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# ONE MODULO $N$-MEAN LABELING

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**Abstract:** In this paper we introduce a new type of labeling known as one modulo $N$ mean labeling (where $N$ is a positive integer). A graph $G$ is said to be one modulo $N$ mean labeling (where $N$ is a positive integer) if there is a bijection $f$ from the set of $G$ to $\{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1\}$ in such a way that (i) $f$ is 1-1 (ii) $f$ induces a bijection $f^*$ from the edge set of $G$ to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ where

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

A graph that admits a one modulo $N$ mean labeling is called a one modulo $N$ mean graph. Here we investigate one modulo $N$ mean behaviour of some standard graphs.

**AMS Subject Classification:** 05C78
**Key Words:** labeling, one modulo $N$ mean labeling, one modulo $N$ mean graph

## 1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with $p$ vertices and $q$ edges. For notations and terminology we follow [1].

Path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$. $K_{1,m}$ is called a star and it is denoted by $S_m$. The bistar $B_{m,n}$ is the graph obtained from $K_2$ by identifying the central vertices of $K_{1,m}$ and $K_{1,n}$ at the end vertices of $K_2$ respectively. $B_{m,m}$ is often denoted by $B(m)$. The product $P_m \times P_n$ is called a planar grid and $P_n \times P_2$ is called a ladder, denoted by $L_n$.

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The $H$-graph of a path $P_n$, denoted by $H_n$ is the graph obtained from two copies of $P_n$ with vertices $v_1, v_2, v_3, \ldots, v_n$ and $u_1, u_2, u_3, \ldots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if $n$ is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if $n$ is even. If $m$ number of pendant vertices are attached at each vertex of $G$, then the resultant graph obtained from $G$ is the graph $G \odot mK_1$. When $m = 1$, $G \odot K_1$ is the corona of $G$. The graph $P_n \odot K_1$ is called a comb. Let $G$ be a graph. For each vertex $u$ of $G$, take a new vertex $u'$. Join $u'$ to those vertices of $G$ adjacent to $u$. The graph thus obtained is called the splitting graph of $G$ and it is denoted by $S'(G)$.

The concept of mean labeling was introduced and meanness of some standard graphs was studied by S. Somasundaram and R. Ponraj [4]. Further some more results on mean graphs are discussed in [5, 6, 7]. A graph $G$ is said to be a mean graph if there exists an injective function $f$ from $V(G)$ to $\{0, 1, 2, \ldots, q\}$ such that the induced map $f^*$ from $E(G)$ to $\{1, 2, 3, \ldots, q\}$ defined by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection.

In [2], K. Manickam and M. Marudai introduced odd mean labeling of a graph. A graph $G$ is said to be odd mean if there exists an injective function $f$ from $V(G)$ to $\{0, 1, 2, 3, \ldots, 2q-1\}$ such that the induced map $f^*$ from $E(G)$ to $\{1, 3, 5, \ldots, 2q-1\}$ defined by $f^*(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ is a bijection.

In [3], V. Ramachandran and C. Sekar introduced the concept of one modulo $N$ graceful where $N$ is any positive integer. A graph $G$ is said to be a one modulo $N$ graceful (where $N$ is a positive integer) if there is a function $\phi$ from the vertex set of $G$ to $\{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1\}$ in such a way that (i) $\phi$ is 1-1 (ii) $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$.

Motivated by the work of these authors, we introduce a new concept called a one modulo $N$ mean labeling of graphs. A graph $G$ is said to be one modulo $N$ mean labeling (where $N$ is a positive integer) if there is a bijection $f$ from the set of $G$ to $\{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1\}$ in such a way that (i) $f$ is 1-1 (ii) $f$ induces a bijection $f^*$ from the edge set of $G$ to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ where

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

A graph that admits a one modulo $N$ mean labeling is called a one modulo $N$ mean graph. In the case $N = 2$, the labeling is an odd mean labeling and in the case $N = 1$, the labeling is a mean labeling.

A one modulo 11 mean labeling of $B_{4,4}$ is shown in Figure 1.


Figure 1.

**Remark 1.1.** The cube $Q_3$ is a mean graph and odd mean graph but not a one modulo $N$ mean graph.

**Remark 1.2.** If $G$ is a one modulo $N$ mean graph with more than two edges then $0, 1, N(q-1)$ and $N(q-1)+1$ must be the vertex labels.

## 2. Main Results

**Theorem 2.1.** *The path $P_n$ is a one modulo $N$ mean graph if $n$ is even.*

*Proof.* Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the path $P_n$.

