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QUATERNION-VALUED GENERALIZED METRIC SPACES AND m-QUATERNION-VALUED G-ISOMETRIC MAPPINGS

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Abstract: In this work, we introduce and study the notion of quaternion-generalized metric spaces which they called Quaternion-valued G-metric spaces. This is a generalization of both real and complex generalized metric spaces. In the second part, we study a new class of mappings on a quaternion-valued generalized metric space. We call these mappings in this class,m-quaternion-valued G-isometric mappings. This generalized the class of m-isometric mappings in metric space introduced by T. Bermúdes et al. in [4].

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1. Introduction

Many authors generalized and extended the notion of a metric spaces to complex metric spaces and quaternion metric spaces. The introduction of the concept of generalized metric spaces (or G-metric spaces) by Mustafa and Sims in [11, 12] yielded a flow of papers generalizing this concept to complex generalized metric spaces. In 2011, Azam et al. [3] introduced the notion of complex valued metric space which is a generalization of the classical metric space. They established some fixed point results for mappings satisfying a rational inequality. After that several authors studied many common fixed point results on complex-valued metric spaces (see [1, 5, 6]).

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In this framework, we are interested to introducing a new concept of generalized metric space, this is the quaternion generalized metric spaces on the field of quaternion numbers. The contents of the paper are the following. In Section 1, we give notation and results about the concepts of real-valued and complex-valued G-metric spaces. In Section 2, we introduce the concept of quaternion-valued G-metric spaces and study some of their basic properties. In Section 3, we give a natural extension of the definition of $(m.q)$-isometric mappings in metric spaces to the m-quaternion-valued G-isometric mappings in quaternion-valued G-metric spaces.

2. Complex-Valued Metric Spaces and Complex-valued G-Metric Spaces

Definition 2.1. ([12]) Let E be an non-empty set and let $G_{\mathbb{R}} : E \times E \times E \longrightarrow \mathbb{R}_+$ be a function satisfying the following conditions

1. $G_{\mathbb{R}}(x, y, z) = 0$ if $x = y = z$
2. $0 < G_{\mathbb{R}}(x, x, y)$ for all $x, y \in E$ with $x \neq y$
3. $G_{\mathbb{R}}(x, x, y) \leq G_{\mathbb{R}}(x, y, z)$ for all $x, y, z \in E$ with $y \neq z$.
4. $G_{\mathbb{R}}(x, y, z) = G_{\mathbb{R}}(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables)
5. $G_{\mathbb{R}}(x, y, z) \leq G_{\mathbb{R}}(x, a, a) + G(a, y, z)$ (for all $x, y, z, a \in E$. (rectangle inequality)

   Then the function $G_{\mathbb{R}}$ is called a real- valued generalized metric or, more specifically, a G-metric on E and the pair $(E, G_{\mathbb{R}})$ is called a real G-metric space.

In [3] the authors introduced the concept of complex valued metric space. They define a partial order $\preccurlyeq$ over the set of complex numbers $\mathbb{C}$ as follows: let $z_1, z_2 \in \mathbb{C}$

$$z_1 \preccurlyeq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2),$$

or equivalently

$$z_1 \preccurlyeq z_2$$

if one of the following conditions is satisfied:

$$(h_1)\quad Re(z_1) = Re(z_2);\quad Im(z_1) < Im(z_2),$$

$$(h_2)\quad Re(z_1) < Re(z_2);\quad Im(z_1) = Im(z_2),$$

$$(h_3)\quad Re(z_1) < Re(z_2);\quad Im(z_1) < Im(z_2),$$

$$(h_4)\quad Re(z_1) = Re(z_2);\quad Im(z_1) = Im(z_2).$$

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(h_1), (h_2)$, and (h_3) is satisfied and we will write $z_1 \prec z_2$ if only $(h3)$ is satisfied. In [3], Azam et al. introduced and studied complex valued metric spaces and established some fixed point results for maps satisfying a rational inequality. The idea of complex valued metric spaces is simply to replace $\mathbb{R}$ with the usual order by $\mathbb{C}$ with certain order.

Definition 2.2. (see [1]). Let E be a nonempty set. A function $d_{\mathbb{C}} : E \times E \to \mathbb{C}$ is called a complex-valued metric on E, if it satisfies the following conditions:
(b_1) $0 \precsim d_{\mathbb{C}}(x, y)$ for all $x, y \in E$ and $d_{\mathbb{C}}(x, y) = 0$, if and only if $x = y$,
(b_2) $d_{\mathbb{C}}(x, y) = d_{\mathbb{C}}(y, x)$; for all $x, y \in E$,
(b_3) $d_{\mathbb{C}}(x, y) \precsim d_{\mathbb{C}}(x, z) + d_{\mathbb{C}}(z, y)$ for all $x, y, z \in E$.

Here, the pair $(E, d_{\mathbb{C}})$ is called a complex-valued metric space.

Definition 2.3. ([10]) Let E be an non-empty set and let $G_{\mathbb{C}} : E \times E \times E \to \mathbb{C}$ be a function satisfying the following conditions

1. $G_{\mathbb{C}}(x, y, z) = 0$ if $x = y = z$
2. $0 \prec G_{\mathbb{C}}(x, x, y)$ for all $x, y \in E$ with $x \neq y$
3. $G_{\mathbb{C}}(x, x, y) \precsim G(x, y, z)$ for all $x, y, z \in E$ with $y \neq z$.
4. $G_{\mathbb{C}}(x, y, z) = G_{\mathbb{C}}(x, z, y) = G_{\mathbb{C}}(y, z, x) = ...$ (symmetry in all three variables)
5. $G_{\mathbb{C}}(x, y, z) \precsim G_{\mathbb{C}}(x, a, a) + G(a, y, z)$ (for all $x, y, z, a \in E$. (rectangle inequality)

Then the function $G_{\mathbb{C}}$ is called a complex-valued generalized metric or, more specifically, a complex-valued G-metric on E and the pair $(E, G_{\mathbb{C}})$ is called a complex valued G-metric space.

2.1. Quaternion-Valued Metric Space

We recall some basic notions about quaternionic spaces (see for example [9, 13]). The space of quaternions $\mathbf{H}$ is the four dimensional real algebra with unity. We denote by $\mathbf{0_H}$ the null element of $\mathbf{H}$ and by $\mathbf{1_H}$ the multiplicative identity of $\mathbf{H}$. The space $\mathbf{H}$ includes three so-called imaginary units, which we indicate by i, j, k. By definition, they satisfy:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ kj = -jk = i \text{ and } ki = -ik = j.$$

The elements $1, i, j, k$ are assumed to form a real vector basis of $\mathbf{H}$, so that any $q \in \mathbf{H}$ in the form $q = x_0 + x_1 i + x_2 j + x_3 k$; where x_0, x_1, x_2, and x_3 belong to $\mathbb{R}$.

Given $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbf{H}$, we recall that:

- $\overline{q} := x_0 - x_1 i - x_2 j - x_3 k$ is the conjugate quaternion of q.
- $|q| := \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \in \mathbb{R}$ is the norm of q.
- $Re(q) := \frac{1}{2}(q + \overline{q}) = x_0 \in \mathbb{R}$ is the real part of q.
- $Im(q) := \frac{1}{2}(q - \overline{q}) = x_1 i + x_2 j + x_3 k$ is the imaginary part of q.

The element $q \in \mathbf{H}$ is said to be real if $q = Re(q)$. It is easy to see that q is real if if and only if $q = \overline{q}$. If $\overline{q} = -q$ or, equivalently, $q = Im(q)$, then q is said to be imaginary.

