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METHOD OF LINES AND PSEUDOSPECTRAL SOLUTIONS OF THE FORCED KORTEWEG-DE VRIES EQUATION WITH VARIABLE COEFFICIENTS ARISES IN ELASTIC TUBE

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Abstract: In this paper, we solved the forced Korteweg-de vries (FKdV) with variable coefficient arises in nonlinear wave propagation in an elastic tube with a symmetrical stenosis filled with an inviscid fluid by two numerical methods, namely method of lines and pseudospectral method. We then compared both numerical solutions with its progressive wave solution. Both methods solve FKdV equation with maximum absolute errors of 10^{-4}.

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1. Introduction

Patoine and Warn [1] showed that the interaction of long, quasi-stationary, baroclinic waves with topography is given by inhomogeneous form of Korteweg-de Vries (KdV) equation (1)

$$U_t + \alpha U_x + \beta U U_x + \gamma U_{xxx} = f'(x), \tag{1}$$

where $U(x,t)$ represents the free surface displacement from its undisturbed position at time t, α represents the long-wave speed, β represents the strength of the nonlinearity, $\gamma < 0$ and $f'(x)$ represents the forcing term.

Akylas [2] showed the long nonlinear water waves by moving pressure is governed by a forced Korteweg-de Vries (FKdV) equation (2) subjects to appropriate asymptotic initial conditions as shown below:

$$\eta_t + \gamma\eta_x - 2\eta\eta_x - \frac{1}{6}\eta_{xxx} = \pi p(0)\delta'(x), \tag{2}$$

where $\eta(x,t)$ represents the difference between free surface elevation and uniform depth of water, x is spatial variable, t is temporal variable, $p(0)$ is initial pressure and $\delta'(x)$ is derivative of delta function.

Ghidaglia [3] showed that the FKdV equation (3) in the long time behavior of the infinite dimensional dynamical system

$$U_t + UU_x + U_{xxx} + \gamma U = f, \tag{3}$$

where f is an external excitation can be time-independent or time-periodic, subjects to space periodic solutions

$$U(x+L,t) = U(x,t), \forall x \in R, \forall t > 0$$

can be described by a finite dimensional attractor.

The one-dimensional stationary FKdV equation (4) was derived by Shen [4] as follows:

$$\gamma U_t - \frac{3}{2}UU_x - \frac{1}{6}U_{xxx} = \frac{1}{2}h_x \tag{4}$$

where $U(x,t)$ represents free surface elevation for the long nonlinear water waves flowing over a bump, $\lambda > 0$ and h represent the forcing term caused by the bump. Recently, the analytical solution of (3) which is a certain form of forcing term has been solved by Zhao and Guo [5] by using Hirota Direct method.

Gaik [6] studied weakly nonlinear wave propagation in a prestressed elastic tube (treated as artery) with a symmetrical stenosis filled with an inviscid fluid (treated as blood) by using reductive perturbation in the long wave approximation. By introducing a set of stretch coordinates of boundary-value problem and expand the field quantities into the asymptotic series of ϵ , the governing equations reduces to the forced Kortweg-de Vries (FKdV) equation with variable coefficient

$$U_\tau + \mu_1 U U_\xi + \mu_3 U_{\xi\xi\xi} + \mu_4(\tau) U_\xi = \mu(\tau), \tag{5}$$

where U is radial displacement of the tube, while the coefficients μ_1, μ_3, $\mu_4(\tau)$ and $\mu(\tau)$ are nonlinear, dispersive, variable coefficient and forcing terms. The variable coefficient and forcing terms are due to the effect of stenosis. Here, τ and ξ are spatial and temporal variables respectively due to the stretch coordinates of boundary-value problem. The coefficient μ_1, μ_3, $\mu_4(\tau)$ and $\mu(\tau)$ are defined by

