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ON AN ORIENTATION-DEPENDENT CAHN-HILLIARD/ALLEN-CAHN SYSTEM

José Luiz Boldrini[1], Patrícia Nunes da Silva[2 §]
[1]UNICAMP-IMECC
P.O. Box 6065, Campinas, SP, 13083-859, BRAZIL
[2]IME-UERJ
Office 6016D, São Francisco Xavier Street
Rio de Janeiro, RJ, 20550-900, BRAZIL

Abstract: We analyse a family of orientation dependent systems consisting of a Cahn-Hilliard and several Allen-Cahn type equations. These systems are similar to one proposed by Fan, L.-Q. Chen, S. Chen and Voorhees (1998) for modelling Ostwald ripening of anisotropic crystals in a two-phase systems. They describe Ostwald ripening by taking several crystallographic orientations into account, considering both the evolution of the compositional field and of the crystallographic orientations. Fan *et al.* presented several numerical experiments to validate their modelling of the coarsening dynamics of one physical phase dispersed in the matrix of another The aim of the present article is to rigorously prove the existence and the uniqueness of solutions for such systems; for this, we firstly consider a suitable family of auxiliary approximate problems; we then deduce certain estimates for their corresponding solutions, and, by using compactness arguments, we extract subsequences that converge to a solution of the original problem.

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[§]Correspondence author

1. Introduction

Our objective in this work is to perform a rigorous mathematical analysis of the following orientation-dependent Cahn-Hilliard/Allen-Cahn system:

$$\begin{cases} \partial_t c = \nabla \cdot \left[D \nabla \left(\partial_c \mathcal{F} - \kappa_c \Delta c \right) \right], & (x,t) \in \Omega_T \\ \partial_t \theta_i = -L_i \left(\partial_{\theta_i} \mathcal{F} - \kappa_i \Delta \theta_i \right), & (x,t) \in \Omega_T \\ \partial_{\mathbf{n}} c = \partial_{\mathbf{n}} \left(\partial_c \mathcal{F} - \kappa_c \Delta c \right) = \partial_{\mathbf{n}} \theta_i = 0, & (x,t) \in S_T \\ c(x,0) = c_0(x), \quad \theta_i(x,0) = \theta_{i0}(x), & x \in \Omega \end{cases} \tag{1}$$

for $i = 1, \ldots, p$.

Here, Ω denotes the physical region where the Ostwald ripening process is occurring; $0 < T < +\infty$ is the final time of interest; $\Omega_T = \Omega \times (0,T)$; $S_T = \partial\Omega \times (0,T)$; $\mathbf{n}$ denotes the unitary exterior normal vector and $\partial_{\mathbf{n}}$ is the exterior normal derivative at the boundary. The unknown $c(x,t)$, for $t \in [0,T]$, $0 < T < +\infty$, $x \in \Omega$, is the compositional field (fraction of the solute with respect to the mixture); $\theta_i(x,t)$, for $i = 1, \ldots, p$, are the crystallographic orientations fields; D, κ_c, L_i, κ_i are positive constants related to the material properties. The function $\mathcal{F} = \mathcal{F}(c, \theta_1, \ldots, \theta_p)$ is the local orientation-dependent free energy density whose exact form will be presented in the next section.

This mathematical problem is related to a family of Ostwald ripening models for anisotropic crystals presented by Fan *et al.* in [20], where several numerical experiments were presented to validate their modelling of the coarsening dynamics of one physical phase dispersed in the matrix of another.

Before describing our results, let us briefly say something on the phase-field methodology as contrasted to more conventional approach, usually known as the sharp-interface methodology, to model microstructure evolution. In the sharp-interface methodology the regions separating the compositional or structural domains are treated as mathematically sharp interfaces, for which a certain regularity is usually required; then one must have evolution equations relating the relevant physical variable in each regions and also evolution equations for the sharp interfaces. Although such an interface-tracking approach can be successful in one dimensional systems, it becomes very difficult to apply in two or three spatial dimensions since interfaces with complicated geometry separating different microstructures usually develop during Ostwald ripening processes.

By contrast, in the past twenty years, the phase-field approach has emerged as one of the most powerful methods for modelling many types of microstructure evolution processes. It is based on a diffuse-interface description developed more than a century ago by van der Waals [32] and almost fifty years ago independently by Cahn and Hilliard [9]. The main idea behind phase-field models is to replace the singular macroscopic treatment of a sharp-interface (usually a discontinuity surface for some variables) by a regularized description. To this end, one or more auxiliary fields, called phase-fields, varying smoothly across the interface are introduced. Phase-field model may be associated to conserved or nonconserved field variables; the first case usually leads to nonlinear Cahn-Hilliard type equations, see for instance Cahn [9], while the second case usually leads to nonlinear Allen-Cahn (time-dependent Ginzburg-Landau) type equations, see for instance Allen and Cahn [1]. With the fundamental thermodynamic and kinetic information as the input, the phase-field method is able to predict the evolution of arbitrary morphologies and complex microstructures without explicitly tracking the position of the interfaces. Moreover, phase-field models describe microstructure phenomena at the mesoscale and contain the corresponding sharp- or thin-interface descriptions as a particular limit.

Phase-field models have been used in a large variety of problems; we just mention a few articles to give an idea of their applicability. They were used for instance for simulation of diffusion limited crystal growth (see Collins and Levine [16]), extended by Kobayashi [23] (and later by Karma and Rappel [24]) to analyse dendritic growth; Kassner and Misbah [25] used it to study stress-induced instabilities in solids. In fluid mechanics, Marangoni convection (see Borcia and Bestehorn [6]), droplet and vesicle dynamics (see Beaucourt *et al.* [4]) and polymer blends (see Roths *et al.* [28]) are examples of applications of these models. Fan *et al.* [20] have used a phase-field model to describe an Ostwald ripening phenomenon of anisotropic crystals. Boldrini *et al.* [5] analysed a phase field model for the solidification/melting of a metallic alloy when two different kinds of crystallization are possible. In [3], Barrett *et al.* have studied the sharp interface limit of an Allen-Cahn/Cahn-Hilliard system which can be viewed as a phase-field system modelling the electromigration of intergranular voids. Such model was introduced by the Barrett *et al.* [2] as an extension of the work by Mahadevan [27] and Cahn and Novick-Cohen [10, 11]. Garcke *et al.* [21] analysed a general class of evolution equations that model phase transitions in anisotropic multiphase systems and have determined the singular limit of the anisotropic multi-phase Allen-Cahn system when the interfacial thickness tends to zero.

