International Journal of Pure and Applied Mathematics
Volume 117 No. 3 2017, 503-519
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v117i3.13

ijpam.eu

A DIFFERENTIAL GAME OF FINANCIAL INTERMEDIARIES: BANK'S PRICING BEHAVIOR

Antonio Castellanos-Luis[1], Francisco Venegas-Martínez[§2],
Salvador Cruz-Aké[3]
[1,2,3]Instituto Politécnico Nacional
Escuela Superior de Economía, MEXICO

Abstract: This paper deals with a non-cooperative duopoly differential game where financial intermediaries compete for setting spread interest rates in order to obtain the maximum profit. The game is constrained by two dynamic systems: a benchmark interest rate and macroeconomic expectation. We characterized the Nash equilibrium in both deterministic and stochastic frameworks.

AMS Subject Classification: 91A80, 49N70
Key Words: differential games, duopoly, optimal control, Hamilton-Jacobi-Bellman, interest rate

1. Introduction

Availability of credits is essential for entrepreneurs starting new projects, and households to consume in the current time in order to increase their quality of life. For obtaining credits firms have, usually, three possibilities: (i) become public in a stock market, and sell stocks, (ii) grant corporate bonds, and (iii) look for a banking loan. For many firms, the first possibility is not sometimes viable because it involves costs and time to become public, the same happens for issuing bonds. It seems to be that the best possibility is to look for financial intermediaries, [10] states that banking loans are the most important source for

Received: October 5, 2017
Revised: November 21, 2017
Published: January 15, 2018

[§]Correspondence author

financing business activities (52 % in the US). This is the main reason why we will focus on this kind of credits throughout this paper in a non-cooperative duopoly differential game.

The amount of credits that a bank grants depends on many factors and, in particular, on macroeconomic variables. Therefore, before granting loans every bank carries out an analysis about these variables, as well as the many risks involving the financial sector. Moreover, banks have to take into account the behavior of their competitors. That is, if there are $N < \infty$ banks in the economy and the bank $i \in N$ does not want to grant a loan, for any reason, there exists another bank $j \in N$ with $j \neq i$ that will do it. An example of the kind of behavior caused the credit crisis that took place in 2008 since banks were concerned about their future access to capital markets, and they started hoarding funds even when the creditworthiness of the borrows does not change [1].

In this paper, the analysis is made from the supply side. We are interested in knowing the incentives that a bank has for granting credits when the constrains are given by macroeconomic variables and a benchmark interest rate. Thus, the main goal of this paper is to examine the strategic behavior of banks in order to understand the mechanism of credit supply in a differential game approach. For achieving this, we will focus on the case where $N = 2$, *i.e.*, a duopoly model. The literature concerning lending channels and pricing behavior is vast, and the approach using dynamic games is quite new; see, for instance, [8] and [9]. That is why we use dynamic game theory because it helps us to understand the strategic behavior of the banks, and to characterize the Nash equilibrium. One of the applications of game theory to model banks behavior is developed in [4], but the model is static. This research solves a game in the framework similar to that of a linear quadratic models. This allows us to find an explicit solution for control variables; for more details about LQ systems see [3] and [6]. In this paper we aim to address the following questions.

1. How do the state variables affect the level of spread chosen for banks to grant more loans?

2. What is the strategic behavior of a bank under duopoly competition?

3. How does different solution methods affect the dynamics of the state variables?

Our results show that given the non-linearity of the model, all the solutions are feedback strategies. That is, all strategies depend on t and the states $r_f(t)$ and $M(t)$. We find that a bank has incentives to increase (decrease) its spread interest rate inversely or in accordance of its competitors as a function of a collusion (or competition) factor κ. Also, when the benchmark interest rate rises the banks will raise its spread interest rate as well by some factor.