$$\begin{aligned}
\mu_1 &= \frac{5}{2\lambda_\theta} + \frac{\beta_2}{\beta_1}, \quad \mu_3 = \frac{m}{4\lambda_z} + \frac{\gamma_\theta^2}{16} - \frac{\beta_0}{2\beta_1}, \\
\mu_4(\tau) &= \frac{\lambda_\theta \gamma_2}{\beta_1} G(\tau) - \left[\frac{\beta_2}{\beta_1} + \frac{1}{2\lambda_\theta}\right] g(\tau), \\
\mu(\tau) &= \frac{1}{2} g^{'}(\tau) - \frac{\lambda_\theta \gamma_1}{2\beta_1} G^{'}(\tau),
\end{aligned} \tag{6}$$

where:

$$\begin{aligned}
\gamma_0 &= \frac{1}{\lambda_\theta \lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2}\right) F(\lambda_\theta, \lambda_z), \\
\gamma_1 &= \frac{1}{\lambda_\theta \lambda_z}\left[1 + \frac{3}{\lambda_\theta^4 \lambda_z^2} + 2\alpha\left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2}\right)^2\right] F(\lambda_\theta, \lambda_z), \\
\gamma_2 &= \frac{1}{2\lambda_\theta \lambda_z}\left[-\frac{12}{\lambda_\theta^5 \lambda_z^2} + 6\alpha\left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2}\right)\left(1 + \frac{3}{\lambda_\theta^4 \lambda_z^2}\right) + 4\alpha^2\left(\lambda_\theta - \frac{1}{\lambda_\theta^3 \lambda_z^2}\right)^3\right] \\
&\quad \times F(\lambda_\theta, \lambda_z), \\
\beta_0 &= \frac{1}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2 \lambda_z^3}\right) F(\lambda_\theta, \lambda_z), \quad \beta_1 = \gamma_1 - \frac{\gamma_0}{\lambda_\theta}, \quad \beta_2 = \gamma_2 - \frac{\beta_1}{\lambda_\theta},
\end{aligned} \tag{7}$$

given that $F(\lambda_\theta, \lambda_z) = \exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2 \lambda_z^2} - 3\right)\right]$, $\alpha = 1.948$, $\lambda_\theta = \lambda_z = 1.6$, $c = 15.391$, $m = 0.1$, $G(\tau) = 0$ and $g(\tau) = \text{sech}(0.01\tau)$. Here α refers to material constant, λ_θ is the initial circumferential stretch ratio, λ_z is the initial axial stretch ratio, m is mass of artery and c is the scale parameter.

Yu [7] studied the bilinear form of the variable generalized variable-coefficient forced Korteweg-de Vries equation (8) in fluids of the form

$$U_t + a(t)UU_{xx} + b(t)U_{xxx} + c(t)U + d(t)U_x = f(t) \tag{8}$$

where $U(x,t)$ is a function of spatial coordinate x and temporal coordinate t, $a(t)$, $b(t)$, $c(t)$, $d(t)$ and $f(t)$ represent the nonlinear, dispersive, line-damping, dissipative, and external-force coefficients, respectively.

Tay et.al [8] studied the numerical solutions of the forced perturbed Korteweg-de Vries (FpKdV) equation [9]

$$U_\tau + \mu_1 UU_\xi - \mu_3 U_{\xi\xi\xi} - \mu_4(\tau)U_\xi + \mu_5 U = \mu(\tau), \tag{9}$$

where U is the radial displacement, ξ is a temporal variable, τ is a spatial variable, and μ_1, μ_3, $\mu_4(\tau)$, μ_5 and $\mu(\tau)$ are the coefficients of nonlinearity, dispersion, variable coefficient, perturbed and forcing terms respectively.

When $\mu_5 = 0$, Equation (9) reduces to FKdV equation (5). When $\mu_4(\tau)$ and $\mu(\tau) = 0$, Equation (5) reduces to the standard Korteweg-de Vries (KdV) equation below:

$$U_\tau + \mu_1 UU_\xi + \mu_3 U_{\xi\xi\xi} = 0 \tag{10}$$

The KdV equation (10) was first introduced by Korteweg-and Vries [10] to describe the evolution of long, one-dimensional shallow water waves with small but finite amplitude. In 1965, Zabusky and Kruskal [11] discovered the concept of the solitons while studying the results of a numerical computation on the KdV equation (10). Since then, the KdV equation (10) has been found to describe many physical phenomena, including long internal waves in ocean, magneto hydrodynamics waves in warm plasma, ion acoustic waves in a plasma, acoustic waves on a crystal lattice and wave propagation in an elastic tube filled with an inviscid fluid.