We consider two quaternions:

$$q_1 = x_0 + x_1 i + x_2 j + x_3 k \text{ and } q_2 = y_0 + y_1 i + y_2 j + y_3 k.$$

The sum and the difference of q_1 and q_2 are then defined as

$$q_1 + q_2 = (x_0 + y_0) + i(x_1 + y_1) + j(x_2 + y_2) + k(x_3 + y_3)$$

and

$$q_1 - q_2 = (x_0 - y_0) + i(x_1 - y_1) + j(x_2 - y_2) + k(x_3 - y_3).$$

Multiplication by a scalar β gives

$$\beta q_1 = \beta x_0 + \beta x_1 i + \beta x_2 j + \beta x_3 k$$

The authors in [7] have defined a partial order on $\mathbf{H}$ as follows:

$$q_1 \preccurlyeq q_2 \text{ if and only if } Re(q_1) \le Re(q_2),\ Im_s(q_1) \le Im_s(q_2),$$

$$q_1, q_2 \in \mathbf{H};\ s = i, j, k; \text{where } Im_i = x_1,\ Im_j = x_2, Im_k = x_3.$$

In [7] it was observed that $q_1 \preccurlyeq q_2$. if one of the following conditions is satisfied:

(i)	$Re(q_1) = Re(q_2);\ Im_{s_1}(q_1) = Im_{s_1}(q_2);$ where $s_1 = j, k;\ Im_i(q_1) = Im_i(q_2);$
(ii)	$Re(q_1) = Re(q_2);\ Im_{s_2}(q_1) = Im_{s_2}(q_2);$ where $s_2 = i, k;\ Im_j(q_1) = Im_j(q_2);$
(iii)	$Re(q_1) = Re(q_2);\ Im_{s_3}(q_1) = Im_{s_3}(q_2);$ where $s_3 = i, j;\ Im_k(q_1) = Im_k(q_2);$
(iv)	$Re(q_1) = Re(q_2);\ Im_{s_1}(q_1) < Im_{s_1}(q_2);\ Im_i(q_1) = Im_i(q_2);$
(v)	$Re(q_1) = Re(q_2);\ Im_{s_2}(q_1) < Im_{s_2}(q_2);\ Im_j(q_1) = Im_j(q_2);$
(vi)	$Re(q_1) = Re(q_2);\ Im_{s_3}(q_1) < Im_{s_3}(q_2);\ Im_k(q_1) = Im_k(q_2);$
(vii)	$Re(q_1) = Re(q_2);\ Im_s(q_1) < Im_s(q_2);$
$(viii)$	$Re(q_1) < Re(q_2);\ Im_s(q_1) = Im_s(q_2);$;
(iX)	$Re(q_1) < Re(q_2);\ Im_{s_1}(q_1) = Im_{s_1}(q_2);$ where $s_1 = j, k;\ Im_i(q_1) = Im_i(q_2);$
(X)	$Re(q_1) < Re(q_2);\ Im_{s_2}(q_1) = Im_{s_2}(q_2);\ ;\ Im_j(q_1) < Im_j(q_2);$
(Xi)	$Re(q_1) < Re(q_2);\ Im_{s_3}(q_1) = Im_{s_3}(q_2);\ ;\ Im_k(q_1) < Im_k(q_2);$
(Xii)	$Re(q_1) < Re(q_2);\ Im_{s_1}(q_1) < Im_{s_1}(q_2);\ Im_i(q_1) = Im_i(q_2);$
$(Xiii)$	$Re(q_1) < Re(q_2);\ Im_{s_2}(q_1) < Im_{s_2}(q_2);\ Im_j(q_1) = Im_j(q_2);$
(Xiv)	$Re(q_1) < Re(q_2);\ Im_{s_3}(q_1) < Im_{s_3}(q_2);\ Im_k(q_1) = Im_k(q_2);$
(Xv)	$Re(q_1) < Re(q_2);\ Im_s(q_1) < Im_s(q_2)$
(Xvi)	$Re(q_1) = Re(q_2);\ Im_s(q_1) = Im_s(q_2).$

Remark 2.1. In particular, we note that $q_1 \precsim q_2$ if $q_1 \neq q_2$ and one from (i) to (Xvi) is satisfied. Also, we note that $q_1 \prec q_2$ if only (Xv) is satisfied. It should be remarked that

$$q_1 \preccurlyeq q_2 \Rightarrow |q_1| \le |q_2|.$$

Remark 2.2. The following proprieties hold:
(1) If $a, b \in \mathbb{R}$ with $a \le b$, then $aq \preceq bq$ for all $q \in \mathbb{H}$.
(2) If $0 \preceq q_1 \precnsim q_2$, then $|q_1| < |q_2|$.
(3) If $q_1 \preceq q_2$ and $q_2 \prec q_3$, then $q_1 \prec q_3$

Very recently, Ahmed et al.([7]) introduced the notion of quaternion-valued metric as a generalization of complex-valued metric and proved a common fixed point theorem in the context of quaternion-valued metric space.

Definition 2.4. ([6]) Let E be a nonempty set. Suppose that the mapping $d_{\mathbf{H}} : E \times E \to \mathbf{H}$ satisfies

(1) $\mathbf{0_H} \preccurlyeq d_{\mathbf{H}}(x,y)$ for all $x,y \in E$ and $d_{\mathbf{H}}(x,y) = \mathbf{0_H}$, if and only if $x = y$,
(2) $d_{\mathbf{H}}(x,y) = d_{\mathbf{H}}(y,x)$; for all $x,y \in E$,
(3) $d_{\mathbf{H}}(x,y) \preccurlyeq d_{\mathbf{H}}(x,z) + d_{\mathbf{H}}(z,y)$ for all $x,y,z \in E$.

Then $d_{\mathbf{H}}$ is called a quaternion valued metric on E, and $(E, d_{\mathbf{H}})$ is called a quaternion valued metric space.

3. Quaternion-Valued G-Metric Spaces

In this section, we introduce the notion of quaternion-valued G-metric space and study some of basic properties. Our inspiration cames from the papers [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13].

3.1. Basic Definitions and Properties

Definition 3.1. Let E be an non-empty set and let $G_{\mathbf{H}} : E \times E \times E \to \mathbf{H}$ be a function satisfying the following conditions

1. $G_{\mathbf{H}}(x,y,z) = \mathbf{0_H}$ if $x = y = z$.
2. $\mathbf{0_H} \prec G_{\mathbf{H}}(x,x,y)$ for all $x,y \in E$ with $x \neq y$.
3. $G_{\mathbf{H}}(x,x,y) \preccurlyeq G_{\mathbf{H}}(x,y,z)$ for all $x,y,z \in E$ with $y \neq z$.
4. $G_{\mathbf{H}}(x,y,z) = G_{\mathbf{H}}(x,z,y) = G_{\mathbf{H}}(y,z,x) = ...$ (symmetry in all three variables)
5. $G_{\mathbf{H}}(x,y,z) \preccurlyeq G_{\mathbf{H}}(x,a,a) + G_{\mathbf{H}}(a,y,z)$. (for all $x,y,z,a \in E$, (rectangle inequality).

Then the function $G_{\mathbf{H}}$ is called a quaternion -valued generalized metric or, more specifically, a quaternion-valued $G_{\mathbf{H}}$-metric on E and the pair $(E, G_{\mathbf{H}})$ is called a quaternion-valued G-metric space.

Remark 3.1. From the condition 4. we deduce that the value of $G_{\mathbf{H}}(x,y,z)$ is independent of the order of x, y and z, and is usually known as the symmetry of G_H in them.

Remark 3.2. It is clear that every complex-valued G-metric and every real valued G -metric are quaternion-valued G-metric.