Define $f : V(P_n) \to \{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1 = N(n-2)+1\}$ as follows:
For every positive integer $N$,

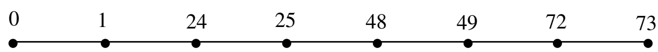
$$f(u_i) = \begin{cases} N(i-1), & 1 \le i \le n \text{ and } i \text{ is odd} \\ N(i-2)+1, & 1 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling $f^*$ is obtained as follows:
For every positive integer $N$,

$$f^*(u_i u_{i+1}) = N(i-1)+1, \quad 1 \le i \le n-1.$$

Thus, $f$ is a one modulo $N$ mean labeling. Hence, the path $P_n$ is a one modulo $N$ mean labeling if $n$ is even.

For example, a one modulo 12 mean labeling of $P_8$ are shown in Figure 2. □




A one modulo 12 mean labeling of $P_8$.
Figure 2.

**Theorem 2.2.** *If $n \geq 3$, the complete graph $K_n$ is not a one modulo $N$ mean graph.*

*Proof.* Suppose $K_n$ is a one modulo $N$ mean graph. To get the edge label 1, there must be two adjacent vertices $u$ and $v$ such that $f(u) = 0$ and $f(v) = 1$. Again to get the edge label $N(q-1)+1$, we must have $N(q-1)$ and $N(q-1)+1$ as the vertex labels of $w$ and $t$ respectively. Now, the edges $ut$ and $vw$ have the same induced label $\frac{N(q-1)+1}{2}$, which should not happen. Hence $K_n$ is not a one modulo $N$ mean graph for $n \geq 3$. □

**Remark 2.3.** The complete graph $K_2$ is a path $P_2$ which is a one modulo $N$ mean graph.

**Theorem 2.4.** *If $n \geq 2$, $K_{1,n}$ is not a one modulo $N$ mean graph.*

*Proof.* Let $\{V_1, V_2\}$ be the bipartition of $K_{1,n}$ with $V_1 = \{u\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$. To get the edge label $N(q-1)+1$, we must have $N(q-1)$ and $N(q-1)+1$ as the label of adjacent vertices. Thus either $N(q-1)$ or $N(q-1)+1$ must be a label of $u$. In both cases, since $n \geq 2$, there will be no edge whose label is 1. This contradiction proves that $K_{1,n}$ is not a one modulo $N$ mean graph for every positive integer $N$ and $n \geq 2$. □

**Remark 2.5.** The star graph $K_{1,1}$ is a path $P_2$. Thus, $K_{1,1}$ is a one modulo $N$ mean graph by Theorem 2.1.

**Theorem 2.6.** *The ladder graph $L_n = P_n \times P_2$ is a one modulo $N$ mean graph if $n$ is odd.*

*Proof.* Let $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the ladder graph $L_n$.

Let the edge set of $L_n$ be $\{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i/1 \leq i \leq n\}$.

Define $f : V(L_n) \to \{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1 = 3N(n-1)+1\}$ as follows:
For every positive integer $N$,

$$f(u_i) = \begin{cases} 3N(i-1)+1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ N(3i-2), & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 3N(i-1), & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ N(3i-4)+1, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling $f^*$ is obtained as follows:

For every positive integer $N$,

$$f^*(u_iu_{i+1}) = N(3i-1)+1, \quad 1 \le i \le n-1$$
$$f^*(v_iv_{i+1}) = N(3i-2)+1, \quad 1 \le i \le n-1$$
$$f^*(u_iv_i) = 3N(i-1)+1, \quad 1 \le i \le n.$$

Thus, $f$ is a one modulo $N$ mean labeling. Hence, the ladder graph $L_n = P_n \times P_2$ is a one modulo $N$ mean graph if $n$ is odd.

For example, a one modulo 17 mean labeling of $L_7$ is shown in Figure 3. □

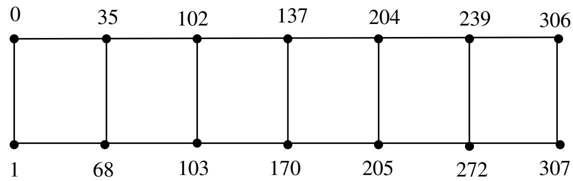



Figure 3. A one modulo 17 mean labeling of $L_7$.

**Theorem 2.7.** *Any comb $P_n \odot K_1$ is a one modulo $N$ mean graph for every positive integer $N$ and $n \ge 1$.*

*Proof.* Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the path $P_n$ and $u'_1, u'_2, u'_3, \ldots, u'_n$ be the vertices joining to $u_1, u_2, u_3, \ldots, u_n$ respectively.