Many studies have been devoted to the numerical solutions of the KdV equation (10), however, none of the literature works dealt with numerical solutions of the FKdV equation (5). Motivated with the works of wave propagation which yielded the FKdV equation (5) as well as numerical methods especially the works of [8], we are going to find numerical solution of the FKdV equation (5) by two numerical methods, namely method of line (MOL) and pseudospectral. The numerical solution of the FKdV equation equation (5) is then compared in terms of its maximum absolute error at certain space with progressive wave solution conducted by [6].

2. The MOL

Method of line is one of the numerical methods used to solve the partial-differential equations (PDEs). By substituting the spatial derivatives by finite-difference approximations, then the PDE will reduce to time dependent system of ordinary differential equation (ODE). The ODE will then be solved by the fourth-order Runge Kutta (RK4) method [12]-[13]. MOL has been widely used to solve the nonlinear evolution equations such as Korteweg-de Vries (KdV) equation [12], extended nonlinear KdV equation, good Boussinesq equation, fifth-order Kaup-Kupershmidt equation and an extended fifth-order Korteweg-de Vries (KdV5) equation [13], delay differential equations [14], two-dimensional sine-Gordon equation [15], the Nwogu one-dimensional extended Boussinesq equation [16].

In this paper, the temporal derivatives in equation (5) are firstly discretized using central finite difference formulae as follows:

$$\begin{aligned} U &\approx \frac{U_{i+1} + U_i + U_{i-1}}{3}, \\ U_\xi &\approx \frac{U_{i+1} - U_{i-1}}{2\Delta\xi}, \\ U_{\xi\xi\xi} &\approx \frac{U_{i+2} - 2U_{i+1} + 2U_{i-1} - U_{i-2}}{2(\Delta\xi)^3}, \end{aligned} \tag{11}$$

i is the index denoting the temporal position along ξ-axis and $\Delta\xi$ is the step size along the axis. The ξ-interval is divided into M points with $i = 1, 2, ..., M - 1, M$. Therefore, the MOL approximation of (5) is given by

$$\begin{aligned} \frac{\partial U_i}{\partial \tau} = &- \frac{\mu_1}{6\Delta\xi} \left(U_{i+1} + U_i + U_{i-1}\right) \left(U_{i+1} - U_{i-1}\right) \\ &- \frac{\mu_3}{2(\Delta\xi)^3} \left(U_{i+2} - 2U_{i+1} + 2U_{i-1} - U_{i-2}\right) \\ &- \frac{\mu_4(\tau)}{2\Delta\xi} \left(U_{i+1} - U_{i-1}\right) + \mu(\tau) \equiv f(U_i). \end{aligned} \tag{12}$$

Since there is only one independent variable, which is τ, hence equation (12) is an ODE. Besides, since i varies from 1, 2, up to M, thus equation (12) represents a system of M equations of ODEs with the initial condition given by

$$U(\xi_i, \tau = 0) = U_0(\xi_i), \quad i = 1, 2, ..., M - 1, M. \tag{13}$$

For the space integration, the RK4 method is applied. Thus, the numerical solution at space τ^{j+1} is

$$U_i^{j+1} = U_i^j + \frac{1}{6}\left(a_i^j + 2b_i^j + 2c_i^j + d_i^j\right), \tag{14}$$

where

$$\begin{aligned} a_i^j &= \Delta\tau f(U_i^j), \\ b_i^j &= \Delta\tau f\left(U_i^j + \frac{1}{2}a_i^j\right), \\ c_i^j &= \Delta\tau f\left(U_i^j + \frac{1}{2}b_i^j\right), \\ d_i^j &= \Delta\tau f(U_i^j + c_i^j). \end{aligned} \tag{15}$$

Here $\Delta\tau$ is the step size of the spatial coordinate.

3. The Pseudospectral Method

Pseudospectral method transforms the spatial derivatives of the PDEs by Fourier transform and substitutes the temporal derivative by finite-difference approximation which yields a 3-level scheme to be solved numerically. It has been used to solve the KdV [17], Burgers [18] and KdV-Burgers equations [19].