As for articles with problems with systems of equations more similar to (1), we mention the following. Gilardi and Rocca [22] established well-posedness of a system that couples a Cahn-Hilliard equation to a second order parabolic equation in higher space dimension. They also analyzed the long time behaviour of its solutions. Boussinot *et al.* [7] presented coupled Allen-Cahn/Cahn-Hilliard systems as 2D and 3D phase-field models for the microstructure evolution in presence of a lattice misfit and with inhomogeneous elastic constants. Dal Passo *et al.* [17] analysed a coupled Allen-Cahn/Cahn-Hilliard system with degenerate mobility with a free energy that involves logarithmic terms. Such terms are important to prove that the phase field variables are bounded. Existence, uniqueness and regularity have been established for an Allen-Cahn/Cahn-Hilliard system with constant mobility and a polynomial free energy with special symmetry relation on the coefficients in Brochet *et al.* [8]. For the free energy associated to the system (1), Silva and Boldrini [29] obtained a generalized solution of (1) such that the composition field $c(x,t)$ takes values in the closed interval $[0,1]$. In [30], Silva and Boldrini used a full discretization of a mixed formulation of (1) to establish the existence and uniqueness solutions, and in [31] the corresponding error estimates were proved.

Finally, we describe the results of the present article.

We prove the existence and the uniqueness of solutions of Problem (1) in a better regularity class than the one obtained in Silva and Boldrini [30] for the compositional field $c(x,t)$, without requiring the symmetries of the coefficients of the free energy that was strongly exploited by Brochet *et al.* in [8].

As for the mathematical arguments to be used to prove our results, we remark that Dal Passo *et al.* [17] proved existence of solutions for a coupled Allen-Cahn/Cahn-Hilliard system with a bounded free energy. Since the free energies we consider in the present work may have polynomial growth, to invoque their results as a first step in our arguments, we introduce a family of auxiliary approximate problems, depending on a positive parameter M, which are obtained by a suitable truncation of the free energy. For such auxiliary problems we then can use the results of Dal Passo *et al.* [17] to obtain approximate solutions; next, we have to be careful to obtain certain estimates for such approximate solutions that are uniform with respect to M. Once this is done, we can extract subsequences that converge as $M \to +\infty$ to a solution of the original problem.

2. Technical Hypotheses and Main Result

Likewise Fan *et al.* [20], we assume that the local free energy $\mathcal{F}$ has the following form:

$$\begin{aligned}\mathcal{F}(c,\theta_1,\dots,\theta_p) = & -\frac{A}{2}(c-c_m)^2 + \frac{B}{4}(c-c_m)^4 + \frac{D_\alpha}{4}(c-c_\alpha)^4 \\ & + \frac{D_\beta}{4}(c-c_\beta)^4 - \gamma\sum_{i=1}^{p} g(c,\theta_i) + \frac{\delta}{4}\sum_{i=1}^{p}\theta_i^4 + \sum_{i=1}^{p}\sum_{i\neq j=1}^{p}\varepsilon_{ij} f(\theta_i,\theta_j).\end{aligned} \tag{2}$$

A, B, D_α, D_β, γ, δ, ε_{ij}, $i \neq j = 1, \dots, p$ are positive constants related to the material properties, c_α and c_β are the solubilities or equilibrium concentrations for the matrix phase and second phase, respectively, and $c_m = (c_\alpha + c_\beta)/2$.

Remark 1. Realistic free energies may be rather intricate. According to Fan *et al.* [20], the main physical requirement on the local free energy $\mathcal{F}$ is having $2p$ degenerate minima at the equilibrium concentration c_β to distinguish the $2p$ orientations differences of the second phase grains in space (see [12, 13, 14, 15]). To attain such requirement, it is possible to choose functions $g(c, \theta_i), f(\theta_i, \theta_j)$ and the parameters A, B, D_α, D_β, γ, δ, ε_{ij}, $i \neq j = 1, \dots, p$ such that $\mathcal{F}(c, \theta_1, \dots, \theta_p)$ has $2p$ degenerate minima of equal depth when $c = c_\beta$ and $(\theta_1, \dots, \theta_p) = (1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$. This kind of situation, specially the existence of many degenerate minima, brings difficulties in realistic numerical simulations of Ostwald ripening process, but they play no role in the mathematical arguments to be used in the present work; that is, our results hold for free energies having or not having such many minima.

Moreover, for technical reasons we also assume that functions f and g satisfy the following: there are constants $F_1, F_2, G_1, G_2 \geq 0$ such that for all $(u, v), (a, b) \in \mathbb{R}^2$

$$f \in C^1(\mathbf{R}^2, \mathbf{R}) \qquad \text{and} \qquad g \in C^2(\mathbf{R}^2, \mathbf{R}),$$

$$\begin{aligned} &|f(a,b) - f(u,v) + \nabla f(u,v) \cdot (u - a, v - b)| \\ &\qquad \leq F_1(u-a)^2 + F_2(v-b)^2 \ (\leq \max\{F_1, F_2\}|(u,v) - (a,b)|^2) \end{aligned} \tag{3}$$

and

$$|g(a,b) - g(u,v) - \nabla g(u,v) \cdot (a - u, b - v)| \leq G_1(u-a)^2 + G_2(v-b)^2. \tag{4}$$

We observe that these last assumptions imply that the difference between $f(a, b)$ and $g(a, b)$ and their Taylor polynomials of degree one at (u, v) are, respectively bounded up to a multiplicative fixed constant by the square of the Euclidean distance between (u, v) and (a, b). This is true for instance if f and g are C^2-functions with bounded second derivatives.

