International Journal of Pure and Applied Mathematics
Volume 116 No. 4 2017, 1075-1080
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v116i4.22

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THE APPLICATION OF THE METHOD OF ENERGY INEQUALITIES FOR EQUATION OF BEAM

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Abstract: In this paper it is shown that under specified conditions on the initial data a certain infinite coupled system of ordinary differential equations has a solution satisfying an auxiliary convergence condition. The infinite system discussed is essentially the Galerkin expansion of the solution to a given quasi-linear wave equation of forth order. The results obtained suffice to prove the existence of a solution to this equation of oscillations of a beam.

AMS Subject Classification: 35G31
Key Words: quasi-linear wave equation, method of energy inequalities, equation of beam

1. Introduction

Consider the problem consisting of the equation

$$w_{tt} + \left(a_0 + a_1 w_{xxx}^2\right) w_{xxxx} = 0 \tag{1}$$

$(a_0 > 0, a_1 > 0)$ together with the following initial, boundary conditions:

$$w\left(x, 0\right) = f\left(x\right) = \sum_{j=1}^{n} \alpha_j \sin jx \ , \tag{2}$$

Received: 2017-08-19
Revised: 2017-11-01
Published: November 23, 2017

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$$w_t\left(x,0\right)=g\left(x\right)=\sum_{j=1}^{n}\beta_j\sin jx \tag{3}$$

$$w\left(0,t\right)=w\left(\pi,t\right)=\ w_{xx}\left(0,t\right)=w_{xx}\left(\pi,t\right)=0. \tag{4}$$

Under a smoothness hypothesis on f and g, similar to Dickey [1] we show solution of problem (1)-(4) is given by

$$w\left(x,t\right)=\sum_{j=1}^{n}T_j\left(t\right)\sin jx\ .$$

where the $T_j\left(t\right)$ satisfy the infinite system of ordinary differential equations

$$\ddot{T}_j+C_0j^4T_j+C_1j\int_0^{\pi}\left(\sum_{i=1}^{\infty}i^3T_i\cos ix\right)^3\cos jxdx=0, \tag{5}$$

together with the initial conditions

$$T_j\left(0\right)=\alpha_j,\dot{T}_j\left(0\right)=\beta_j, \tag{6}$$

and the auxiliary condition

$$\sum_{j=1}^{n}j^8\ T_j^2<\infty. \tag{7}$$

A summary of his method is as follows. Approximate solutions

$$w^{(N)}=\sum_{j=1}^{N}T_{j,N}\sin jx$$

satisfy

$$\ddot{T}_{j,N}+C_0j^2T_{j.N}+C_ij\int_0^{\pi}\left(\sum_{i=1}^{N}i^3T_{i,N}\cos ix\right)^3\cos jxdx=0. \tag{8}$$

and $T_j\left(t\right)=0$ for $j>N$. An elementary energy estimate yields a bound on $T_j\left(t\right)$ and $\dot{T}_j\left(t\right)$ independent of N. Consequently, the Arzela-Ascoli Theorem implies that on any closed subinterval $[0,\ t^{*}]$, there is a subsequence of $\{T_{j,N}\left(t\right)\}$ which converges uniformly to a continuous functions $T_j\left(t\right)$. In order to prove that the functions $T_j(t)$ are solutions of (5) it is necessary to obtain a better estimate on thefunctions $T_{j,N}(t)$ than that furnished by the energy estimate

$$\sum_{j=1}^{N} j^2 \dot{T}_{j,N}^2 + C_0 \sum_{j=1}^{N} j^4 T_{j,N}^2 + \frac{C_1}{2} \int_0^\pi \left(\sum_{j=1}^{N} j^3 T_{j,N} \cos jx \right)^4 dx = h_N,$$

$$h_N = \sum_{j=1}^{N} j^2 \beta_j^2 + C_0 \sum_{j=1}^{N} j^4 \alpha_j^2 + \frac{C_1}{2} \int_0^\pi \left(\sum_{i=1}^{N} i^3 \alpha_i \cos ix \right)^4 dx.$$