The Chan and Kerkhoven [17] scheme is extended for the FKdV equation (5). The FKdV equation (5) is integrated in space τ by the leapfrog finite-difference scheme in the spectral time ξ. The infinite interval is replaced by $-L < \xi < L$ with L sufficiently large such that the periodicity assumptions hold.

By introducing $\zeta = s\xi + \pi$, where $s = \pi/L$, $U(\xi,\tau)$ will be transformed into $V(\zeta,\tau)$ as

$$\frac{\partial V}{\partial \tau} + \mu_1 sV\frac{\partial V}{\partial \zeta} + \mu_3 s^3\frac{\partial^3 V}{\partial \zeta^3} + \mu_4(\tau)s\frac{\partial V}{\partial \zeta} = \mu(\tau). \tag{16}$$

By letting $W(\zeta,\tau) = \frac{1}{2}sV^2$, the nonlinear term in equation (12) reduces to

$$\frac{\partial V}{\partial \tau} + \mu_1\frac{\partial W}{\partial \zeta} + \mu_3 s^3\frac{\partial^3 V}{\partial \zeta^3} + \mu_4(\tau)s\frac{\partial V}{\partial \zeta} = \mu(\tau). \tag{17}$$

In order to obtain the numerical solution of (17), the interval $[0, 2\pi]$ is discretized by $N+1$ equidistant points. Let $\zeta_0 = 0, \zeta_1, \zeta_2, ..., \zeta_N = 2\pi$, so that $\Delta\zeta = \frac{2\pi}{N}$. In this case, N is power of two, let $m = \frac{N}{2}$, the Discrete Fourier Transform (DFT) of $V(\zeta_j, \tau)$ for $j = 0, 1, 2, ..., N-1$, denoted by $\hat{V}(p, \tau)$ is given by

$$\hat{V}(p,\tau) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} V(\zeta_j, \tau) e^{-\left(\frac{2\pi jp}{N}\right)i},$$

where $p = -m, -m+1, -m+2, ..., m-1$ and $i = \sqrt{-1}$, the usual imaginary number. The inverse Fourier transform of $\hat{V}(p, \tau)$ for $p = -m, -m+1, -m+2, ..., m-1$, denoted by $V(\xi_j, \tau)$ can be written as

$$V(\xi_j, \tau) = \frac{1}{\sqrt{N}} \sum_{p=-m}^{m-1} \hat{V}(p,\tau) e^{\left(\frac{2\pi jp}{N}\right)i}$$

where $j = 0, 1, 2, ..., N-1$.
Then, DFT of (17) with respect to ζ gives

$$\begin{aligned} &\frac{\partial \hat{V}(p,\tau)}{\partial \tau} + i\mu_1 p \hat{W}(p,\tau) - i\mu_3 (sp)^3 \hat{V}(p,\tau) \\ &+ i\mu_4(\tau) sp \hat{V}(p,\tau) = \mu(\tau), \end{aligned} \tag{18}$$

where the cap stands for the Fourier transform.

By using the following approximations

$$\begin{aligned} \frac{\partial \hat{V}(p,\tau)}{\partial \tau} &\approx \frac{\hat{V}(p,\tau+\Delta\tau) - \hat{V}(p,\tau-\Delta\tau)}{2\Delta\tau} = \frac{\hat{V}^{k+1} - \hat{V}^{k-1}}{2\Delta\tau}, \\ \hat{V}(p,\tau) &\approx \frac{\hat{V}(p,\tau+\Delta\tau) + \hat{V}(p,\tau-\Delta\tau)}{2} = \frac{\hat{V}^{k+1} + \hat{V}^{k-1}}{2}. \end{aligned} \tag{19}$$

on equation (18),this reduces to the forward scheme given by

$$\hat{V}^{k+1} = \frac{\hat{V}^{k-1}\left[1 + i\Delta\tau\mu_3 s^3 p^3 - i\Delta\tau\mu_4(\tau) sp\right] - 2i\Delta\tau\mu_1 p\hat{W}(p,\tau) + 2\Delta\tau\mu(\tau)}{\left[1 - i\Delta\tau\mu_3 s^3 p^3 + i\Delta\tau\mu_4(\tau) sp\right]}. \tag{20}$$