Example 3.1. Let $E = \mathbf{H}$ be a set of quaternion number. Define a quaternion valued function $G_{\mathbf{H}} : \mathbf{H} \times \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ by

$$
\begin{aligned}
&G_{\mathbf{H}}(q_1,q_2,q_3)\\
&\quad= |x_0^1-x_0^2|+|x_0^1-x_0^3|+|x_0^2-x_2^3|+i\Big(|x_1^1-x_1^2|+|x_1^1-x_1^3|+|x_1^2-x_1^3|\Big)\\
&+j\Big(|x_2^1-x_2^2|+|x_2^1-x_2^3|+|x_2^2-x_2^3|\Big)+k\Big(|x_3^1-x_3^2|+|x_3^1-x_3^3|+|x_3^2-x_3^3|\Big),
\end{aligned}
$$

where $q_r = x_0^r + x_1^r i + x_2^r j + x_3^r k$ for $r = 1,2,3$. Then $(\mathbf{H}, G_{\mathbf{H}})$ is a quaternion valued G-metric space.

Example 3.2. Let $(E_r, d_{\mathbf{H}}^r)$ be a quaternion-valued metric space for $r = 1,2,...,d$ and consider $E = \prod_{r=1}^{d} E_r$. Define the maps ${\mathbf{G}_{\mathbf{H}}}^s : E\times E\times E \to \mathbf{H}$ by

$$G_{\mathbf{H}}^s(x,y,z) \quad = \quad \sum_{r=1}^{d}\Big(d_{\mathbf{H}}^r(x_r,y_r)+d_{\mathbf{H}}^r(y_r,z_r)+d_{\mathbf{H}}^r(z_r,x_r)\Big)$$

for all $\quad x=(x_r)_{1\le r\le d},\ \ y=(y_r)_{1\le r\le d}$ and $z=(z_r)_{1\le r\le d}\in E.$

A simple computation shows that ${\mathbf{G}_{\mathbf{H}}}^s$ is a quaternion-valid G-metric on E .

Example 3.3. Let E_k be a quaternion-valued metric space with the metric $d_{\mathbf{H}}^k$ for $k = 1,2,...,d$ and consider $E = \prod_{k=1}^{d} E_k$. Define the maps ${\mathbf{G}_{\mathbf{H}}}^m : E\times E\times E\to \mathbf{H}$ by

$${\mathbf{G}_{\mathbf{H}}}^m(x,y,z) \quad = \quad \sum_{j=1}^{d}\Bigg(\max\Bigg(\Big|d_{\mathbf{H}}^j(x_j,y_j)\Big|,\Big|d_{\mathbf{H}}^j(y_j,z_j)\Big|,\Big|d_{\mathbf{H}}^j(z_j,x_j)\Big|\Bigg)$$

for all $\quad x=(x_j)_{1\le j\le d},\ \ y=(y_j)_{1\le j\le d}$ and $z=(z_j)_{1\le j\le d}\in E.$

A simple computation shows that ${\mathbf{G}_{\mathbf{H}}}^m$ is a quaternion-valued G-metric on E.

Proposition 3.1. *Let $(E, G_{\mathbf{H}})$ be a quaternion valued G-metric space. Then for all x,y,z and $a\in E$ the following properties hold.*

(1) $G_{\mathbf{H}}(x,y,z)=0 \Rightarrow x=y=z$

(2) $G_{\mathbf{H}}(x,y,z)\preccurlyeq G_{\mathbf{H}}(x,x,y)+G_{\mathbf{H}}(x,x,z)$

(3) $G_{\mathbf{H}}(x,y,y)\preccurlyeq 2G_{\mathbf{H}}(y,x,x)$

(4) $G_{\mathbf{H}}(x,y,z)\preccurlyeq G_{\mathbf{H}}(x,a,z)+G_{\mathbf{H}}(a,y,z)$

(5) $G_{\mathbf{H}}(x,y,z)\preccurlyeq \frac{2}{3}\big(G_{\mathbf{H}}(x,y,a)+G_{\mathbf{H}}(x,a,z)+G_{\mathbf{H}}(a,y,z)\big)$

(6) $G_{\mathbf{H}}(x,y,z)\preccurlyeq 2\Bigg(G_{\mathbf{H}}(x,x,a)+G(y,y,a)+G_{\mathbf{H}}(z,z,a)\Bigg).$

Proof. (2)

$$\begin{aligned} G_{\mathbf{H}}(x,y,z) = G_{\mathbf{H}}(z,x,y) \precsim G_{\mathbf{H}}(z,x,x) + G_{\mathbf{H}}(x,x,y) \\ = G_{\mathbf{H}}(x,x,z) + G_{\mathbf{H}}(x,x,y). \end{aligned}$$

(3) From the inequality (2)

$$G_{\mathbf{H}}(x,y,y) \precsim G_{\mathbf{H}}(x,x,y) + G_{\mathbf{H}}(x,x,y) = 2G_{\mathbf{H}}(y,x,x).$$

(4)

$$\begin{aligned} G_{\mathbf{H}}(x,y,z) &\precsim G_{\mathbf{H}}(x,a,a) + G(a,y,z) \\ &= G_{\mathbf{H}}(a,a,x) + G_{\mathbf{H}}(a,y,z) \\ &\precsim G_{\mathbf{H}}(a,x,z) + G_{\mathbf{H}}(a,y,z). \end{aligned}$$

(5)

$$\begin{cases} G_{\mathbf{H}}(x,y,z) \precsim G_{\mathbf{H}}(a,y,z) + G_{\mathbf{H}}(x,a,z) \\ \\ G_{\mathbf{H}}(z,x,y) \precsim G_{\mathbf{H}}(z,a,y) + G_{\mathbf{H}}(a,x,y) \\ \\ G_{\mathbf{H}}(y,z,x) \precsim G_{\mathbf{H}}(y,a,x) + G(a,z,x) \end{cases}$$

$$\Longrightarrow 3G_{\mathbf{H}}(x,y,z) \leq 2\big(G_{\mathbf{H}}(x,y,a) + G_{\mathbf{H}}(x,a,z) + G_{\mathbf{H}}(a,y,z)\big).$$

(6)

$$\begin{aligned} G_{\mathbf{H}}(,y,z) &\precsim G_{\mathbf{H}}(x,a,a) + G_{\mathbf{H}}(a,y,z) \\ &= G_{\mathbf{H}}(x,a,a) + G_{\mathbf{H}}(y,z,a) \\ &\precsim G_{\mathbf{H}}(x,a,a) + G(y,a,a) + G_{\mathbf{H}}(a,z,a) \\ &\precsim G_{\mathbf{H}}(x,a,a) + G_{\mathbf{H}}(y,a,a) + G_{\mathbf{H}}(z,a,a) \\ &\precsim 2G_{\mathbf{H}}(x,x,a) + 2G_{\mathbf{H}}(y,y,a) + 2G_{\mathbf{H}}(z,z,a). \end{aligned}$$

□

Definition 3.2. Let $(E, G_{\mathbf{H}})$ be a quaternion-valued G-metric space

(1) Any point $x \in E$ is called interior of a set $\mathcal{A} \subset E$, if there exists, $q_0 \in \mathbf{H}:\ 0 \prec q_0$ such that

$$\mathbb{B}_{\mathbf{H}}(x_0, q_0) := \{\ y \in E \ /\ G_{\mathbb{H}}(x_0, y, y) \prec q_0\ \} \subset \mathcal{A}.$$

(2) A subset $\mathcal{A}$ in E is called open whenever each point of $\mathcal{A}$ is an interior point of $\mathcal{A}$.