The paper is organized as follows. Section 2 introduces the base model and set the framework for the differential game. In section 3, we find the optimal solution when the players only take into account one state variable, and we carry out a numerical and sensitivity analysis. In section 4 the players solve their problem with two state variables through the maximum principle and HJB condition. Section 5 is the comparison of the results of the previous games, section 6 deals with a basic model of a stochastic differential game, and, finally, section 7 briefly concludes.

2. The Model

We start out by assuming a duopoly competition where the players are financial intermediaries. In this research, we only consider commercial banks, each one is indexed by $j = 1, 2$. The game is non-cooperative. Also, the time horizon is infinite $[0, T = \infty)$. In the framework of duopoly theory, the game we introduce has some similarities to the one called *Bertrand duopoly*, where each bank chooses the "price" of its product. In this context, we will understand the interest rate as the price of money at which the loans and deposits are priced in the financial market. The difference relies in the fact that our objective functional is nonlinear. It is well know from the literature (see [4]) that banks have many other activities. In order to keep the model as simple as possible, we assume that commercial banks activities are constrained only by taking deposits from households and grant credits to entrepreneurs, and also they have the possibility to borrow and lend money at the benchmark interest rate. Therefore, the bank j has two kind of interest rates: the active interest rate r_L at which the loans are priced and the passive interest rate r_D for pricing deposits. A necessary condition for a bank to be profitable is that $r_L > r_D$. In this way $r_j = r_L - r_D > 0$ is the spread interest rate chosen by bank j, technically this is the control variable. The vector of control variables in this game is defined as follows: $\mathbf{r_j} \in U_j \subset \mathbb{R}^{m_j}$ where U_j is the control space of bank j and $\mathbb{R}^{m_j}$ is the m-dimensional vector space. In order to have a consistent result, we assume that U_j is a compact set. It is worthy to notice that the vector space is different for each bank, *i.e.*, a bank can have more possibilities for choosing the spread rate due to many factors such as the size of the bank, its liquidity and capitalization rate. After we defined the control variable, in table 1, we show the balance sheet of the bank with $L+I = D+C+\sigma(r_f,t)$ where L, I, D and C are measurable functions define on some compact sets, and $\sigma(r_f,t)$ is a measurable function defined on a measurable space $(\Omega, \mathscr{B}(X))$. There is only one risk factor in the model.

Assets	Liabilities	Risk factors
Loans (L)	Deposits (D)	interest rate risk $(\sigma(r_f,t))$
Interbank deposits (I)	Costs (C)	

Table 1: Banks balance sheet

The choice of the spread interest rates of bank j is not only constrained for the action of the remaining $j-1$ banks, but also the state equations. In this research the level of spread interest rate is constrained by two state functions, one which we will call the benchmark interest rate r_f and an implicit function M that measures the expectation that banks have about macroeconomic conditions or economic trends. Both functions r_f and M are defined on a compact set $X \subseteq \mathbb{R}^n$, and they are of class $\mathscr{C}^2$ in order to allow the existence of a maximum for the game. And additional constraint is that $r_f(t) \geq 0 \quad \forall t \in [0,\infty]$. Function $M(t)$ can be positive, negative or zero. If it is positive, we will say that banks and entrepreneurs have better expectations about the economic environment, thus the entrepreneurs have more incentives to ask for credits and banks to grant them. The benchmark interest rate evolves according to the following dynamics:

$$\dot{r_f}(t) = a[d - (r_1 + r_2 - r_f(t))] \tag{1}$$

This is a modified version of a mean reversion model for interest rates, where a is the speed of adjustment and the rates r_1 and r_2 are the long term mean levels, here $a, d, r_1, r_2, \psi \in \mathbb{R}_{++}$. This kind of dynamics is used in the literature for models with sticky prices, see [2].