Equation (20) is a three-level scheme, in which one needs to know the first level, initial condition that is $\hat{V}^{k-1}$ and subsequent second level, $\hat{V}^k$, then one can get the third level, $\hat{V}^{k+1}$. The process is repeated till the desired $\hat{V}^{k+1}$ is obtained. To get the second level, $\hat{V}^k$, the interval between $\hat{V}^{k-1}$ and $\hat{V}^k$ is divided by ten sub intervals. Later, we substitute $\Delta\tau$ in (20) by $\Delta\tau/10$ in order to get the equation for $\hat{V}^k$ as

$$\hat{V}^k = \frac{\hat{V}^{k-1}\left[1 + i\frac{\Delta\tau}{10}\mu_3 s^3 p^3 - i\frac{\Delta\tau}{10}\mu_4(\tau)sp\right] - 2i\frac{\Delta\tau}{10}\mu_1 p\hat{W}(p,\tau) + 2\frac{\Delta\tau}{10}\mu(\tau)}{\left[1 - i\frac{\Delta\tau}{10}\mu_3 s^3 p^3 + i\frac{\Delta\tau}{10}\mu_4(\tau)sp\right]}. \quad (21)$$

Equation (21) is evaluated for ten times to get $\hat{V}^k$ since the interval between $\hat{V}^{k-1}$ and $\hat{V}^k$ is divided by ten sub intervals.

4. Progressive Wave Solution

The progressive wave solution of the FKdV equation (5) is given by [6] as

$$U = a\text{sech}^2\zeta + \frac{1}{2}\left[g(\tau) - \frac{\lambda_\theta\gamma_1}{\beta_1}G(\tau)\right], \quad (22)$$

where a is the amplitude of the solitary wave. The phase function ζ can be expressed as

$$\zeta = \left(\frac{\mu_1 a}{12\mu_3}\right)^{\frac{1}{2}}\left\{\xi - \frac{\mu_1 a}{3}\tau - \int_0^\tau \left[\left(\frac{3}{4\lambda_\theta} - \frac{\beta_2}{2\beta_1}\right)g(s) + \frac{\lambda_\theta}{\beta_1}\left(\gamma_2 - \frac{\mu_1\gamma_1}{2}\right)G(s)\right]ds\right\}. \quad (23)$$

5. Results and Discussion

For both numerical methods, we need the initial condition to start the numerical simulations. By letting $\tau = 0$ and $a = 1$ in (22), we use the initial condition as

$$U = \text{sech}^2\sqrt{\frac{\mu_1}{12\mu_3}}\xi + \frac{1}{2}. \quad (24)$$

To calculate the accuracy of the numerical solution with the progressive wave solution, the maximum absolute errors between the progressive wave and numerical solutions were calculated based on the formula:

$$L_\infty = |U_{\text{progressive}} - U_{\text{numerical}}|_{\max}. \quad (25)$$

By utilizing $\triangle\xi = 0.01$, $\triangle\tau = 1 \times 10^{-6}$ in the MOL scheme, the MOL solution of the FKdV equation with variable coefficient (5) versus time ξ at