(3) Any point $x \in E$, is said a limit point of $\mathcal{A} \subset E$ if for every $0 \prec q \in \mathbf{H}$, we have

$$\mathbb{B}_{\mathbf{H}}(x, q) \cap (\mathcal{A} - \{x\}) \neq \emptyset.$$

(4) Any subset $\mathcal{A}$ of E is said to be a closed if each limit point of $\mathcal{A}$ belongs to $\mathcal{A}$.

The family $\mathcal{F}_{\mathbf{H}} = \{ \mathbb{B}_{\mathbf{H}}(x, q), \quad x \in E, \ 0 \prec q \in \mathbf{H}\}$ is a sub-base for a topology on E. We denote this quaternion topology by $\tau_{\mathbf{H}}$.

3.2. Convergence, Continuity and Completness in Quaternion-Valued G-Metric Spaces

Definition 3.3. Let $(E, \mathbf{G_H})$ be a quaternion-valued G-metric space, and let $(x_n)_n$ be a sequence of points of E.

1. We say that $(x_n)_n$ is quaternion valued G-convergent to $x \in E$ if

$$\lim_{n,m\to\infty} \mathbf{G_H}(x_n, x_m, x) = 0;$$

   or equivalently if

$$\forall\, q_0 \in \mathbf{H} : 0 \prec q_0, \exists\, n_0 \in \mathbb{N} \ / \ G_{\mathbf{H}}(x, x_n, x_m) \prec q_0, \ \ \forall\, n, m \geq n_0.$$

   We call x the limit of the sequence x_n and write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.

2. The sequence $(x_n)_n$ is said to be a quaternion-valued G-Cauchy sequence if, for every $q_0 \in \mathbf{H}$, $0 \prec q_0$, there is a positive integer k such that

$$\mathbf{G_H}(x_n, x_m, x_l) \prec q_0 \quad \text{for all} \ \ n, m, l \geq k.$$

3. A quaternion-valued G-metric space $(E, G_{\mathbf{H}})$ is said to be quaternion-valued complete if every quaternion-valued G-Cauchy sequence is quaternion-valued G-convergent in $(E, G_{\mathbf{H}})$.

Definition 3.4. Let $(E, G_{\mathbf{H}})$ and $(E', G'_{\mathbf{H}})$ be two G-metric spaces, and let
$g : (E, G_{\mathbf{H}}) \to (E', G'_{\mathbf{H}})$ be a function.We said that g is quaternion-valued G-continuous at a point $x \in E$ if $g^{-1}\big(\mathbb{B}_{G'_{\mathbf{H}}}(x, q)\big) \in \tau_{\mathbf{H}}(\mathbf{G_H})$ for all $0 \prec q \in \mathbf{H}$. We say g is quaternion valued G-continuous if it quaternion-valued G-continuous at all points of E, that is, continuous as a function from E with $\tau_{\mathbf{H}}(\mathbf{G_H})$-topology to E with $\tau(\mathbf{G_H}')$- topology.

Proposition 3.2. *Let $(E, G_{\mathbf{H}})$ ba a G-metric space and let $(x_n)_n$ be a sequence of E. If $x_n \to x$ in $(E, G_{\mathbf{H}})$ and $x_n \to y$ in $(E, G_{\mathbf{H}})$, then $x = y$.*

Proof. Assume that $G_{\mathbf{H}}(x, x, x_n) \to 0$ and $G_{\mathbf{H}}(y, y, x_n) \to 0$ as $n \to \infty$. From (6) of Proposition 2.1 it follows that

$$G_{\mathbf{H}}(x, y, y) \preccurlyeq 2\big(G_{\mathbf{H}}(x, x, x_n) + G_{\mathbf{H}}(y, y, x_n) + G_{\mathbf{H}}(y, y, x_n)\big).$$

If $n \to \infty$ we obtain that $G_{\mathbf{H}}(x, y, y) = 0$ and hence $x = y$ as desired. □

Proposition 3.3. *Let $(E, G_{\mathbf{H}})$ be a quaternion-valued G-metric space and $(x_n)_n \subset E$ be a sequence. Then $(x_n)_n$ is a quaternion-valued G-convergent to $x \in E$ if and only if*

$$|G_{\mathbf{H}}(x, x_n, x_m)| \to 0 \text{ as } n, m \to \infty.$$

Proof. Assume that $(x_n)_n$ is quaternion G-convergent to x.For a given $\epsilon > 0$, let
$q_0 = \frac{\epsilon}{2} + \frac{\epsilon}{2}i + \frac{\epsilon}{2}j + \frac{\epsilon}{2}k$. We have that $0_{\mathbf{H}} \prec q_0 \in \mathbf{H}$ and there exists $n_0 > 0$ such that $G_{\mathbf{H}}(x, x_n, x_m) \prec q_0, \quad \forall\ n, m \geq n_0$. Therefore $|G_{\mathbf{H}}(x, x_n, x_m)| < |q_0| = \epsilon$.

Consequently, $|G_{\mathbf{H}}(x, x_n, x_m)| \to 0$ as $n, m \to \infty$.

Conversely, assume that $|G_{\mathbf{H}}(x, x_n, x_m)| \to 0$ as $n, m \to \infty$. Let $q_0 \in \mathbf{H}$: $0_{\mathbf{H}} \prec q_0$. There exists $\alpha > 0$ such that for $q \in \mathbf{H}$ such that $|q| < \alpha \Rightarrow q \prec q_0$.

Since $|G_{\mathbf{H}}(x, x_n, x_m)| \to 0$ as $n, m \to \infty$, we have for $\alpha > 0$ there exists $n_0 \in \mathbb{N}$ for all $n, m \geq n_0 : |G_{\mathbf{H}}(x, x_n, x_m)| < \alpha$. This implies that

$$G_{\mathbf{H}}(x, x_n, x_m) \prec q_0.$$

This means that $(x_n)_n$ is quaternion G-convergent as required. □

Theorem 3.1. *Let $(E, G_{\mathbf{H}})$ be a quaternion-valued G-metric space, $(x_n)_n \subset E$ and $x \in E$. The following properties are equivalent*

(1) $(x_n)_n$ *is quaternion-valued G-convergent to x.*
(2) $|\mathbf{G}_{\mathbf{H}}(x_n, x_n, x)| \to 0$ *as* $n \to \infty$.
(3) $|G_{\mathbf{H}}(x_n, x, x)| \to 0$ *as* $n \to \infty$.
(4) $|G_{\mathbf{H}}(x_m, x_n, x)| \to 0$ *as* $n \to \infty$.

Proof. (1) $\Rightarrow$ (2) obvious.

(2) $\Rightarrow$ (3) follows form (3) of Proposition 2.1 (3) $\Rightarrow$ (4) follows form (2) of Proposition 2.1.