Under the above hypotheses we will prove the following:

Theorem 2. *Let $T > 0$ and $\Omega \subset R^d$, $1 \leq d \leq 3$ be a bounded domain with Lipschitz boundary. For all $c_0, \theta_{i0}, i = 1, \dots, p$, satisfying $c_0,\ \theta_{i0} \in H^1(\Omega)$, there exists a unique $(p+1)$-tuple $(c, \theta_1, \dots, \theta_p)$ such that, for $i = 1, \dots, p$,*

(a) $c \in L^\infty(0, T, H^1(\Omega)) \cap L^2(0, T, H^3(\Omega))$,

(b) $\theta_i \in L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega))$,

(c) $\partial_t c \in L^2(0,T,[H^1(\Omega)]')$, $\partial_t\theta_i \in L^2(\Omega_T)$;

(d) $\partial_c\mathcal{F},(c,\theta_1,\dots,\theta_p)$, $\partial_{\theta_i}\mathcal{F}(c,\theta_1,\dots,\theta_p) \in L^2(\Omega_T)$;

(e) $c(x,0) = c_0(x)$, $\theta_i(x,0) = \theta_{i0}(x)$;

(f) $\partial_{\mathbf{n}} c_{|_{S_T}} = \partial_{\mathbf{n}}\theta_{i|_{S_T}} = 0$ *in* $L^2(S_T)$;

(g) $(c,\theta_1,\dots,\theta_p)$ *satisfies*

$$\int_0^T \langle \partial_t c, \phi\rangle dt = -\iint_{\Omega_T} D\nabla(\partial_c\mathcal{F}(c,\theta_1,\dots,\theta_p) - \kappa_c\Delta c)\nabla\phi, \tag{5}$$

$\forall\phi \in L^2(0,T,H^1(\Omega))$ *and*

$$\iint_{\Omega_T} \partial_t\theta_i\psi_i = -\iint_{\Omega_T} L_i(\partial_{\theta_i}\mathcal{F}(c,\theta_1,\dots,\theta_p) - \kappa_i\Delta\theta_i)\psi_i, \tag{6}$$

$\forall\psi_i \in L^2(\Omega_T), i = 1,\dots,p$, *and where* $\mathcal{F}$ *is given by (2),* $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $H^1(\Omega)$ *and its dual and* $(\cdot,\cdot)$ *denotes the inner product in* $L^2(\Omega)$.

Remark 3. (i) The results of this work apply, for instance, to a family of problems which contains a local free energy density given as in (2) but with

$$g(c,\theta_i) = g_{c_\beta}(c - c_\alpha)g_2(\theta_i) \qquad \text{and} \qquad f(\theta_i,\theta_j) = g_2(\theta_i)g_2(\theta_j)$$

where the functions g_M, $M = 2$ or c_β, are given by

$$g_M(u) = u^2, \text{ for } |u| \le M \quad \text{and} \quad g_M(u) = 6M^2 - \frac{8M^3}{|u|} + \frac{3M^4}{|u|^2}, \text{ for } |u| \ge M.$$

This example coincides in a ball of radius $\min\{c_\beta, 2\}$ with the local free energy density $\mathcal{E}(c,\theta_1,\dots,\theta_p)$ presented by [20], where

$$\begin{aligned}\mathcal{E}(c,\theta_1,\dots,\theta_p) = &-\frac{A}{2}(c-c_m)^2 + \frac{B}{4}(c-c_m)^4 + \frac{D_\alpha}{4}(c-c_\alpha)^4 \\ &+ \frac{D_\beta}{4}(c-c_\beta)^4 - \frac{\gamma}{2}\sum_{i=1}^p (c-c_\alpha)^2\theta_i^2 + \frac{\delta}{4}\sum_{i=1}^p \theta_i^4 + \sum_{i=1}^p\sum_{i\ne j=1}^p \frac{\varepsilon_{ij}}{2}\theta_i^2\theta_j^2.\end{aligned}$$

having therefore the same local minima and satisfying the cited requirement.

(ii) Equation (5) implies that the average of c is conserved.

(iii) Throughout this paper, standard notation will be used for the required functional spaces and we denote by $\overline{f}$ the mean value of f in Ω of a given $f \in L^1(\Omega)$.

3. Auxiliary Truncated Problems

Before presenting our auxiliary truncated problems, present some remarks on an existence result stated by Dal Passo *et al.* [17] for the following system:

$$\begin{cases} \partial_t u = \left[q_1(u,v)\left(f_1(u,v) - \kappa_1 u_{xx}\right)_x\right]_x, & (x,t) \in \Omega_T \\ \partial_t v = -q_2(u,v)\left[f_2(u,v) - \kappa_2 v_{xx}\right], & (x,t) \in \Omega_T \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} u_{xx} = \partial_{\mathbf{n}} v = 0 & (x,t) \in S_T \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega \end{cases} \tag{7}$$

where q_i and f_i satisfy:

(H1) $q_i \in C(\mathbf{R}^2, \mathbf{R}^+)$, with $q_{\min} \le q_i \le q_{\max}$ for some $0 < q_{\min} \le q_{\max}$;

(H2) $f_1 \in C^1(\mathbf{R}^2, \mathbf{R})$ and $f_2 \in C(\mathbf{R}^2, \mathbf{R})$, with $\|f_1\|_{C^1} + \|f_2\|_{C^0} \le F_0$ for some $F_0 > 0$.