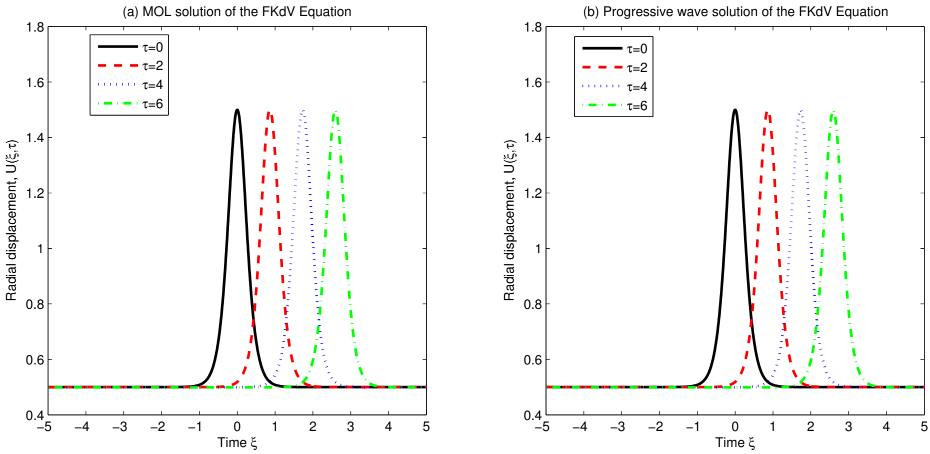

Figure 1: (a) MOL and (b) Progressive Wave Solutions of the FKdV equation (5) versus Time ξ for different Space τ with $\Delta\xi = 0.01$, $\Delta\tau = 1 \times 10^{-6}$

different space τ is shown in Figure 1(a), while the progressive wave solutions of the FKdV equation with variable coefficient (5) versus time ξ at different space τ is displayed in Figure 1(b). Figures 1(a) and 1(b) reveals both MOL and progressive wave solutions are exactly the same from observation.

To confirm if there is any difference for both MOL and progressive wave solutions, both solutions were plotted in the same figure as depicted in Figure 2. It is seen that the both MOL solution and progressive wave solution of the FKdV equation with variable coefficient (5) are overlapped exactly in terms of amplitude and position.

To confirm if there are some errors between MOL and progressive wave solutions even though they are overlapped exactly intuitively, the absolute error between the MOL and progressive wave solution of the FKdV equation with variable coefficient (5) at each space τ were calculated and plotted in Figure 3. It shows the maximum absolute errors are in order of 10^{-4}.

From Figure 3, the maximum absolute errors between the progressive wave and MOL solutions for each temporal point at certain space τ were calculated based on formula (25). Table-1 gives the maximum absolute error between the progressive wave solution and MOL solution for each temporal point at certain space τ. It shows the maximum absolute errors are in order of 10^{-4}.

The pseudospectral solution of the FKdV equation with variable coefficient

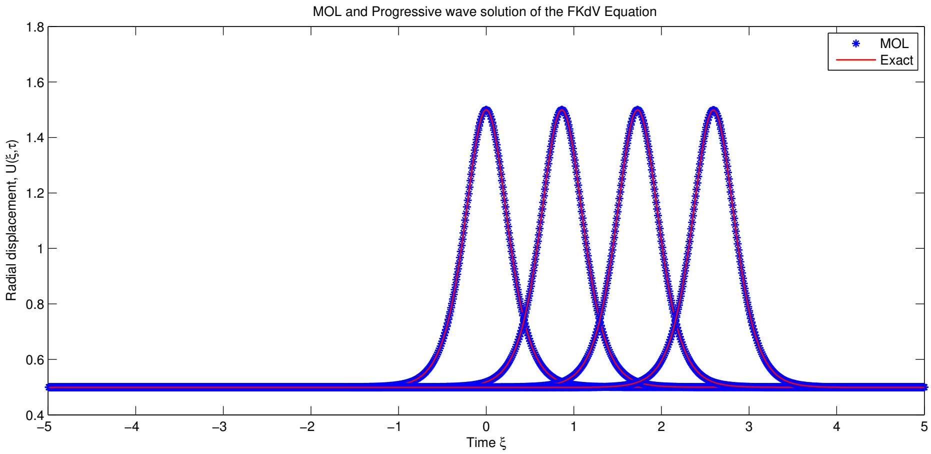

Figure 2: MOL and Progressive Wave Solutions of the FKdV equation (5) versus Time ξ for different Space τ with $\Delta\xi = 0.01$, $\Delta\tau = 1\times10^{-6}$

