty set, since $a * 1 = 1 \in I$ and $a * a = 1 \in I$. The following theorem is straight forward.

Theorem 4.1. *Let A be a distributive implication groupoid and $I \in \mathcal{I}(A)$. Then (i) $I_a \in \mathcal{I}(A)$ $\quad(ii)$ $(\{a\} \cup I] = I_a, \quad$ for all $a \in A$.*

Definition 4.2. A proper ideal M of A is called maximal if it is not properly contained in any other proper ideal of A.

Theorem 4.3. *Let I be an ideal of A and $I \neq A$. Then The following are equivalent:*

(1) I is maximal ideal

(2) If $a \notin I$, then $(I \cup \{a\}] = A, \quad$ for all $a \in A$.

Proof. (1) $\Rightarrow$ (2) : Assume (1) and $a \notin I$. Then clearly $I \subseteq (I \cup \{a\}]$ and $I \neq (I \cup \{a\}]$. Therefore $(I \cup \{a\}] = A$, since I is maximal. (2) $\Rightarrow$ (1) : Assume (2). Suppose there is a proper ideal G of A such that $I \subseteq G$ and $I \neq G$. Hence there exists $a \in G \setminus I$. Then, by (2), $(I \cup \{a\}] = A$. Since $(I \cup \{a\}] \subseteq G$, it follows $G = A$. □

Theorem 4.4. *Let I be a maximal ideal of A. Then, for any $x, y \in A$, $x * y \in I$ or $y * x \in I$.*

Proof. Let $x, y \in A$. If $x \in I$ or $y \in I$ then $x * y \in I$ or $y * x \in I$. Suppose $x \notin I$, $y \notin I$ and $x * y \notin I$. Then $I_{x \square y} = \{z \in A \mid (x * y) * z \in I\}$ is an ideal containing I and $x * y$. Since I is maximal, we have $I_{x \square y} = A$. Therefore $(x * y) * (y * x) \in I$ and hence $y * x = y * ((x * y) * x) \in I$. □

Converse of the above theorem need not be true. For example,

Example 4.5. Let $A = \{1, a, b, c, d\}$ with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	1	1	b	b	1

Then $(A, *, 1)$ is a distributive implication groupoid. Clearly $I = \{1, b\}$ is an ideal of A satisfying $x * y \in I$ or $y * x \in I$. But I is not maximal ideal of A, since $I = \{1, a, b, c\}$ is an ideal of A.

Definition 4.6. An ideal I of A is said to be an obstinate ideal($\mathcal{O}$-ideal) if $x, y \notin I$ imply $x * y \in I$ and $y * x \in I$.

We can observe that, in Example 4.5, $\{1, a\}$ is an ideal of A but not an $\mathcal{O}$-ideal, since $b, d \notin I$ and $d = b * d \notin I, b = d * b \notin I$.

Theorem 4.7. *Every $\mathcal{O}$-ideal of A is an $\mathcal{I}-$ideal of A.*

Proof. Let I be an $\mathcal{O}$-ideal of A. Suppose I is not $\mathcal{I}-$ideal. Then there is $x, y \in A$ such that $x \notin I$ and $(x * y) * x \in I$. If $y \in I$, then $x * y \in I$ and hence $x \in I$, which is a contradiction. If $y \notin I$. Then $x * y \in I$, a contradiction. Hence I is an $\mathcal{I}-$ideal of A. □

Converse of the above theorem need not be true. For example

Example 4.8. Let $A = \{1, a, b, c, d, e, f, g\}$ with the following table:

$*$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	1	1	1	1	1	1
b	1	c	1	c	g	1	1	g
c	1	f	f	1	f	1	f	1
d	1	c	e	c	1	e	1	1
e	1	a	f	f	d	1	f	g
f	1	c	e	c	g	e	1	g
g	1	a	b	c	f	e	f	1

Then $(A, *, 1)$ is a distributive implication groupoid. Clearly $I = \{1, b, e, f\}$ is an $\mathcal{I}-$ideal of A but not an $\mathcal{O}$-ideal of A, since $c, d \notin I$ but $d * c = c \notin I$.

Theorem 4.9. *Let I be an ideal of A. Then I is an $\mathcal{O}$-ideal if and only if I is maximal ideal of A.*

Proof. Assume that I is an $\mathcal{O}$-ideal of A. Let J be an ideal of A such that $I \subseteq J$. If $I \neq J$, then there is $x \in J$ such that $x \notin I$. Thus $(I \cup \{x\}] \subseteq J$. Let y be an arbitrary element of A. If $y \in I$, then $y \in (I \cup \{x\}] \subseteq J$. If $y \notin I$, then $x * y \in I$, since I is an obstinate filter and $x \notin I$. Therefore $y \in (I \cup \{x\}]$. Thus $A \subseteq (I \cup \{x\}]$ and so $J = A$. Therefore I is a maximal ideal. Conversely, let I be a maximal ideal and $x, y \notin I$. Thus $(I \cup \{x\}] = A$ and $(I \cup \{x\}] = I_x = A$. Thus $y \in I_x$ and so $x * y \in I$. Similarly $(I \cup \{y\}] = I_y = A$ and so $x \in I_y$. Thus $y * x \in I$. Therefore I is an $\mathcal{O}$-ideal of A. □

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