(4) $\Rightarrow$ (1) obvious. □

Proposition 3.4. *Let $(E, G_{\mathbf{H}})$ ba a G-metric space and let $(x_n)_n$ be a sequence of E. If $x_n \to x$ in $(E, G_{\mathbf{H}})$ and $x_n \to y$ in $(E, G_{\mathbf{H}})$, then $x = y$.*

Proof. Assume that $G_{\mathbf{H}}(x, x, x_n) \to 0$ and $G_{\mathbf{H}}(y, y, x_n) \to 0$ as $n \to \infty$. From (6) of Proposition 2.1 it follows that

$$G_{\mathbf{H}}(x, y, y) \preccurlyeq 2\big(G_{\mathbf{H}}(x, x, x_n) + G_{\mathbf{H}}(y, y, x_n) + G_{\mathbf{H}}(y, y, x_n)\big).$$

If $n \to \infty$ we obtain that $G_{\mathbf{H}}(x, y, y) = 0$ and hence $x = y$ as desired. □

Proposition 3.5. *Let $(E, G_{\mathbf{H}})$ be a quaternion-valued G-metric space and $(x_n)_n \subset E$ be a sequence. Then $(x_n)_n$ is a quaternion-valued G-Cauchy sequence if and only if*

$$|G_{\mathbf{H}}(x_l, x_n, x_m)| \to 0 \quad \textit{as} \quad l, n, m \to \infty.$$

Proof. Assume that $(x_n)_n$ is quaternion-valued G-Cauchy sequence. For a given $\epsilon > 0$, let $q_0 = \frac{\epsilon}{2} + \frac{\epsilon}{2}i + \frac{\epsilon}{2}j + \frac{\epsilon}{2}k$. We have that $0_{\mathbf{H}} \prec q_0 \in \mathbf{H}$ and there exists $n_0 > 0$ such that

$$G_{\mathbf{H}}(x_l, x_n, x_m) \prec q_0, \quad \forall\ n, m, l \geq n_0.$$

Therefore $|G_{\mathbf{H}}(x_l, x_n, x_m)| < |q_0| = \epsilon$. Consequently, $|G_{\mathbf{H}}(x, x_n, x_m)| \to 0$ as $n, m \to \infty$.

Conversely, assume that $|G_{\mathbf{H}}(x_l, x_n, x_m)| \to 0$ as $n, m, l \to \infty$. Let $q_0 \in \mathbf{H}$: $0_{\mathbf{H}} \prec q_0$. There exists $\alpha > 0$ such that for $q \in \mathbf{H}$ such that $|q| < \alpha \Rightarrow q \prec q_0$.

Since $|G_{\mathbf{H}}(x, x_n, x_m)| \to 0$ as $n, m \to \infty$, we have for $\alpha > 0$ there exists $n_0 \in \mathbb{N}$ for all $n, m \geq n_0 : |G_{\mathbf{H}}(x, x_n, x_m)| < \alpha$. This implies that

$$G_{\mathbf{H}}(x_l, x_n, x_m) \prec q_0.$$

This means that $(x_n)_n$ is quaternion-valued G-Cauchy sequence as required. □

4. m-Quaternion-Valued G-Isometric Mappings

Inspired by the above definitions of m-isometric mapping on metric space and quaternion-valued **G**-metric space, we introduce the class of m-isometric mappings on quaternion-valued G-metric spaces which generalizes the class of m isometric mappings in metric spaces. We extend many results proved in [4] to the case of quaternion-valued G-metric spaces.

Let $(E, G_{\mathbf{H}})$ be a quaternion-valued G- metric space where $\mathbf{H}$ is the skew field of quaternion number q that is,

$$\mathbf{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k,\ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4\}.$$

We define the map

$$\mathcal{L}^T_{(l,\,\mathbf{H})}(.,.,.) = \mathcal{L}^T_l(.,.,.) : E \times E \times E \to \mathbf{H}$$

by

$$\mathcal{L}^T_{(l,\,\mathbf{H})}(x,y,z) := \sum_{r=0}^{l} (-1)^{l-r} \binom{l}{r} G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big),\ \forall\ \ x,y,z \in E. \tag{4.1}$$

Definition 4.1. ([4]) Let E be a metric space. A map $T : E \to E$ is called an (m,q)-isometry ($m \geq 1$ integer, $q > 0$ real) if, for all $x, y \in E$

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} d\big(T^r x, T^r y\big)^q = 0. \tag{4.2}$$

For $m \geq 2$. T is a strict (m,q)-isometry if it is (m,q)-isometry, but not an $(m-1,q)$-isometry.

Remark 4.1. For $m = 1$ the equation (3.2) is equivalent to $d(Tx, Ty) = d(x,y)$.

Definition 4.2. Let $(E, G_{\mathbb{R}})$ be a real-valued G-metric space. A map $T : E \to E$ is called an (m,q)-G-isometry for some positive integer m and $q > 0$ if, for all $x, , y, z \in E$

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbb{R}}\big(T^r x, T^r y, T^r z\big)^q = 0., \tag{4.3}$$

which is equivalent to the condition $\mathcal{L}^T_{(m,\,\mathbb{R})}(x,y,z) = 0$ for all $x, y, z \in E$.

For $m \geq 2$, T is a strict (m,q)-G-isometry if it is an (m,q)-G-isometry, but is not an $(m-1,q)$-G-isometry.

Definition 4.3. Let $(E, G_{\mathbb{C}})$ be a complex-valued G-metric space. A map $T : E \to E$ is called an m-complex-valued G-isometric for some positive integer m if, for all $x, , y, z \in E$

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbb{C}}\big(T^r x, T^r y, T^r z\big) = \mathbf{0}_{\mathbb{C}}, \tag{4.4}$$

which is equivalent to the condition $\mathcal{L}^T_{(m,\,\mathbb{C})}(x, y, z) = \mathbf{0}_{\mathbb{C}}$ for all $x, y, z \in E$.

For $m \geq 2$, T is a strict m-complex G isometry if it is an m-complex-G-isometry, but is not an $(m-1)$-complex-G-isometry.

Definition 4.4. Let $(E, G_{\mathbf{H}})$ be a quaternion-valued G-metric space. A map $T : E \to E$ is called an m-quaternion-valued G-isometric map for some positive integer m if, for all $x, , y, z \in E$

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big) = \mathbf{0}_{\mathbf{H}}, \tag{4.5}$$

which is equivalent to the condition $\mathcal{L}^T_{(m,\,\mathbb{H})}(x, y, z) = \mathbf{0}_{\mathbf{H}}$ for all $x, y, z \in E$.

For $m \geq 2$, T is a strict m-quaternion-G-isometry if it is an m-quaternion-G-isometry, but is not an $(m-1)$-quaternion-G-isometry.

Example 4.1. Let $(E, G_{\mathbf{H}})$ be quaternion-valued metric space and define the map $T : E \to E$ by $Tx = x$. It is easy to see that T is an m-quaternion-valued G-isometry for all $m \geq 1$. Since

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{k} G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big) = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{k} G_{\mathbf{H}}\big(x, y, z\big) = \mathbf{0}_{\mathbf{H}}.$$

Example 4.2. Let $(\mathbf{H}, G_{\mathbf{H}})$ be the quaternion-valued G-metric space given in Example 2.1. Consider the mapping $T : \mathbf{H} \to \mathbf{H}$ defined by $Tq = q+1$.

A simple calculation shows that

$$G_{\mathbf{H}}\big(T^2 q_1, T^2 q_2, T^2 q_3\big) - 2G_{\mathbf{H}}\big(Tq_1, Tq_2, Tq_3\big) + G_{\mathbf{H}}\big(q_1, q_2, q_3\big) = 0_{\mathbf{H}}.$$

Hence, T is an 2-quaternion-valued G-isometric mapping.