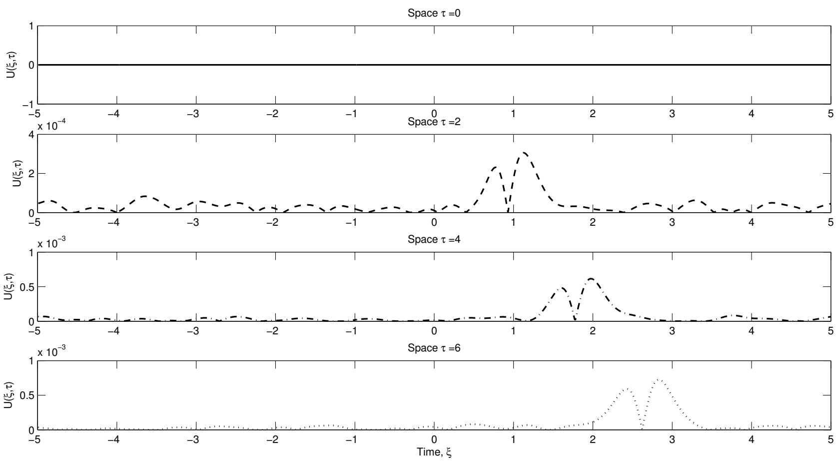

Figure 3: Absolute error between MOL and Progressive Wave Solutions of the FKdV equation (5) versus Time ξ for different Space τ

(5) is depicted in Figure 4 (a), whereas the progressive wave solution is shown in Figure 4(b). Note that, both Figures 4(a) and 4(b) are indistinguishable.

To check if there is any difference between pseudospectral and progressive

Table 1: Maximum absolute error of the FKdV equation (5) for different space τ by MOL method

Space, τ	**0**	**2**	**4**	**6**
L_∞	0	3.0557×10^{-4}	6.1515×10^{-4}	7.2554×10^{-4}

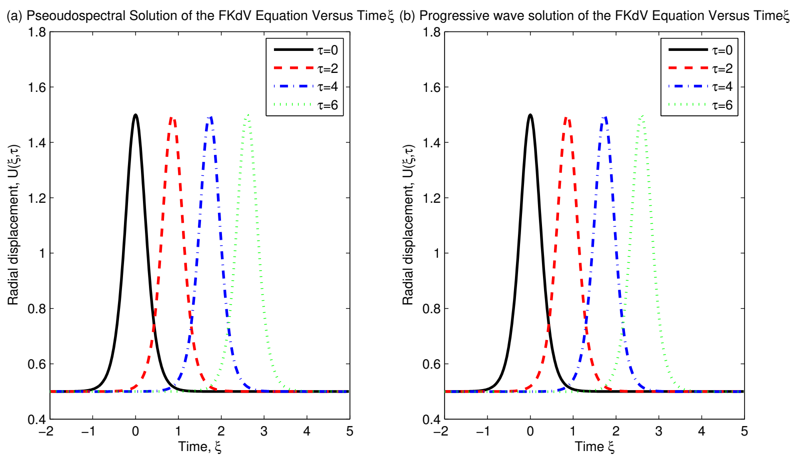

Figure 4: (a) Pseudospectral and (b) Progressive Wave Solutions of the FKdV equation versus Time ξ for different Space τ

wave solutions, both pseudospectral and progressive wave solutions were plotted in the same figure as given in Figure 5. It is seen that both pseudospectral and progressive wave solutions are ovelapped exactly in terms of amplitude and position.

To check if there are some errors numerically even though both pseudospectral and progressive wave solutions are overlapped exactly intuitively, the absolute error between pseudo spectral and progressive wave were calculated and depicted in Figure 6.

From Figure 6, the maximum absolute error of the FKdV equation (5) between the pseudospectral and progressive wave solutions for each discretized temporal point at certain space τ were calculated based on formula (25). Table 2 gives the maximum absolute error of the FKdV equation (5) between the pseudospectral and progressive wave solutions. It is noticed that the error is bigger if compared to the MOL solution.