Proposition 4. *Assuming (H1), (H2) and* $u_0,\ v_0 \in H^1(\Omega)$, *there exists a pair of functions* (u,v) *such that:*

1. $u \in L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^3(\Omega)) \cap C([0,T];H^\lambda(\Omega))$, $\lambda < 1$
2. $v \in L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega)) \cap C([0,T];H^\lambda(\Omega))$, $\lambda < 1$
3. $\partial_t u \in L^2(0,T,[H^1(\Omega)]')$, $\quad \partial_t v \in L^2(\Omega_T)$
4. $u(0) = u_0$ *and* $v(0) = v_0$ *in* $L^2(\Omega)$
5. $\partial_{\mathbf{n}} u_{|S_T} = \partial_{\mathbf{n}} v_{|S_T} = 0$ *in* $L^2(S_T)$
6. (u,v) *solves (7) in the following sense:*

$$\int_0^t \langle \partial_t u, \phi \rangle = -\iint_{\Omega_t} q_1(u,v)(f_1(u,v) - \kappa_1 u_{xx})_x \phi_x, \quad \forall \phi \in L^2(0,T,H^1(\Omega))$$

$$\iint_{\Omega_t} \partial_t v \psi = -\iint_{\Omega_t} q_2(u,v)(f_2(u,v) - \kappa_2 v_{xx})\psi, \quad \forall \phi \in L^2(\Omega_T).$$

Remark 5. **(i)** Dal Passo *et al.* [17] stated Proposition 4 for $n = 1$, but, as they pointed out (see [17, Remark 2.2]) their result is still valid in any space dimension. In [17], Proposition 4 is in fact an auxiliary result for dealing with a coupled Allen-Cahn/Cahn-Hilliard system with degenerate mobility, which brings difficulties to consider higher dimensions. Since our problem does not have a degenerate mobility, we can apply Proposition 4 in cases of higher dimensions.

(ii) Furthermore, to prove Proposition 4, Dal Passo *et al.* apply a Galerkin approximation that can be straightforward extended to systems with multiple Allen-Cahn equations coupled to a Cahn-Hilliard equation. Thus, Proposition 4 still holds in such situations.

We stress that, for the sake of simplicity of exposition, without loss of generality, we present in this work the proof for the case of dimension one with only two orientation field variables, that is, when Ω is a bounded open interval and p is equal to two. In this case, the local free energy density is reduced to

$$\begin{aligned}\mathcal{F}(c,\theta_1,\theta_2) = &-\frac{A}{2}(c-c_m)^2 + \frac{B}{4}(c-c_m)^4 + \frac{D_\alpha}{4}(c-c_\alpha)^4 + \frac{D_\beta}{4}(c-c_\beta)^4 \\ &- \gamma g(c,\theta_1) - \gamma g(c,\theta_2) + \frac{\delta}{4}\theta_1^4 + \frac{\delta}{4}\theta_2^4 + \varepsilon_{12} f(\theta_1,\theta_2) + \varepsilon_{21} f(\theta_2,\theta_1).\end{aligned} \tag{8}$$

Now, we construct a family of auxiliary systems depending on a positive parameter M which controls a truncation of the local free energy $\mathcal{F}$.

For each $M > 0$, we consider a suitable truncation of the free energy density. We take

$$\mathcal{F}_M(c,\theta_1,\theta_2) = \mathcal{F}(c,\theta_1,\theta_2), \quad -M \le c,\theta_1,\theta_2 \le M, \tag{9}$$

and outside $[-M,M]^3$, we extend $\mathcal{F}_M$ to satisfy

$$\|\partial_c \mathcal{F}_M\|_{C^1(\mathbf{R}^2,\mathbf{R})} \le U_0(M) \text{ and } \|\partial_{\theta_i}\mathcal{F}_M\|_{C(\mathbf{R}^2,\mathbf{R})} \le V_0(M), \tag{10}$$

$$|\partial_c \mathcal{F}_M|^2 \le K\left[c^6 + \sum_{i=1}^{2}\theta_i^6 + 1\right] \text{ and } |\partial_{cc}\mathcal{F}_M|^2 \le K\left[c^4 + \sum_{i=1}^{2}\theta_i^4 + 1\right], \tag{11}$$

$$|\partial_{\theta_i}\mathcal{F}_M|^2 \le K\left[c^6 + \theta_i^6 + 1\right] \text{ and } |\partial_{c\theta_i}\mathcal{F}_M|^2 \le K\left[c^4 + \theta_i^4 + 1\right], \tag{12}$$

$$|\mathcal{F}_M|^2 \le K\left[c^4 + \sum_{i=1}^{2}\theta_i^4 + 1\right] \text{ and } \mathcal{F}_M \ge m_{\mathcal{F}}, \tag{13}$$

$\forall M > 0$, $\forall c,\theta \in \mathbf{R}$, where $K > 0$ and $m_{\mathcal{F}} \in \mathbb{R}$ are constants independent of M and $U_0(M)$ and $V_0(M)$ are finite constants that may depend on M.

These $\mathcal{F}_M$ can be obtained for instance as follows: fix a cutoff function $\zeta \in C^2(\mathbf{R})$ such that $\zeta(z) = 1$ for $z \le 1$ and $\zeta(z) = 0$ for $z \ge 2$; then define $\mathcal{F}_M(c, \theta_1, \theta_2) := \mathcal{F}(c, \theta_1, \theta_2)\zeta(|(c, \theta_1, \theta_2)| - M)$, where $|\cdot|$ denotes the standard euclidian norm. Then, the definition (8) of $\mathcal{F}$ and conditions (3) and (4) guarantee that properties (10) – (13) are satisfied.

These properties of the truncated free energy density $\mathcal{F}_M$ allow us to apply an existence result of Dal Passo *et al.* [17] and to obtain in the next section estimates that are uniform with respect to M.

The auxiliary truncated systems are given by

$$\begin{cases} \partial_t c = D\left(\partial_c \mathcal{F}_M(c, \theta_1, \theta_2) - \kappa_c c_{xx}\right)_{xx}, & (x,t) \in \Omega_T \\ \partial_t \theta_i = -L_i\left[\partial_{\theta_i} \mathcal{F}_M(c, \theta_1, \theta_2) - \kappa_i(\theta_i)_{xx}\right], & (x,t) \in \Omega_T \\ \partial_{\mathbf{n}} c = \partial_{\mathbf{n}}(\partial_c \mathcal{F}_M(c, \theta_1, \theta_2) - \kappa_c c_{xx}) = \partial_{\mathbf{n}} \theta_i = 0 & (x,t) \in S_T \\ c(x,0) = c_0(x), \quad \theta_i(x,0) = \theta_{i0}(x), & x \in \Omega \end{cases} \tag{14}$$

Now, since (10) holds, for each $M > 0$, we can use Proposition 4, together with Remark 5, to guarantee that there exists a solution $(c_M, \theta_{1M}, \theta_{2M})$ to Problem (14) in the following sense

$$\int_0^t \langle \partial_t c_M, \phi \rangle = -\iint_{\Omega_t} D(\partial_c \mathcal{F}_M(c_M, \theta_{1M}, \theta_{2M}) - \kappa_c [c_M]_{xx})_x \phi_x, \tag{15}$$

for $\phi \in L^2(0, T, H^1(\Omega))$ and

$$\iint_{\Omega_t} \partial_t \theta_{iM} \psi_i = -\iint_{\Omega_t} L_i(\partial_{\theta_i} \mathcal{F}_M(c_M, \theta_{1M}, \theta_{2M}) - \kappa_i [\theta_{iM}]_{xx}) \psi_i, \tag{16}$$

for $\psi_i \in L^2(\Omega_T)$.