$$\dot{M}(t) = \gamma M(t) - (r_1 + r_2)\, s \tag{2}$$

This differential equation models the way the economy is driven by macroeconomic variables measured in an implicit function M. The quantity $s > 0$ is the speed of adjustment when banks decide to change their interest rate spread. The banks in the game behave in the business-as-usual fashion. That is, they are profit maximizers. The Revenue of the bank $j = \{1, 2\}$ depends on the difference between the amount of credit granted to entrepreneurs and the deposits supply from households, and the spread interest rate. The bank j also have some cost [1] and the possibility to invest money at the benchmark interest rate in the financial market. The demand for credits is given by the following equation:

$$L(r_f(t), M(t), r_1, r_2) = \alpha r_f(t) + \nu M(t) - \beta r_2 + \frac{\kappa r_1}{2} \tag{3}$$

This is linear demand function where $\alpha, \nu, \beta, \kappa$ measure demand's sensitivity to a change in variables such as the benchmark interest rate, the macroeconomic environment and the interest rate spread set by the banks. In particular, we are interested in the sign of κ (collusion factor). If it is negative, the banks behave inversely (if one bank increases the spread interest rate, then another will decrease it) and *viceversa*. The amount of deposits supply by households is given by:

$$D(r_f(t), M(t), r_1, r_2) = -\theta r_f - \frac{\eta M}{2} - \frac{1}{2}(\kappa r_1) + \mu r_2. \tag{4}$$

This is a linear function where $\theta, \eta, \kappa, \mu$ measure the marginal changes in deposits supply. Finally, the cost and investment function are:

$$C(r_j) = r_j z + w, \quad j = \{1, 2\} \tag{5}$$

There is a cost associated with the interest rate spread (z), which can be small, and a fixed cost w.

$$I(r_f) = \frac{r_f}{2} \tag{6}$$

The investment function is just a linear function of the benchmark interest rate. We shall omit the time argument from now on. With all the elements already set, the bank j optimization problem is given by:

$$\underset{r_j \in U_j}{\text{Maximize}}\ \pi_j = \int_0^\infty e^{-\rho_j t} \left([L(\cdot) - D(\cdot) - C(\cdot)] r_j + I(\cdot) r_f \right) dt$$

$$j = \{1, 2\} \tag{7}$$

subject to equations (1) and (2) and:

$$r_f(0) = r_f^0 \geq 0; \quad M(0) = M_0; \quad r_j \in U_j \tag{8}$$

Replacing equations (3)-(6) into (7), we will observe that the objective functional contains the product of the state a control variables, *i.e.*, $r_f r_j$ and $r_j M$, this fact makes the problem a nonlinear differential game.

[1]This costs are associated only to administrative issues, is not a cost a production

3. Characterization of Nash Equilibrium, One State Variable

In order to have a point of comparison, we are going to solve the banking game when the state variable is one-dimensional (M is a constant), and characterize the Nash equilibrium. We start by assuming that both banks follow Markovian (Feedback) strategies. The goal is to find a function $\phi^*(t, r_f)$ as the best response for the bank 1 to the best response strategy of bank 2. The problem is therefore reduced to:

$$\underset{r_j \in U_j}{\text{Maximize}}\ \pi_j = \int_{t=0}^{\infty} e^{-\rho_j t} \left\{ [L(\cdot) - D(\cdot) - C(\cdot)] r_j + I(\cdot) r_f \right\} dt \qquad j = \{1, 2\} \quad (9)$$

subject to equation (1) and $r_f(0) = r_f^0 \geq 0$. The value function for bank j is $V_j(t, r_f)$ and the Hamilton-Jacobi-Bellman equation associated with bank j optimizatdion problem is:

$$\rho_j V_j(t, r_f) - \frac{\partial V_j}{\partial t} = \max_{r_j \in U_j} \left\{ \left(L(\cdot) - D(\cdot) - C(\cdot) \right) r_j + I(\cdot) r_f + \frac{\partial V_j}{\partial r_f} (\dot{r_f}) \right\} \quad (10)$$