2. Existence Theorem

The necessary estimate follows upon multiplying (8) by $j^4 \dot{T}_j(t)$ and summing over j. The resulting expression may be written

$$\frac{1}{2}\frac{d}{dt} \left(\sum_{j=1}^{N} j^6 \dot{T}_{j,N}^2 + C_0 \sum_{j=1}^{\infty} j^8 T_{j,N}^2 \right) + C_1 \left\langle \left(w_{xxx}^{(N)} \right)^3, w_{xxxxxxxt}^{(N)} \right\rangle = 0, \tag{9}$$

$$\langle \mu, \nu \rangle = \int_0^\pi \mu(x) \nu(x) \, dx.$$

After two integrations by parts it is found that

$$\left\langle w_{xxx}^{(N)^3}, w_{xxxxxxxt}^{(N)} \right\rangle = \frac{3}{2} (d/dt) \left\langle w_{xxx}^{(N)^2}, w_{xxxxx}^{(N)^2} \right\rangle - 3 \left\langle w_{xxx}^{(N)} w_{xxxt}^{(N)}, w_{xxxxx}^{(N)^2} \right\rangle$$
$$+6 \left\langle w_{xxx}^{(N)} w_{xxxx}^{(N)^2}, w_{xxxxxt}^{(N)} \right\rangle. \tag{10}$$

Equations (9) and (10) imply that solutions of (6) satisfy the identity

$$(d/dt) E_n = 6 \left\langle w_{xxx}^{(N)} w_{xxxt}^{(N)}, w_{xxxxxx}^{(N)^2} \right\rangle - 12 \left\langle w_{xxx}^{(N)} w_{xxxx}^{(N)^2}, w_{xxxxxt}^{(N)} \right\rangle \tag{11}$$

$$E_n = \sum_{j=1}^{N} j^6 \dot{T}_{j,N}^2 + C_0 \sum_{j=1}^{N} j^8 T_{j,N}^2 +$$

$$+3C_1 \int_0^\pi \left(\sum_{j=1}^{N} j^3 T_{j,N} \cos jx \right)^2 \left(\sum_{j=1}^{N} j^5 T_{j,N} \cos jx \right)^2 dx =$$

$$= (2/\pi)\left\langle w^{(N)}_{xxxt}, w^{(N)}_{xxxt}\right\rangle + (2C_0/\pi)\left\langle w^{(N)}_{xxxx}, w^{(N)}_{xxxx}\right\rangle + 3C_1\left\langle w^{(N)^2}_{xxx}, w^{(N)^2}_{xxxxx}\right\rangle.$$

The object now is to estimate the right side of (11) in terms of E_N. Note that, since $w^{(N)}(0,t) = w^{(N)}(\pi,t) = w^{(N)}_{xx}(0,t) = w^{(N)}_{xx}(\pi,t) = 0$, then Rolle's theorem implies the existence of a points $\zeta = \zeta(t)$ $\eta = \eta(t)$ such that $w^{(N)}_{x}(\zeta,t) = 0$, $w^{(N)}_{xxx}(\eta,t) = 0$. Therefore

$$\left|w^{(N)}_{xxx}\right| \leq \left|\int_\eta^x w^{(N)}_{xxxx}dx\right| \leq \int_0^\pi \left|w^{(N)}_{xxxx}\right|dx \leq \left(\pi\int_0^\pi w^{(N)^2}_{xxxx}dx\right)^{\frac{1}{2}} \tag{12}$$

In addition $w^{(N)}_{xx}(0,t) = 0$ and $w^{(N)}_{xxxx}(0,t) = 0$ so that

$$\left|w^{(N)}_{xxxx}\right| = \left|\int_0^x w^{(N)}_{xxxxx}dx\right| \leq \int_0^\pi \left|w^{(N)}_{xxxxx}\right|dx \leq \left(\pi\int_0^\pi w^{(N)^2}_{xxxxx}dx\right)^{\frac{1}{2}} \tag{13}$$

Similarly it is easily shown that

$$\left|w^{(N)}_{xxxt}\right| \leq \left(\pi\int_0^\pi w^{(N)^2}_{xxxxt}dx\right)^{\frac{1}{2}} \tag{14}$$

The inequalities (12)-(14) yield pointwise estimates on$w^{(N)}_{xxx}, w^{(N)}_{xxxx}$ $w^{(N)}_{xxxt}$ in terms of E_n. Thus

$$\left|w^{(N)}_{xxx}\right| \leq \left(\pi^2/(2C_0)^2\right)E_N^{\frac{1}{2}},\ \left|w^{(N)}_{xxxx}\right| \leq \left(\pi/(2C_0)^2\right)E_N^{\frac{1}{2}},$$

$$\left|w^{(N)}_{xxxt}\right| \leq \left(\pi/\sqrt{2}\right)E_N^{\frac{1}{2}}.$$

The first term on the right of (10) can be estimated by

$$\left|\left\langle w^{(N)}_{xxx}w^{(N)}_{xxxt}, w^{(N)^2}_{xxxxx}\right\rangle\right| \leq \frac{\pi^3}{2C_0^{1/2}}E_N\int_0^\pi w^{(N)^2}_{xxxxx}dx \leq \frac{\pi^4}{4C_0^{\frac{3}{2}}}E_N^2. \tag{15}$$