Moreover, equation (15) implies that the mean value of c_M in Ω is given by

$$\overline{c_M(t)} = \overline{c_0} \tag{17}$$

4. Estimates for the Solutions of the Truncated Problems

In this section we obtain some *a priori* estimates that are uniform with respect to M. The first estimates are the following:

Lemma 6. *There exists a constant C_1 independent of M such that*

1. $\|c_M\|_{L^\infty(0,T,H^1(\Omega))} \leq C_1$

2. $\|\theta_{iM}\|_{L^\infty(0,T,H^1(\Omega))} \leq C_1$

3. $\|(\partial_c \mathcal{F}_M - \kappa_c (c_M)_{xx})_x\|_{L^2(\Omega_T)} \leq C_1$

4. $\|\partial_{\theta_i} \mathcal{F}_M - \kappa_i (\theta_{iM})_{xx}\|_{L^2(\Omega_T)} \leq C_1$

5. $\|\partial_t c_M\|_{L^\infty(0,T,[H^1(\Omega)]')} \leq C_1$

6. $\|\partial_t \theta_{iM}\|_{L^2(\Omega_T)} \leq C_1$

7. $\|\mathcal{F}_M(c_M, \theta_{1M}, \theta_{2M})\|_{L^\infty(0,T,L^1(\Omega))} \leq C_1$

Proof. To obtain items 3, 4 and 7, we argue as Dal Passo *et al.* [17] and [18]. First, we observe that by the regularity of c_M and θ_{iM}, we could take

$$\partial_c \mathcal{F}_M - \kappa_c (c_M)_{xx} \quad \text{and} \quad \partial_{\theta_i} \mathcal{F}_M - \kappa_i (\theta_{iM})_{xx}$$

as test functions in the equations (15) and (16), respectively, to obtain

$$\begin{aligned} &\int_0^t \langle \partial_t c_M, \partial_c \mathcal{F}_M - \kappa_c (c_M)_{xx} \rangle + \sum_{i=1}^{2} \iint_{\Omega_t} \partial_t \theta_{iM} \partial_{\theta_i} \left[\mathcal{F}_M - \kappa_i (\theta_{iM})_{xx}\right] \\ &= - \iint_{\Omega_t} D[(\partial_c \mathcal{F}_M - \kappa_c (c_M)_{xx})_x]^2 - \sum_{i=1}^{2} \iint_{\Omega_t} L_i [\partial_{\theta_i} \mathcal{F}_M - \kappa_i (\theta_{iM})_{xx}]^2. \end{aligned} \tag{18}$$

Also, given a $h > 0$ sufficiently small, we consider functions defined as

$$c_{Mh}(t,x) = \frac{1}{h} \int_{t-h}^{t} c_M(\tau, x) d\tau,$$

where we set $c_M(t,x) = c_0(x)$ for $t \leq 0$. Since $\partial_t c_{Mh}(t,x) \in L^2(\Omega_T)$, we have

$$\begin{aligned} &\int_0^T \langle (c_{Mh})_t, [\partial_c \mathcal{F}_M(c_{Mh}, \theta_{1M}, \theta_{2M}) - \kappa_c (c_{Mh})_{xx}] \rangle dt \\ &\quad + \sum_{i=1}^{2} \iint_{\Omega_T} (\theta_{iM})_t [\partial_{\theta_i} \mathcal{F}_M(c_{Mh}, \theta_{1M}, \theta_{2M}) - \kappa_i (\theta_{iM})_{xx}] \\ &= \int_\Omega \left[\frac{\kappa_c}{2} |[c_{Mh}(t)]_x|^2 + \sum_{i=1}^{2} \frac{\kappa_i}{2} |[\theta_{iM}]_x(t)|^2 + \mathcal{F}_M(c_{Mh}, \theta_{1M}, \theta_{2M}) \right] \\ &\qquad - \int_\Omega \left[\frac{\kappa_c}{2} |[c_0]_x|^2 + \sum_{i=1}^{2} \frac{\kappa_i}{2} |[\theta_{i0}]_x|^2 + \mathcal{F}_M(c_0, \theta_{10}, \theta_{20}) \right]. \end{aligned}$$

By taking the limit as h tends to zero in the above expression and by using (18), we obtain

$$\iint_{\Omega_t} D[(\partial_c \mathcal{F}_M(c_M, \theta_{1M}, \theta_{2M}) - \kappa_c (c_M)_{xx})_x]^2$$
$$+ \sum_{i=1}^{2} \iint_{\Omega_t} L_i[\partial_{\theta_i} \mathcal{F}_M(c_M, \theta_{1M}, \theta_{2M}) - \kappa_i (\theta_{iM})_{xx}]^2$$
$$+ \frac{\kappa_c}{2} \|[c_M]_x(t)\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \frac{\kappa_i}{2} \|[\theta_{iM}]_x(t)\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{F}_M(c_M, \theta_{1M}, \theta_{2M})$$
$$= \frac{\kappa_c}{2} \|[c_0]_x\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \frac{\kappa_i}{2} \|[\theta_{i0}]_x\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{F}_M(c_0, \theta_{10}, \theta_{20})$$

for almost every $t \in (0, T]$. By using the regularity of the initial conditions (see Theorem 2) and (13), there exists a constant $C_1 > 0$, depending only on the initial conditions, κ_c and κ_i, such that for all $M > 0$

$$\iint_{\Omega_T} D[(\partial_c \mathcal{F}_M - \kappa_c (c_M)_{xx})_x]^2 + \sum_{i=1}^{2} \iint_{\Omega_T} L_i[\partial_{\theta_i} \mathcal{F}_M - \kappa_i (\theta_{iM})_{xx}]^2$$
$$+ \frac{\kappa_c}{2} \|[c_M]_x(t)\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \frac{\kappa_i}{2} \|[\theta_{iM}]_x(t)\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{F}_M(t) \leq C_1 \tag{19}$$

which implies items 3, 4 and 7 since by (13), we have $\mathcal{F}_M \geq m_{\mathcal{F}}$. By using the Poincaré inequality and (17), item 1 is also verified.