The game has infinite horizon. Therefore, the strategies are stationary ([6]), which implies that $\partial V_j / \partial t = 0$. Also, given the similarity with a L-Q system, we can ensure that the optimality conditions are satisfied for the existence of a maximum $\left(\partial^2 RHS / \partial r_j^2 \leq 0 \right)$. [2] Assuming an interior solution, the first order condition for the maximization of problem (10) is given by the feedback strategy:

$$r_j^* = \frac{-2aV_j'(r_f) + 2(\alpha + \theta) r_f(t) + (\eta + 2\nu) M + 2\kappa r_{j-1} - 2w}{4(\beta + \mu + z)} \quad (11)$$

where $\phi^*(t, r_f) = r_j^*$. We can conclude from the last equation the following claims:

Claim 1. *If $\kappa < 0$ in (11) the relationship between the interest rate spread chosen by bank 1 is negative with the one chosen by bank 2. This implies that if a bank does not want to grant more credits there exists always other bank that will be able to grant credits with a lower interest rate level.*

Claim 2. *If the benchmark rate r_f increases, then bank 1 has incentives to increase the spread rates by a proportion $2(\alpha + \theta)$*

[2]The Legendre condition is satisfied, for further information see [14]

The following proposition characterizes the feedback Nash Equilibrium for this game:

Proposition 3. *If there exists a value function $V_j(r_f)$ and the strategy r_j^* satisfying the equilibrium condition in equation (11), then r_j^* is a Feedback Nash equilibrium for bank j.*

Proof. By the nature of the problem, we assume symmetry in the players[3], *i.e.*, we substitute $r_1 = r_2$, therefore, equation (11) becomes:

$$r_2 = \frac{-2aV_2'(r_f) + 2(\alpha + \theta)r_f(t) + \eta M + 2\nu M - 2w}{4\beta - 2\kappa + 4\mu + 4z} \tag{12}$$

Given the similarity with a LQ system, we make the informed guess for the value function:

$$V_1(r_f, t) = c_1 + c_2 r_f(t) + \frac{1}{2} c_3 r_f(t)^2 \tag{13}$$

$$\frac{\partial V}{\partial r_f} = c_3 r_f(t) + c_2 \tag{14}$$

where c_i are constants to be determined. Replacing equations (14) and (12) in the RHS of (10), and by using the method of undetermined coefficients, we get the value of coefficients c_i $\{1, 2, 3\}$ by solving a system of algebraic equations. The solution of this system of equations gives the trajectory of the value function of bank 1 □

After solving the banking game for symmetric players, the optimal control variable only depends on the value of parameters and the state variable:

$$r_1^* = \frac{2r_f(t)\left(-ac_3 + \alpha + \theta\right) - 2ac_2 + (\eta + 2\nu)M - 2w}{4\beta - 2\kappa + 4\mu + 4z} \tag{15}$$

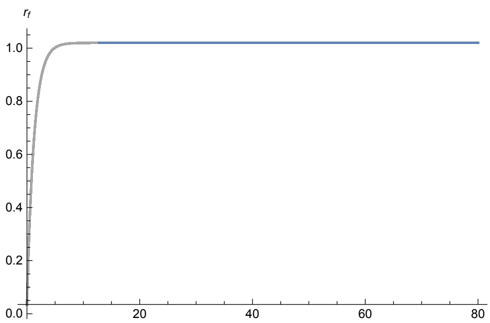
In order to show the equilibrium trajectory of the state variable, we will use numerical methods.