For the second term on the right of (10) integrate once by parts so that

$$\left|\left\langle w^{(N)}_{xxx}w^{(N)^2}_{xxxx}, w^{(N)}_{xxxxxt}\right\rangle\right| \leq \frac{\pi^3}{(2C_0)^{\frac{3}{2}}}E_N^{\frac{3}{2}}\left(\pi\int_0^\pi w^{(N)^2}_{xxxxt}dx\right)^{\frac{1}{2}} +$$

$$+\frac{\pi^3}{C_0}E_N\left(\int_0^\pi w^{(N)^2}_{xxxxx}dx\int_0^\pi w^{(N)^2}_{xxxxt}dx\right)^{\frac{1}{2}} \leq \left(\pi^4/4C_0^{\frac{3}{2}}\right)E_N^2 + \left(\pi^4/2C_0^{\frac{3}{2}}\right) \tag{16}$$

In view of the inequalities (15) and (16) the identity (6) can be replaced by

$$\frac{dE_n}{dt} \le \left(21\pi^4/2C_0^{\frac{3}{2}}\right) E_N^2). \tag{17}$$

The inequality (17) yields the desired estimate on E_N.

Lemma 1. Assume

$$\lim_{N\to\infty} E_N(0) = e = \sum_{j=1}^{\infty} j^6\beta_j^2 + C_0\sum_{j=1}^{\infty} j^8 a_j^2 +$$

$$+3C_1\int_0^{\pi}\left(\sum_{j=1}^{\infty} j^3\alpha_j \cos jx\right)^2\left(\sum_{j=1}^{\infty} j^5\alpha_j \cos jx\right)^2 dx < \infty, \tag{18}$$

i.e. assume $E_N(0)$ converges as $N \to \infty$. Then E_N is uniformly bounded independent of N on any closed subinterval $0 \le t \le t^* < t_c$ where $t_c = 2C_0^{\frac{3}{2}}/21\pi^4 e$.

Proof. The inequality (17) implies that

$$E_N(t) \le \frac{E_N(0)}{1-\left(21\pi^4/2C_0^{\frac{3}{2}}\right)E_N(0)\,t},$$

$$0 \le t < 2C_0^{\frac{3}{2}}/21\pi^4 E_N(0).$$

The lemma follows after taking the limit as $N \to \infty$.

The bound on E_N furnished by Lemma 1 is the key feature in provingthe functions T_j, i.e. the limits of the subsequence $T_{j,N}$, are solutions of (5). In fact this result is a consequence of the following two lemmas which we give without proofs.

Lemma 2. If $e < \infty$ (cf. (17)) the infinite series $\sum_{j=1}^{\infty} j^8 T_j^2$ converges in the interval $0 \le t \le t^* < t_c$.

Lemma 3. If $e < \infty$, the functions $w^{(N_i)}$ and $w_x^{(N_i)}$converge to w and w_x, $w = \sum_{j=1}^{\infty} T_j \sin jx$, as $N_i \to \infty$ for t in the interval $0 \le t \le t^* < t_c$.

Theorem 1. The functions T_j are a solution of (5) satisfying the initial conditions (6) and auxiliary condition (7) in the interval $0 \le t \le t^*$ if $e < \infty$.

Proof. The functions T_{j,N_i} satisfy the Volterra integral equation

$$T_{j,N_i} = \alpha_j + \beta_j t - \int_0^t (t-\tau)\left\{C_0 j^4 T_{j,N_i} + C_1 j\left\langle {w_x^{(N_i)}}^3, \cos jx\right\rangle\right\} d\tau =$$

$$= \alpha_j + \beta_j t - G_j w_x^{(N_{i})}$$

for $j = 1, 2, \ldots, N_i$. The object is to show that the functions T_j satisfya similar equation. For this purpose write ($\|*\| = max_{0 \le t \le t^*} |*|$)

$$|T_j - \alpha_j - \beta_j t + G_j w_x| = \left| T_j - T_{j,N_i} - G_j w_x^{(N_{i})} + G_j w_x \right| \le$$

$$\le \|T_j - T_{j,N_i}\| + C_0 j^2 t^* \|T_j - T_{j,N_i}\| + C_0 j^2 t^* \left\| \left\langle w_x^{(N_i)^3} - w_x^3, \cos jx \right\rangle \right\| .$$

The right side of the above inequality approaches zero as $N_i \to \infty$. Therefore T_j is a solution of

$$T_j = \alpha_j + \beta_j t - G_j w_x. \tag{19}$$

The theorem follows on differentiation of (19) .

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