To prove item 6, we choose $\psi_i = \partial_t \theta_M$ as a test function in (16), which yields

$$\iint_{\Omega_T} [\partial_t \theta_{iM}]^2 = - \iint_{\Omega_T} (\partial_{\theta_i} \mathcal{F}_M - \kappa_i (\theta_{iM})_{xx}) \partial_t \theta_{iM}$$
$$\leq \left(\iint_{\Omega_T} (\partial_{\theta_i} \mathcal{F}_M - \kappa_i (\theta_{iM})_{xx})^2 \right)^{1/2} \left(\iint_{\Omega_T} [\partial_t \theta_{iM}]^2 \right)^{1/2}.$$

Since we have (19), we obtain

$$\int_\Omega \theta_{iM}^2 \leq 2 \int_\Omega \theta_{i0}^2 + 2t \iint_{\Omega_T} (\partial_t \theta_{iM})^2 d\tau \leq C_2,$$

and item 2 is verified. Finally, item 5 follows since

$$\left|\int_0^T \langle \partial_t c_M, \phi\rangle\right| \leq \left(\iint_{\Omega_T} [(\partial_c \mathcal{F}_M - \kappa_c (c_M)_{xx})_x]^2\right)^{1/2} \left(\iint_{\Omega_T} (\phi_x)^2\right)^{1/2}$$

for all $\phi \in L^2(0,T,H^1(\Omega))$. □

Remark 7. From (19), using (13), we obtain

$$\iint_{\Omega_T} D[(\partial_c \mathcal{F}_M - \kappa_c (c_M)_{xx})_x]^2 + \sum_{i=1}^{2} \iint_{\Omega_T} L_i [\partial_{\theta_i} \mathcal{F}_M - \kappa_i (\theta_{iM})_{xx}]^2 + \frac{\kappa_c}{2} \|[c_M(t)]_x\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \frac{\kappa_i}{2} \|[\theta_{iM}]_x(t)\|^2_{L^2(\Omega)} \leq C_1. \tag{20}$$

Lemma 8. *For $M > 0$, there exists a constant C_3 independent of M such that*

1. $\|\partial_c \mathcal{F}_M\|_{L^2(0,T,H^1(\Omega))} \leq C_3$,
2. $\|\partial_{\theta_i} \mathcal{F}_M\|_{L^2(\Omega_T)} \leq C_3$,
3. $\|[c_M]_{xx}\|_{L^2(\Omega_T)} \leq C_3$,
4. $\|[\theta_{iM}]_{xx}\|_{L^2(\Omega_T)} \leq C_3$.

Proof. First, we prove items 2 and 4. From Lemma 6, item 4, we have

$$\iint_{\Omega_T} (\partial_{\theta_i} \mathcal{F}_M)^2 - 2\kappa_i \iint_{\Omega_T} \partial_{\theta_i} \mathcal{F}_M [\theta_{iM}]_{xx} + \kappa_i^2 \iint_{\Omega_T} [\theta_{iM}]^2_{xx} \leq C_1. \tag{21}$$

By using (12), we obtain

$$2\kappa_i \partial_{\theta_i} \mathcal{F}_M [\theta_{iM}]_{xx} \leq \frac{\kappa_i^2}{2} [\theta_{iM}]^2_{xx} + C[c_M^6 + \theta_{iM}^6 + 1].$$

Thus, from Lemma 6, items 1 and 2, it follows from (21) that

$$\iint_{\Omega_T} (\partial_{\theta_i} \mathcal{F}_M)^2 + \frac{\kappa_i^2}{2} \iint_{\Omega_T} [\theta_{iM}]^2_{xx} \leq C_3. \tag{22}$$

Next, we prove item 3. By defining $H_M = \partial_c \mathcal{F}_M - \kappa_c [c_M]_{xx}$, since $[c_M]_{x|S_T} = 0$, we have

$$\iint_{\Omega_T} H_M = \iint_{\Omega_T} \partial_c \mathcal{F}_M,$$

and from Lemma 6, item 3,

$$\iint_{\Omega_T} [H_M]_x^2 \leq C_1.$$

We also have

$$\iint_{\Omega_T} H_M^2 = \iint_{\Omega_T} (\partial_c \mathcal{F}_M)^2 - 2\iint_{\Omega_T} (\partial_c \mathcal{F}_M)[c_M]_{xx} + \kappa_c^2 \iint_{\Omega_T} [c_M]_{xx}^2.$$

On the other hand, we can write

$$\begin{aligned}\iint_{\Omega_T} H_M^2 &= \iint_{\Omega_T} [H_M - \overline{H_M}]^2 + \iint_{\Omega_T} \overline{H_M}^2 \\ &\leq C_P \iint_{\Omega_T} [H_M]_x^2 + \iint_{\Omega_T} (\partial_c \mathcal{F}_M)^2\end{aligned}$$

where C_P denotes the Poincaré constant. Now, item 3 follows from (12) and Lemma 6, items 1 and 2.

Finally, by using again (12) and Lemma 6, items 1 and 2, we obtain

$$\|\partial_c \mathcal{F}_M\|_{L^2(\Omega_T)}^2 \leq C_3.$$

Lemma 8, items 3 and 4, and (12) imply that $\|[\partial_c \mathcal{F}_M]_x\|_{L^2(\Omega_T)}^2$ is also bounded by a constant. Thus, item 1 is proved. □

5. Existence and Uniqueness of Solutions

Next, we state an existence result, still in the one-dimensional case.