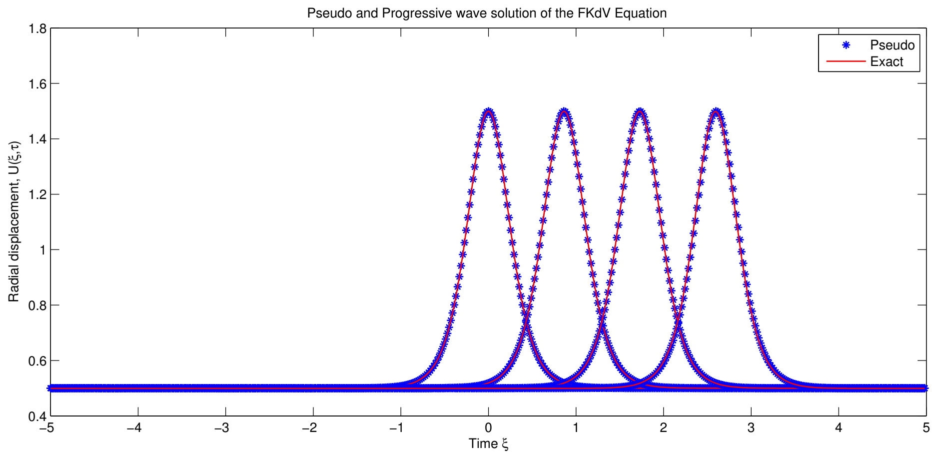

Figure 5: Pseudo spectral and Progressive Wave Solutions of the FKdV equation (5) versus Time ξ for different Space τ

Table 2: Maximum absolute error of the FKdV equation (5) for different space τ by pseudospectral method

Space, τ	**0**	**2**	**4**	**6**
L_∞	0	9.1546×10^{-4}	0.0066	0.0214

6. Conclusion

We have solved the FKdV equation with variable coefficient (5) by using two numerical methods, namely MOL and pseudospectral method. Both methods can solve the FKdV equation (5) pretty well, however the maximum absolute error for MOL is smaller if compared to the pseudospectral method.

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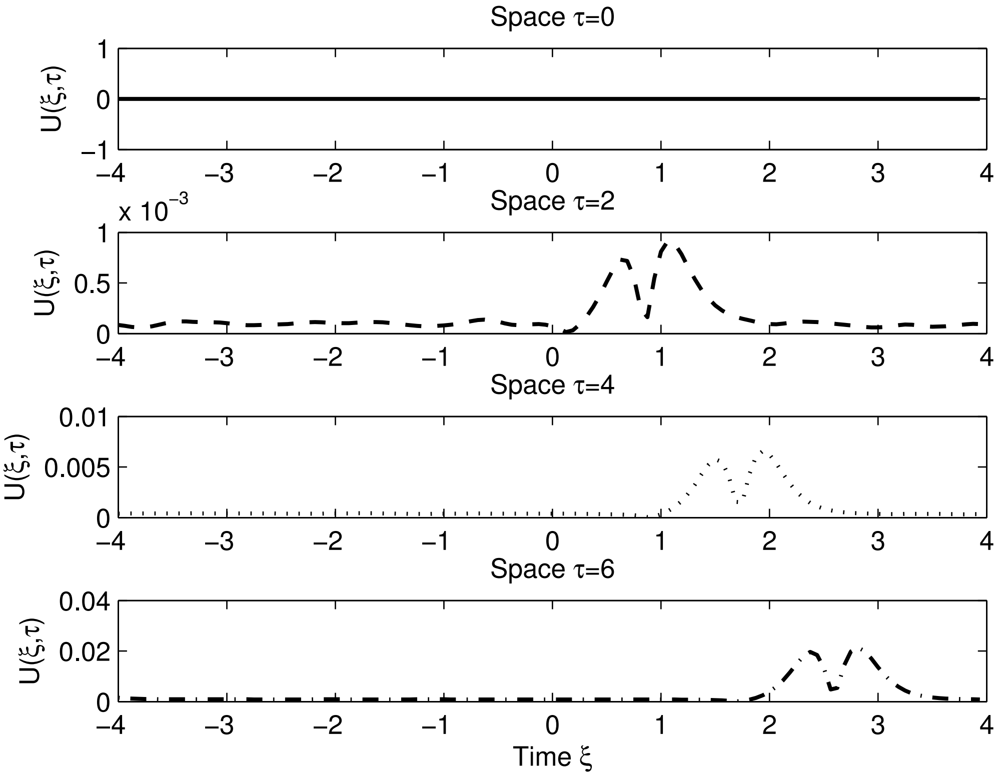

Figure 6: Absolute error between pseudospectral and Progressive Wave Solutions of the FKdV equation versus Time ξ for different Space τ

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