Example 4.3. If $(E, d_{\mathbb{R}})$ be a real metric space and let $T : E \to E$ be an $(m, 1)$-isometric mapping on $(E, d_{\mathbb{R}})$.Then T is an m-quaternion-valued G-isometric on $(X, G_{\mathbf{H}})$ where $G_{\mathbf{H}}$ is a quaternion-valued metric given by $G_{\mathbf{H}} : E \times E \times E \to \mathbf{H}$:

$$G_{\mathbf{H}}(x, y, z) = d_{\mathbb{R}}(x, y) + i d_{\mathbb{R}}(x, z) + j d_{\mathbb{R}}(y, z)$$

In fact

$$\begin{aligned}
&\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big) \\
= \;&\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{k} d_{\mathbb{R}}(T^r x, T^r y) + i \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} d_{\mathbb{R}}(T^r x, T^r z)
\end{aligned}$$

$$
\begin{aligned}
& +j\sum_{k=0}^{m}(-1)^{m-r}\binom{m}{r}d_{\mathbb{R}}(T^r y, T^r z) \\
= \quad & 0
\end{aligned}
$$

Proposition 4.1. *Let $(E, \mathbf{G_H})$ be a quaternion-valued G-metric space and $T : E \to E$ is a map. For positive integer $p \geq 1$ and $x, y, z \in E$, the following identity holds*

$$\mathcal{L}_{p+1}^T(x, y, z) = \mathcal{L}_p^T(Tx, Ty, Tz) - \mathcal{L}_p^T(\, x, y, z). \tag{4.6}$$

In particular, if T is an m-quaternion-valued G-isometry then T is an n-quaternion-valued G-isometry for $n \geq m$.

Proof. By the standard formula $\binom{p+1}{r} = \binom{p}{r} + \binom{p}{r-1}$ for binomial coefficients we have the equalities

$$
\begin{aligned}
& \mathcal{L}_{p+1}^T(x, y, z) \\
= \quad & \sum_{r=0}^{p+1}(-1)^{p+1-r}\binom{p+1}{r}G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big) \\
= \quad & (-1)^{p+1}G_{\mathbf{H}}\big(x, y, z\big) + \sum_{r=1}^{p}(-1)^{p+1-r}\binom{p+1}{r}G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big) \\
& +G_{\mathbf{H}}\big(T^{p+1}x, T^{p+1}y, T^{p+1}z\big) \\
= \quad & (-1)^{p+1}G_{\mathbf{H}}\big(x, y, z\big) + \sum_{r=1}^{p}(-1)^{p+1-r}\left(\binom{p}{r} + \binom{p}{r-1}\right)G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big) + \\
& +G_{\mathbf{H}}\big(T^{p+1}x, T^{p+1}y, T^{p+1}z\big) \\
= \quad & -\sum_{r=0}^{p}(-1)^{p-r}\binom{p}{r}G_{\mathbf{H}}(x, y, z) + \sum_{r=0}^{p}(-1)^{p-r}\binom{p}{r}G_{\mathbf{H}}\big(T^r Tx, T^r Ty, T^r Tz\big) \\
= \quad & \mathcal{L}_p^T(Tx, Ty, Tz) - \mathcal{L}_p^T(x, y, z).
\end{aligned}
$$

□

Theorem 4.1. *Let $(X, G_{\mathbf{H}})$ be quaternion valued G-metric space and let $T : X \to X$ be a mapping. If T is an bijective m-quaternion-valued G-isometric, then T^{-1} is an m-quaternion-valued G-isometric.*

Proof. By assumption we have for all $x, y, z \in E$

$$\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big) = 0_{\mathbf{H}}.$$

replacing x by $T^{-m}x$, y by $T^{-m}y$ and z by $T^{-m}z$ we get

$$\begin{aligned}
0_{\mathbf{H}} &= \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}G_{\mathbf{H}}\big(T^{-(m-r)}x, T^{-(m-r)}y, T^{-(m-r)}z\big)\\
&= \sum_{p=0}^{m}(-1)^{p}\binom{m}{m-p}G_{\mathbf{H}}\big(T^{-p}x, T^{-p}y, T^{-p}z\big)\\
&= (-1)^m\sum_{p=0}^{m}(-1)^{m-p}\binom{m}{p}G_{\mathbf{H}}\big(T^{-p}x, T^{-p}y, T^{-p}z\big)\\
&\quad \left(\text{since } \binom{m}{m-p} = \binom{m}{p}\right)\\
&= (-1)^m\mathcal{L}_m^{T^{-1}}(x, y, z).
\end{aligned}$$

Hence T^{-1} is an m-quaternion-valued G-isometry. $\square$

Lemma 4.1. ([4], *Lemma* 2.1) *Let $(e_p)_{p\geq 0}$ and $(d_r)_{r\geq 0}$ be sequences of real numbers and let $(c_{p,r})_{p,r\geq 0}$ be a double sequence of real numbers. Then*

$$\sum_{p=0}^{n} e_p \sum_{r=0}^{p} c_{p,r}d_r = \sum_{r=0}^{n} d_r \sum_{p=r}^{n} c_{p,r}e_p. \tag{4.7}$$

Theorem 4.2. *Let $(E, G_{\mathbf{H}})$ be a quaternion-valued metric space and $T : E \to E$ be a map. The following statements hold.*

(1) *For every integer $n \in \mathbb{N}$ and $x, y, z \in E$, we have*

$$G_{\mathbf{H}}\big(T^n x, T^n y, T^n z\big) = \sum_{r=0}^{n}\binom{n}{r}\mathcal{L}_r^T(x, y, z). \tag{4.8}$$

(2) *T is an m-quaternion-valued G-isometric mapping if and only if, for every integer $n \geq 1$ and all $x, y, z \in E$, we have*

$$\mathbf{G}_{\mathbf{H}}\big(T^n x, T^n y, T^n z\big) = \sum_{j=0}^{m-1}\binom{n}{r}\mathcal{L}_r^T(x, y, z). \tag{4.9}$$

(3) *If T is an m-quaternion-valued G-isometric, then we have*

$$\mathcal{L}_{m-1}^{T}(x,y,z)=\lim_{n\to\infty}\frac{1}{\binom{n}{m-1}}G_{\mathbf{H}}\big(T^{n}x,T^{n}y,T^{n}z\big). \tag{4.10}$$

Proof. (1) We proceed by induction. It is easy to see that (3.8) ie true for n $=0$ and n $=1$. Now assume that (3.8) is true for any $r=0,1,2,....n$. We shall deduce it at step $n+1$. By virtue of (3.1) and (3.8) we have

$$\begin{aligned}
&\mathbf{G}_{\mathbf{H}}\big(T^{n+1}x,T^{n+1}y,T^{n+1}z\big)\\
=\;&\mathcal{L}_{n+1}^{T}(x,y,z)-\sum_{r=0}^{n}(-1)^{n+1-r}\binom{n+1}{r}\mathbf{G}_{\mathbf{H}}\big(T^{r}x,T^{r}y,T^{r}z\big)\\
=\;&\mathcal{L}_{n+1}^{T}(x,y,z)-\sum_{r=0}^{n}(-1)^{n+1-r}\binom{n+1}{r}\Big(\sum_{l=0}^{r}\binom{r}{l}\mathcal{L}_{l}^{T}(x,y,z)\Big)\\
=\;&\mathcal{L}_{n+1}^{T}(x,y,z)-\sum_{l=0}^{n}\mathcal{L}_{l}^{T}(x,y,z)\Big(\sum_{r=l}^{n}(-1)^{n+1-r}\binom{n+1}{r}\binom{r}{l}\Big) \qquad \text{(by 3.7)}\\
=\;&\mathcal{L}_{n+1}^{T}(x,y,z)-\sum_{l=0}^{n}\binom{n+1}{l}\mathcal{L}_{l}^{T}(x,y,z)\Big(\sum_{r=l}^{n}(-1)^{n+1-r}\underbrace{\binom{n+1}{r}\binom{r}{l}\frac{1}{\binom{n+1}{l}}}\Big)\\
=\;&\mathcal{L}_{n+1}^{T}(x,y,z)-\sum_{l=0}^{n}\binom{n+1}{l}\mathcal{L}_{l}^{T}(x,y,z)\Big(\sum_{r=l}^{n}(-1)^{n+1-r}\underbrace{\binom{n+1-l}{r-l}}\Big)\\
=\;&\mathcal{L}_{n+1}^{T}(x,y,z)-\sum_{l=0}^{n}\binom{n+1}{l}\mathcal{L}_{l}^{T}(x,y,z)\Big(-1\Big)\\
=\;&\sum_{l=0}^{n+1}\binom{n+1}{l}\mathcal{L}_{l}^{T}(x,y,z).
\end{aligned}$$

(2) If T is an m-quaternion-valued G-isometric, then $\mathcal{L}_{r}^{T}(x,y,z)=0$ for all $r\geq m$. Hence we drive (3.9) from (3.8). On the other hand, if (3.9) holds for all $n\geq 1$. Then $\mathcal{L}_{r}^{T}(x,y,z)=0$ for $r\geq m$ by (3.8), so T is an m-quaternion-valued G-isometry.