Proposition 9. *There exists a triple* (c, θ_1, θ_2) *such that:*

1. $c \in L^\infty(0, T, H^1(\Omega)) \cap L^2(0, T, H^3(\Omega))$
2. $\theta_i \in L^\infty(0, T, H^1(\Omega)) \cap L^2(0, T, H^2(\Omega))$
3. $\partial_t c \in L^2(0, T, [H^1(\Omega)]')$, $\quad \partial_t \theta_i \in L^2(\Omega_T)$
4. $\partial_c \mathcal{F}(c, \theta_1, \theta_2)$, $\partial_{\theta_i} \mathcal{F}(c, \theta_1, \theta_2) \in L^2(\Omega_T)$
5. $c(0) = c_0$ *and* $\theta_i(0) = \theta_{i0}$ *in* $L^2(\Omega)$

6. $[c]_{x|_{S_T}} = [\theta_i]_{x|_{S_T}} = 0$ *in* $L^2(S_T)$

7. (c, θ_1, θ_2) *solves the system (14) in the following sense:*

$$\int_0^T \langle \partial_t c, \phi \rangle = -\iint_{\Omega_T} D[\partial_c \mathcal{F}(c, \theta_1, \theta_2) - \kappa_c(c)_{xx}]_x \phi_x \tag{23}$$

for all $\phi \in L^2(0, T, H^1(\Omega))$*, and*

$$\iint_{\Omega_T} \partial_t \theta \psi_i = -\iint_{\Omega_T} L_i(\partial_{\theta_i} \mathcal{F}(c, \theta) - \kappa_i(\theta)_{xx})\psi_i \tag{24}$$

for all $\psi_i \in L^2(\Omega_T)$*, and* $\mathcal{F}$ *is given by (8).*

Proof. The proof will be done by considering solutions of the approximate truncated problem for each M and then letting $M \to +\infty$.

For this, let us firstly observe that from Lemma 6, item 3 and Lemma 8, item 1, the norm of $[c_M]_{xxx}$ in $L^2(\Omega_T)$ is bounded by a constant which does not depend on M. This fact and estimates of Lemmas 6 and 8, together with a compactness argument, imply that there exists a subsequence (which for simplicity of notation we still denote by $\{(c_M, \theta_{1M}, \theta_{2M})\}$) such that as M goes to infinity

c_M	weakly-* converges to	c	in	$L^\infty(0, T, H^1(\Omega))$,
θ_{iM}	weakly-* converges to	θ_i	in	$L^\infty(0, T, H^1(\Omega))$,
c_M	converges weakly to	c	in	$L^2(0, T, H^3(\Omega))$,
θ_{iM}	weakly converges to	θ_i	in	$L^2(0, T, H^2(\Omega))$,
$\partial_t c_M$	weakly converges to	$\partial_t c$	in	$L^2(0, T, [H^1(\Omega)]')$,
$\partial_t \theta_{iM}$	weakly converges to	$\partial_t \theta_i$	in	$L^2(\Omega_T)$
c_M	strongly converges to	c	in	$L^2(\Omega_T)$
θ_{iM}	strongly converges to	θ_i	in	$L^2(\Omega_T)$.

By recalling Lemmas 6 and 8, Proposition 9, items 1–3, now follow.

Next, items 1 and 2 of Lemma 8 imply that

$$\partial_c \mathcal{F}_M(c_M, \theta_{1M}, \theta_{2M}) \quad \text{weakly converges to} \quad \mathcal{G} \quad \text{in} \quad L^2(\Omega_T),$$
$$\partial_{\theta_i} \mathcal{F}_M(c_M, \theta_{1M}, \theta_{2M}) \quad \text{weakly converges to} \quad \mathcal{H} \quad \text{in} \quad L^2(\Omega_T).$$

Since the strong convergence of the sequence (c_M) implies that (at least for a subsequence) $\partial_c \mathcal{F}_M(c_M, \theta_{iM})$ converges pointwise in Ω_T, it follows from Lemma 1.3 from [26], p. 12, that $\mathcal{G} = \partial_c \mathcal{F}(c, \theta_1, \theta_2)$. Similarly, we have $\mathcal{H} = \partial_{\theta_i} \mathcal{F}(c, \theta_1, \theta_2)$.

Thus item 4 is proved.

Item 5 is straightforward. Now, by compactness we have that

$$c_M \text{ converges to } c \text{ in } L^2(0,T,H^{2-\rho}(\Omega)), \quad \text{for any sufficiently small } \rho>0,$$
$$\theta_{iM} \text{ converges to } \theta \text{ in } L^2(0,T,H^{2-\rho}(\Omega)), \quad \text{for any sufficiently small } \rho>0,$$

which imply item 6.

To prove item 7, by using the previous convergences, we pass to the limit as M goes to infinity in the equations (15) and (16). □

Proof of Theorem 2. We start with the proof of the existence of solutions of (5)–(6) in the general case since this requires only slight modifications as compared to the previously situation, that is the one space dimension with $p=2$.

In fact, when the spatial dimension is $2 \le d \le 3$, as we already observed, for the coupling of the Cahn-Hilliard equation with multiple Allen-Cahn equations, we still can use Proposition 4 in Dal Passo *et al.* [17].

Moreover, the extra terms that appear in the local free energy (2), as compared to the ones in the simpler previously discussed case, can be handled in a straightforward way; we then obtain exactly the same uniform estimates for the solutions of the truncated problems just by using arguments of elliptic regularity for the Laplacian to obtain estimates in $L^2(0,T,H^2(\Omega))$ and in $L^2(0,T,H^3(\Omega))$. Then, we can pass to the limit as $M\to+\infty$ exactly as we did in the previous proposition and get a solution of the original problem.

As for the uniqueness of solutions of (5)–(6), we argue as in Elliott and Luckhaus [19].