Define $f : V(P_n \odot K_1) \to \{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1 = 2N(n-1)+1\}$ as follows:

$$f(u_i) = \begin{cases} 2N(i-1), & 1 \le i \le n \text{ and } i \text{ is odd} \\ 2N(i-1)+1, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$
$$f(u'_i) = \begin{cases} 2N(i-1)+1, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 2N(i-1), & 1 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling $f^*$ is obtained as follows:

$$f^*(u_iu_{i+1}) = N(2i-1)+1, \quad 1 \le i \le n-1$$
$$f^*(u_iu'_i) = 2N(i-1)+1, \quad 1 \le i \le n.$$

Thus, the graph $P_n \odot K_1$ is a one modulo $N$ mean labeling. Hence, the comb graph $P_n \odot K_1$ is a one modulo $N$ mean graph for every positive integer $N$.

For example, a one modulo 5 mean labeling of $P_8 \odot K_1$ is shown in Figure 4. □

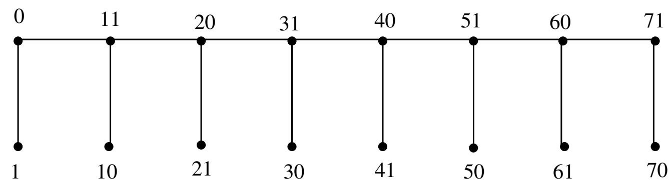



Figure 4. A one modulo 5 mean labeling of $P_8 \odot K_1$.

**Theorem 2.8.** *The graph $S'(P_{2n})$ is a one modulo $N$ mean graph for every positive integer $N$ and $n \geq 1$.*

*Proof.* Let $u_1, u_2, u_3, \ldots, u_n$ and $u'_1, u'_2, u'_3, \ldots, u'_n$ be the vertices of $S'(P_{2n})$.

Define $f : V(S'(P_{2n})) \to \{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1 = N(3n-4)+1\}$ as follows:

$$f(u_i) = \begin{cases} N(2n+i-3), & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ N(i-2)+1, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(u'_i) = \begin{cases} N(i-1), & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ N(2n+i-4)+1, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling $f^*$ is obtained as follows:

$$f^*(u_i u_{i+1}) = N(n+i-2)+1, \quad 1 \leq i \leq n-1$$

$$f^*(u'_i u_{i+1}) = \begin{cases} N(i-1)+1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ N(2n+i-3)+1, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(u_i u'_{i+1}) = \begin{cases} N(2n+i-3)+1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ N(i-1)+1, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

Thus, $f$ is a one modulo $N$ mean labeling. Hence, the graph $S'(P_{2n})$ is a one modulo $N$ mean graph for every positive integer $N$ and $n \geq 1$. For example, a one modulo 8 mean labeling of $S'(P_8)$ is shown in Figure 5. □

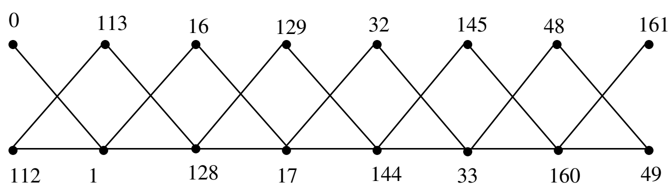



Figure 5. A one modulo 8 mean labeling of $S'(P_8)$.

**Theorem 2.9.** *The $H$-graph $G$ is a one modulo $N$ mean graph for every positive integer $N$.*

*Proof.* Let $v_1, v_2, v_3, \ldots, v_n$ and $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the $H$-graph $G$.

Define $f : V(G) \to \{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1 = 2N(n-1)+1\}$ as follows:
For $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} N(i-1), & i \text{ is odd} \\ N(i-2)+1, & i \text{ is even} \end{cases}$$

$$f(u_i) = \begin{cases} N(n+i-2)+1, & i \text{ is odd and } n \text{ is odd} \\ N(n+i-1), & i \text{ is even and } n \text{ is odd} \\ N(n+i-1), & i \text{ is odd and } n \text{ is even} \\ N(n+i-2)+1, & i \text{ is even and } n \text{ is even.} \end{cases}$$

The induced edge labeling $f^*$ is obtained as follows:

$$f^*(v_i v_{i+1}) = N(i-1)+1, \quad 1 \leq i \leq n-1$$
$$f^*(u_i u_{i+1}) = N(n+i-1)+1, \quad 1 \leq i \leq n-1$$
$$f^*\left(v_{\frac{n+1}{2}} u_{\frac{n+1}{2}}\right) = N(n-1)+1 \text{ if } n \text{ is odd}$$
$$f^*\left(v_{\frac{n}{2}+1} u_{\frac{n}{2}}\right) = N(n-1)+1 \text{ if } n \text{ is even.}$$

Thus, $f$ is a one modulo $N$ mean labeling. Hence the $H$-graph $G$ is a one modulo $N$ mean graph for every positive integer $N$.