(3) One first has to observe that, by (3.9) if T is an m-quaternion-valued G-isometric mapping

$$G_{\mathbf{H}}\big(T^{n}x,T^{n}y,T^{n}z\big)=\sum_{r=0}^{m-2}\binom{n}{r}\mathcal{L}_{r}^{T}(x,y,z)+\binom{n}{m-1}\mathcal{L}_{m-1}^{T}(x,y,z).$$

Dividing both sides by $\binom{n}{m-1}$ we gate

$$\frac{1}{\binom{n}{m-1}} G_{\mathbf{H}}(T^n x, T^n y, T^n z) = \sum_{r=0}^{m-2} \frac{\binom{n}{r}}{\binom{n}{m-1}} \mathcal{L}_r^T(x,y,z) + \mathcal{L}_{m-1}^T(x,y,z).$$

Since $\frac{\binom{n}{r}}{\binom{n}{m-1}} \to 0$ for $0 \le r \le m-2$, the desired result follows by taking $n \to \infty$. $\square$

Lemma 4.2. *Let m be nonnegative integer, then the following identities hold.*

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^j = 0 \tag{4.11}$$

for $j = 0, 1, ..., m-1$ and

$$\sum_{l=0}^{m} (-1)^{m-k} \binom{m}{k} k^m = m!. \tag{4.12}$$

Proof. For (3.11), using induction on j.

(i) Case $j = 0$, we have $\sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} = 0$ so that (3.11) holds.

(ii) Assume that (3.11) is true for certain $j < m$ and we prove that it is also true for $j+1 < m$. Indeed observe that $\binom{m}{k} = \frac{m}{k} \binom{m-1}{k-1}$ and it follows that

$$\begin{aligned}
\sum_{0 \le k \le n} (-1)^{m-k} \binom{m}{k} k^{j+1} &= \sum_{1 \le k \le m} (-1)^{m-k} \binom{m}{k} k^{j+1} \\
&= -m \sum_{1 \le k \le m} (-1)^{m-(k-1)} \binom{m-1}{k-1} k^j \\
&= m \sum_{0 \le k \le m} (-1)^{m-1-k} \binom{m-1}{k} (k+1)^j.
\end{aligned}$$

Since $(k+1)^j = \sum_{r=0}^{j} \binom{j}{r} k^r$, the desired result follows from the hypothesis of induction.

For (3.12). We introduce the auxiliary $f(x) := e^x - 1$ from which it follows that for $k \in \{0, 1, ..., n\}$ we have

$$\frac{d^m}{dx^m}\left[f(x)^m\right] = \frac{d^{m-k}}{dx^{m-k}}\left[\sum_{j=0}^{k} a_j(k) f(x)^{m-j} (f')^j(x)\right],$$

where $a_j(k)$ are real constants with $a_k(k) = m(m-1)...(m-k+2)(m-k+1)$. Now observe that, if $k = m$ we get

$$\frac{d^m}{dx^m}\left[f(x)^m\right] = \sum_{0 \le j \le m} a_j(m)(e^x - 1)^{m-j} e^{jx}.$$

Thus implies that

$$\sum_{0 \le k \le m} \binom{m}{k} (-1)^{m-k} k^m e^{kx} = \sum_{0 \le j \le m} a_j(m)(e^x - 1)^{m-j} e^{jx}.$$

By setting $x = 0$ in the last equation we obtain

$$\sum_{0 \le k \le m} \binom{m}{k} (-1)^{m-k} k^m = a_m(m) = m!.$$

So the proof is achieved. $\square$

Lemma 4.3. ([4]) *Let n, m, r be integers such that $0 \le r \le m - 1 < n$. Then*

$$\sum_{l=r}^{m-1} (-1)^{l-r} \binom{n}{l} \binom{l}{r} = (-1)^{m-r-1} \frac{n(n-1) - - - - \overbrace{(n-r)} .... (n-m+1)}{r!(m-r-1)!)} \tag{4.13}$$

where $\overbrace{(n-r)}$ denotes the factor $(n-r)$ is omitted.

Proposition 4.2. *Let $(E, \mathbf{G_H})$ be a quaternion-valued G-metric space and $T : E \to E$ be a mapping. Then T is an m-quaternion G-isometry if and only if for any non negative integer n and every $x, y, z \in E$*

$$G_{\mathbf{H}}(T^n x, T^n y, T^n z) = \sum_{r=0}^{m-1} \left(\sum_{l=r}^{m-1} (-1)^{l-r} \binom{n}{l} \binom{l}{r} \right) \mathbf{G_H}(T^r x, T^r y, T^r z) \tag{4.14}$$

Proof. Assume that T is an m-quaternion-G-isometry. it follows from (3.8) that

$$G_{\mathbf{H}}\big(T^n x, T^n y, T^n z\big) = \sum_{l=0}^{m-1} \binom{n}{l} \mathcal{L}_l^T(x,y,x).$$

By (3.1) we deduce that

$$\begin{aligned} G_{\mathbf{H}}\big(T^n x, T^n y, T^n z\big) &= \sum_{l=0}^{m-1} \binom{n}{l} \sum_{r=0}^{l} (-1)^{l-r} \binom{l}{r} G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big) \\ &= \sum_{r=0}^{m-1} \Bigg(\sum_{l=r}^{m-1} (-)^{l-r} \binom{n}{l} \binom{l}{r} \Bigg) G_{\mathbf{H}}(T^r x, T^r y, T^r z). \end{aligned}$$

Conversely, if we assume that (3.14) holds , we obtain by Lemma 3.3 that $G_{\mathbf{H}}\big(T^n x, T^n y, T^n z\big)$ is a polynomial in n of degree $\leq m-1$;

$$G_{\mathbf{H}}\big(T^n x, T^n y, T^n z\big) = a_0(x,y,z) + a_1(x,y,z)n + \ldots + a_{m-1}(x,y,z)n^{m-1}.$$

Applying (3.11) of Lemma 3.2 we obtain that

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbf{H}}\big(T^r x, T^r y, T^r z\big) = 0.$$

Thus T is an m-quaternion G-isometry. □

The following Lemma is inspired from [2].

Lemma 4.4. *Let $(X, \mathbf{G_H})$ be quaternion-valued $\mathbf{G}$-metric space and $T : X \to X$ be a map. If T is an m-isometric and $n \geq m$ is a nonnegative integer. Then for $x, y, z \in X$ we have the following*

$$\sum_{0 \leq k \leq n} (-1)^k \binom{n}{k} k^i \mathbf{G_H}\big(T^{n-k}x, T^{n-k}y, T^{n-k}z\big) = 0 \tag{4.15}$$

for $i = 0, 1, \ldots, n-m$.