For this, we introduce the following Green's operator $\mathcal{G}$: given $f\in[H^1(\Omega)]'_{null} = \{f\in[H^1(\Omega)]',\ \langle f,1\rangle=0\}$, we define $\mathcal{G}f\in H^1(\Omega)$ as the unique solution of

$$\int_\Omega \nabla\mathcal{G}f\cdot\nabla\psi = \langle f,\psi\rangle, \quad \forall\psi\in H^1(\Omega) \qquad \text{and} \qquad \int_\Omega \mathcal{G}f = 0. \tag{25}$$

Now, let $z^c = c_1 - c_2$ and $z^{\theta_i} = \theta_{i1}-\theta_{i2},\ i=1,\dots,p$ be the differences of two pair of solutions of (5)–(6) as in Theorem 2. Let

$$z^{\mathcal{F}} = D[\partial_c\mathcal{F}(c_1,\theta_{11},\dots,\theta_{p1}) - \partial_c\mathcal{F}(c_2,\theta_{12},\dots,\theta_{p2}) - \kappa_c\Delta z^c].$$

Since equation (5) implies that the mean value of the composition field in Ω is conserved, we have that $(z^c,1)=0$, and we find from (5) that

$$-\mathcal{G}z^c_t = \overline{z^{\mathcal{F}}}.$$

The definition of the Green operator and the fact that $(z^c, 1) = 0$ give

$$-(\nabla \mathcal{G} z_t^c, \nabla \mathcal{G} z^c) = -(\mathcal{G} z_t^c, z^c) = (\overline{z^{\mathcal{F}}}, z^c) = (z^{\mathcal{F}}, z^c).$$

Thus

$$\frac{1}{2}\frac{d}{dt}|\nabla \mathcal{G} z^c|^2 + (D[\partial_c \mathcal{F}(c_1, \theta_{11}, \ldots, \theta_{p1}) - \partial_c \mathcal{F}(c_2, \theta_{12}, \ldots, \theta_{p2}) - \kappa_c \Delta z^c], z^c) = 0.$$

We find from (6) that

$$\frac{D}{2L_i}\frac{d}{dt}|z^{\theta_i}|^2 + D\kappa_i |\nabla z^{\theta_i}|^2 + D\big(\partial_{\theta_i} \mathcal{F}(c_1, \theta_{11}, \ldots, \theta_{p1}) \\ - \partial_{\theta_i} \mathcal{F}(c_2, \theta_{12}, \ldots, \theta_{p2}), z^{\theta_i}\big) = 0.$$

By adding the above equations, using the convexity of the function $[\mathcal{F} + \mathcal{R}](c, \theta_1, \ldots, \theta_p)$ with

$$\mathcal{R}(c, \theta_1, \ldots, \theta_p) = \frac{A}{2}(c - c_m)^2 + \frac{\gamma}{2}\sum_{i=1}^{p} g(c, \theta_i) - \sum_{i=1}^{p}\sum_{i \neq j=1}^{p} \frac{\varepsilon_{ij}}{2} f(\theta_i, \theta_j),$$

and by integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\nabla \mathcal{G} z^c|^2 + \kappa_c D |\nabla z^c|^2 + \sum_{i=1}^{p}\left[\frac{D}{2L_i}\frac{d}{dt}|z^{\theta_i}|^2 + D\kappa_i |\nabla z^{\theta_i}|^2\right] \\ &\leq \big(\nabla(\mathcal{R}(c_1, \theta_{11}, \ldots, \theta_{p1}) - \mathcal{R}(c_2, \theta_{12}, \ldots, \theta_{p2})) \cdot (z^c, z^{\theta_1}, \ldots, z^{\theta_p}), 1\big) \end{aligned} \tag{26}$$

In order to estimate the term at the right hand side of the above inequality, we observe that (3) and (4) imply that

$$\varepsilon_{ij}(\nabla(f(\theta_{i1}, \theta_{j1}) - f(\theta_{i2}, \theta_{j2})) \cdot (z^{\theta_i}, z^{\theta_j}), 1) \leq 2\varepsilon_{ij} F_1 |z^{\theta_i}|^2 + 2\varepsilon_{ij} F_2 |z^{\theta_j}|^2$$

and

$$\gamma(\nabla(g(c_1, \theta_{i1}) - g(c_2, \theta_{i2})) \cdot (z^c, z^{\theta_i}), 1) \leq 2\gamma G_1 |z^c|^2 + 2\gamma G_2 |z^{\theta_i}|^2.$$

The above inequalities, together with (26), imply that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\nabla \mathcal{G} z^c|^2 + \frac{\kappa_c D}{2}|\nabla z^c|^2 + \sum_{i=1}^{p}\left[\frac{D}{2L_i}\frac{d}{dt}|z^{\theta_i}|^2 + \frac{D\kappa_i}{2}|\nabla z^{\theta_i}|^2\right] \\ \leq C[\|z^c\|^2_{L^2(\Omega)} + \|z^{\theta_i}\|^2_{L^2(\Omega)} + \|z^{\theta_j}\|^2_{L^2(\Omega)}]. \end{aligned}$$

From the definition of the Green operator, we have that $|z^c|^2 = (\nabla \mathcal{G} z^c, \nabla z^c)$. Using the Hölder inequality, we can rewrite the above inequality as

$$\frac{1}{2}\frac{d}{dt}|\nabla \mathcal{G} z^c|^2 + \frac{\kappa_c D}{4}|\nabla z^c|^2 + \sum_{i=1}^{p}\left[\frac{D}{2L_i}\frac{d}{dt}|z^{\theta_i}|^2 + \frac{D\kappa_i}{2}|\nabla z^{\theta_i}|^2\right]$$
$$\leq C[\|\nabla \mathcal{G} z^c\|^2_{L^2(\Omega)} + \|z^{\theta_i}\|^2_{L^2(\Omega)} + \|z^{\theta_j}\|^2_{L^2(\Omega)}].$$

Then, a standard Gronwall argument yields $\nabla \mathcal{G} z^c = 0$ and $z^{\theta_i} = 0$ for $i = 1, \ldots, p$, since $\mathcal{G} z^c(0) = 0$ and $z^{\theta_i}(0) = 0$ for $i = 1, \ldots, p$. Then we have uniqueness since $|z^c|^2 = (\nabla \mathcal{G} z^c, \nabla z^c) = 0$. □

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