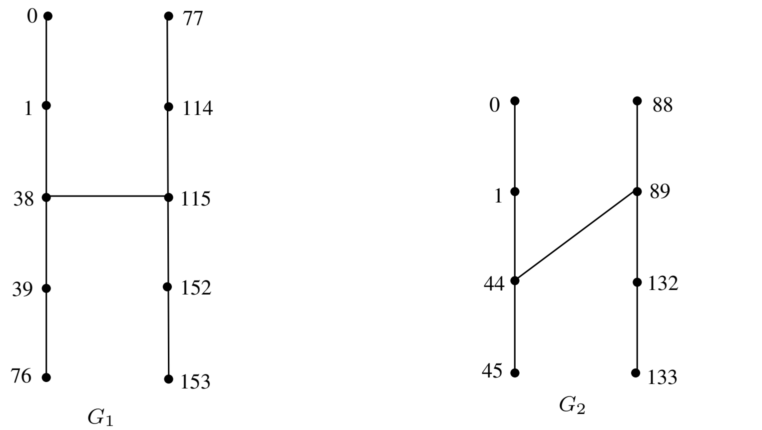
For example, a one modulo 19 mean labeling of $G_1$ and a one modulo 22 mean labeling of $G_2$ are shown in Figure 6. □

**Theorem 2.10.** *For a $H$-graph $G$, $G \odot K_1$ is a one modulo $N$ mean graph for every positive integer $N$.*

*Proof.* Let $v_1, v_2, v_3, \ldots, v_n$ and $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the $H$-graph $G$ and let $v_i', u_i' (1 \leq i \leq n)$ be the vertices of $G \odot K_1$.

Let $V(G \odot K_1) = V(G) \cup \{v_1', v_2', \ldots, v_n'\} \cup \{u_1', u_2', \ldots, u_n'\}$
and $E(G \odot K_1) = E(G) \cup \{v_i v_i', u_i u_i' : 1 \leq i \leq n\}$

Define $f : V(G \odot K_1) \to \{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1 = 2N(2n-1)+1\}$ as follows:




A one modulo 19 mean labeling of $G_1$ A one modulo 22 mean labeling of $G_2$

Figure 6.

For $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} 2N(i-1)+1, & i \text{ is odd} \\ 2N(i-1), & i \text{ is even} \end{cases}$$

$$f(v_i') = \begin{cases} 2N(i-1), & i \text{ is odd} \\ 2N(i-1)+1, & i \text{ is even} \end{cases}$$

For $1 \leq i \leq n$ and $n$ is odd,

$$f(u_i) = \begin{cases} 2N(n+i-1), & i \text{ is odd} \\ 2N(n+i-1)+1, & i \text{ is even} \end{cases}$$

$$f(u_i') = \begin{cases} 2N(n+i-1)+1, & i \text{ is odd} \\ 2N(n+i-1), & i \text{ is even} \end{cases}$$

For $1 \leq i \leq n$ and $n$ is even,

$$f(u_i) = \begin{cases} 2N(n+i-1)+1, & i \text{ is odd} \\ 2N(n+i-1), & i \text{ is even} \end{cases}$$

$$f(u_i') = \begin{cases} 2N(n+i-1), & i \text{ is odd} \\ 2N(n+i-1)+1, & i \text{ is even.} \end{cases}$$

The induced edge labeling $f^*$ is obtained as follows:

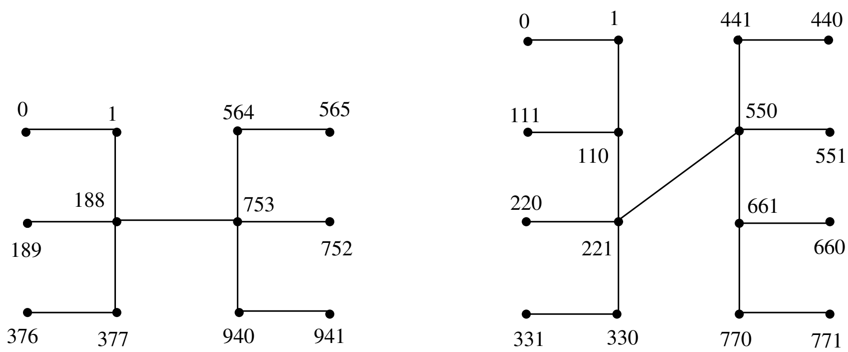
$$f^*(v_iv_i') = 2N(i-1)+1, \quad 1 \leq i \leq n$$

$$f^*(u_iu_i') = 2N(n+i-1)+1, \quad 1 \le i \le n$$
$$f^*(v_iv_{i+1}) = N(2i-1)+1, \quad 1 \le i \le n-1$$
$$f^*(u_iu_{i+1}) = N(2n+2i-1)+1, \quad 1 \le i \le n-1$$
$$f^*\left(v_{\frac{n+1}{2}}u_{\frac{n+1}{2}}\right) = N(2n-1)+1 \quad \text{if } n \text{ is odd}$$

$$f^*\left(v_{\frac{n}{2}+1}u_{\frac{n}{2}}\right) = N(2n-1)+1 \quad \text{if } n \text{ is even.}$$

Thus, $f$ is a one modulo $N$ mean labeling. Hence, the graph $G \odot K_1$ is a one modulo $N$ mean graph for every positive integer $N$.

For example, a one modulo 94 mean labeling of $G_1 \odot K_1$ and one modulo 55 mean labeling of $G_2 \odot K_1$ are shown in Figure 7. □




$G_1 \odot K_1$ $G_2 \odot K_1$

Figure 7.

**Theorem 2.11.** *The graph $P_n \odot mK_1$ is a one modulo $N$ mean graph if $n$ is even and $m \ge 1$.*

*Proof.* Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the path $P_n$ and let $u_i^j, 1 \le j \le m$ be the vertices attached to the vertex $u_i, 1 \le i \le n$ of $P_n$.

Define $f : V(P_n \odot mK_1) \to \{0, 1, N, N+1, 2N, 2N+1, \ldots, N(q-1), N(q-1)+1 = N(n(m+1)-2)+1\}$ as follows:

For every positive integer $N$,

$$f(u_i) = \begin{cases} N(i-1)(m+1), & 1 \le i \le n \text{ and } i \text{ is odd,} \\ N[i(m+1)-2]+1, & 1 \le i \le n \text{ and } i \text{ is even,} \end{cases}$$

$$f\left(u_i^j\right) = \begin{cases} N(m+1)(i-1)+2N(j-1)+1, & 1 \le i \le n, 1 \le j \le m \text{ and } i \text{ is odd,} \\ N(m+1)(i-2)+2Nj, & 1 \le i \le n, 1 \le j \le m \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labeling $f^*$ is obtained as follows:

$$f^*(u_i u_{i+1}) = N[(m+1)i-1]+1, \quad 1 \le i \le n-1$$
$$f^*\left(u_i u_i^j\right) = N(m+1)(i-1)+N(j-1)+1, \quad 1 \le i \le n \text{ and } 1 \le j \le m.$$

Thus, $f$ is a one modulo $N$ mean labeling. Hence, $P_n \odot mK_1$ is a one modulo $N$ mean graph if $n$ is even.

For example, a one modulo 7 mean labeling of $P_4 \odot 5K_1$ is shown in Figure 8. □

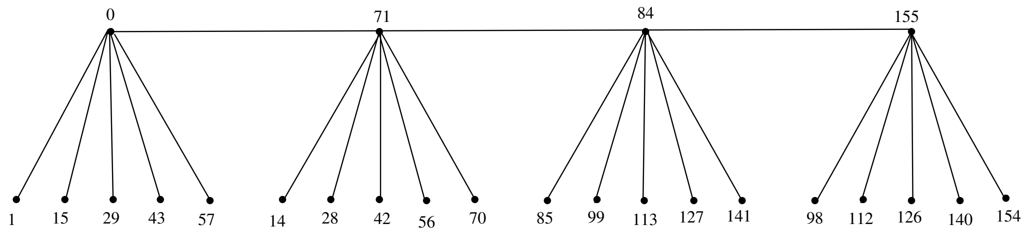



Figure 8. A one modulo 7 mean labeling of $P_4 \odot 5K_1$.

**Corollary 2.12.** *The Bistar $B_{n,n}$ are one modulo $N$ mean graph.*

*Proof.* It follows from Theorem 2.11. □

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