Proof. Since T is an m-isometry, it is know that T is an n-isometry for each $n \geq m$. Thus, for $i = 0$ the proof of (3.15) is immediate. Assume that $i \geq 1$ and prove (3.15) by indication on n . The result is true for $n = m$ Suppose that (3.15) is true for $i \in \{1, 2, \ldots, n-m\}$ and prove it for $i \in \{1, 2, \ldots, n-m+1\}$. By the induction hypothesis, we obtain

$$\sum_{0 \leq k \leq n+1} (-1)^k \binom{n+1}{k} k^i \mathbf{G_H}\big(T^{n-k+1}x, T^{n-k+1}y, T^{n-k+j}z\big)$$

$$
\begin{aligned}
&= \sum_{1\le k\le n+1} (-1)^k \binom{n+1}{k} k^i \mathbf{G_H}\big(T^{n-k+1}x, T^{n-k+1}y, T^{n-k+j}z\big)\\
&= \sum_{0\le k\le n} (-1)^{k+1} \binom{n+1}{k+1} (k+1)^i \mathbf{G_H}\big(T^{n-k}x, T^{n-k}y, T^{n-k}z\big)\\
&= -(n+1) \sum_{0\le k\le n} (-1)^k \frac{n!}{k!(n-k)!} (k+1)^{i-1} \mathbf{G_H}\big(T^{n-k}x, T^{n-k}y, T^{n-k}\big)\\
&= -(n+1) \sum_{0\le k\le n} (-1)^k \binom{n}{k} \Big(\sum_{0\le j\le i-1} \binom{i-1}{j} k^j \Big) \mathbf{G_H}\big(T^{n-k}x, T^{n-k}y, T^{n-k}\big)\\
&= -(n+1) \sum_{0\le j\le i-1} \binom{i-1}{j} \Big(\sum_{0\le k\le n} (-1)^k \binom{n}{k} k^j \mathbf{G_H}\big(T^{n-k}x, T^{n-k}y, T^{n-k}z\big)\Big)\\
&= 0.
\end{aligned}
$$

□

Theorem 4.3. *Let $T : X \longrightarrow X$ be a mapping on quaternion-valued G-metric space.If T is an m- isometry then so is T^n for each n.*

Proof. Assume that T is an m-quaternion-valued G-isometry then we have by (3.7) that

$$G_H\big(T^{nr}x, T^{rn}y, T^{rn}z\big) = \sum_{l=0}^{m-1} \binom{rn}{l} \mathcal{L}_l^T(x,y,z)$$

and

$$
\begin{aligned}
\mathcal{L}_m^{T^n}(x,y,z) &= \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} G_{\mathbf{H}}\big(T^{nr}x, T^{nr}y, T^{nr}z\big)\\
&= \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} \Big(\sum_{l=0}^{m-1} \binom{rn}{l} \mathcal{L}_l^T(x,y,z)\Big)\\
&= \sum_{l=0}^{m-1} \Big(\sum_{r=0}^{m} (-1)^{m-k} \binom{m}{r} \binom{nr}{l} \Big) \mathcal{L}_l^T(x,y,z)\\
&= 0 \ \text{(by Lemma 3.2)}.
\end{aligned}
$$

Hence, T^n is m-quaternion-valued G-isometric mapping. □

In the following theorem, we generalized [4, Theorem 2.14].

Theorem 4.4. *Let $(X, G_{\mathbf{H}})$ be quaternion-valued* **G***-metric space and $T, S : X \to X$ be a map such that $TS = ST$. If T is an m-quaternion-valued G-isometry and S is an n-quaternion-valued-G-isometry, then TS is an $(m+n-1)$-quaternion-valued G-isometry.*

Proof. The outline of the proof is similar to [2, Theorem 3].

We need to prove that $\mathcal{L}^{TS}_{m+n-1}(x,y,z) = 0$ for all $x, y, z \in E$.

In fact, we have

$$\begin{aligned}
&\mathcal{L}^{TS}_{m+n-1}(x,y,z)\\
= &\sum_{r=0}^{m+n-1} (-1)^r \binom{m+n-1}{r}\\
&G_{\mathbf{H}}\big((TS)^{m+n-1-r}x, (TS)^{m+n-1-r}y, (TS)^{m+n-1-r}z\big)\\
= &\sum_{r=0}^{m+n-1} (-1)^r \binom{m+n-1}{r}\\
&G_{\mathbf{H}}\big(T^{m+n-1-r}S^{m+n-1-r}x, T^{m+n-1-r}S^{m+n-1-r}y, T^{m+n-1-r}S^{m+n-1-r}z\big)
\end{aligned}$$

On the other hand since T is an m-quaternion-valued G-isometric, it follows by Proposition 3.2 that for all $x, y, z \in E$

$$\begin{aligned}
&G_{\mathbf{H}}\big(T^{m+n-1-r}S^{m+n-1-r}x, T^{m+n-1-r}S^{m+n-1-r}y, T^{m+n-1-r}S^{m+n-1-r}z\big)\\
= &\sum_{l=0}^{m-1} \left(\sum_{q=l}^{m-1} (-1)^{q-l} \binom{m+n-1-r}{q} \binom{q}{l} \right)\\
&G_{\mathbf{H}}\big(S^{m+n-1-r}T^l x, S^{m+n-1-r}T^l y, S^{m+n-1-r}T^l z\big)
\end{aligned}$$

$\square$

As

$$\sum_{q=l}^{m-1} (-1)^{q-l} \binom{m+n-1-r}{q} \binom{q}{l} = \sum_{\alpha=0}^{m-1} b_\alpha r^\alpha$$

we deduce that

$$\begin{aligned}
&\mathcal{L}^{TS}_{m+n-1}(x,y,z)\\
= &\sum_{r=0}^{m+n-1} \sum_{l=0}^{m-1} \sum_{\alpha=0}^{m-1} b_\alpha (-1)^r \binom{m+n-1}{r} r^\alpha\\
&G_{\mathbf{H}}\big(S^{m+n-1-r}T^l x, S^{m+n-1-r}T^l y, S^{m+n-1-r}T^l z\big).
\end{aligned}$$

To prove that $\mathcal{L}^{TS}_{m+n-1}(x,y,z)=0$ it suffices to prove that for $l\in\{0,1,...,m-1\}$ we have

$$\sum_{r=0}^{m+n-1}\sum_{\alpha=0}^{m-1} b_\alpha(-1)^r\binom{m+n-1}{r}r^\alpha$$
$$G_{\mathbf{H}}\big(S^{m+n-1-r}T^lx, S^{m+n-1-r}T^ly, S^{m+n-1-r}T^lz\big)=0.$$

Since S is an $(m+n-1)$-quaternion-valued G-isometric it follows by Lemma 3.4 that

$$\sum_{r=0}^{m+n-1}(-1)^r\binom{m+n-1}{r}r^\alpha$$
$$G_{\mathbf{H}}\big(S^{m+n-1-r}T^lx, S^{m+n-1-r}T^ly, S^{m+n-1-r}T^lz\big)=0$$

for $\alpha\in\{0,1,...,m-1\}$.

Corollary 4.1. *Let $(X,G_{\mathbf{H}})$ be quaternion-valued* $\mathbf{G}$*-metric space and $T,S:X\to X$ be a map such that $TS=ST$. If T is an m-quaternion-valued G-isometry and S is an n-quaternion-valued-G-isometry, then T^pS^q is an $(m+n-1)$-quaternion-valued G-isometry for $p,q=1,2,.....$*

Proof. It is a consequence of Theorem 3.3 and Theorem 3.4. □

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