3.1. Numerical and Sensitivity Analysis

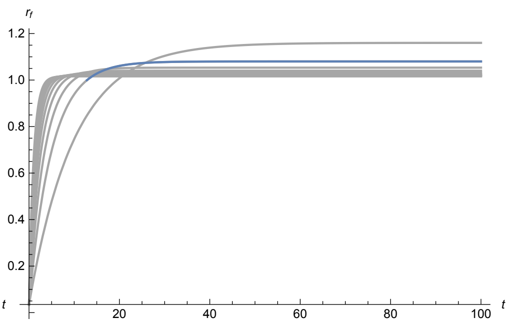
Our interest relies on modeling the equilibrium trajectory of the state variable, we use the functionality of NDSolve by the software Wolfram Mathematica to get the results. We assign arbitrarily the value of the parameters.
We can see from figure 1a that the steady state is achieved in a short period of

[3]Symmetry means that the control space is the same for all the players. An example of this kind of games in classical game theory is the prisoner's dilemma, see [11]

time. We do not know the real value of the parameters. In order to see how a change in each of the parameters affect the dynamics of the benchmark interest rate, we use the functionality of ParametricNDSolve by Wolfram Mathematica to see this sensitivity. From figure 1b we can observe that the state variable is only sensitive to changes in the speed of reversion ($a \in [0.1, 1]$), meanwhile for any change in $\alpha \in [-1, 1]$ and the remaining parameters, the equilibrium trajectory is the same. In particular $\dot{r_f}$ is not sensitive to changes in $\kappa \in [-1, 1]$, so the claim 1 is satisfied. And banks always compete against each other.



(a)



(b)

Figure 1: (a) Equilibrium trajectory of $\dot{r_f}$ with fixed parameters (b) Sensitivity of $\dot{r_f}$ w.r.t. a

4. Characterization of Nash Equilibrium, Two State Variables

So far, in the last section we assume that banks only take into account the dynamics of the benchmark interest rate, however, [5] states that macroeconomics conditions are important for setting the interest rates. The author tests his hypothesis with econometric methods. Through this section, we will work with a differential game approach. The optimization problem for bank j is now:

$$\underset{r_j \in U_j}{\text{Maximize}}\ \pi_j = \int_{t=0}^{\infty} e^{-\rho_j t} \left\{ [L(\cdot) - D(\cdot) - C(\cdot)] r_j + I(\cdot) r_f \right\} dt$$

$$j = \{1, 2\} \quad (16)$$

subject to equations (1) and (2) with $r_f(0) = r_f^0 \geq 0$, $M(0) = M_0$ and $r_j \in U_j$. In this scenario, the game has two kinematic equations. We will solve this game in two scenarios when both players follow Markovian Strategies. Given the non-linear structure of the objective functional, we will solve the game with two mathematical tools: the Pontryagin Maximum principle and the Hamilton-Jacobi-Bellman equations. The goal is then to characterize the Nash equilibrium in both scenarios in order to compare them.

4.1. Feedback Nash Equilibrium with Pontryagin Maximum Principle

In order to have a concave function and to guarantee the existence of a maximum, we assume now that the parameter κ is negative. This assumption is justified by the fact that the model is not sensitive to changes in some parameters including κ. The Hamiltonian for bank j is given by:

$$\mathbb{H}_j(r_j, r_f, M, \lambda_1, \lambda_2) = I(\cdot)r_f + r_j(L(\cdot) - P(\cdot)) - C(\cdot)r_j + \lambda_1 \dot{r_f} + \lambda_2 \dot{M} \quad (17)$$

where λ_i stand for the adjoint equations. Assuming an interior solution, we apply the optimality conditions of optimal control:

$$\frac{\partial \mathbb{H}_j}{\partial r_1} = 0; \quad \frac{\partial \mathbb{H}_j}{\partial \lambda_1} = \dot{r_f}; \quad \frac{\partial \mathbb{H}_j}{\partial \lambda_2} = \dot{M} \quad (18)$$

$$\dot{\lambda_1} = \lambda_1 \rho - \frac{\partial \mathbb{H}_1}{\partial r_f}; \quad \dot{\lambda_2} = \lambda_2 \rho - \frac{\partial \mathbb{H}_1}{\partial M} \quad (19)$$

and transversality conditions

$$\lim_{t \to \infty} e^{-\rho t} \lambda_1(t) = 0 \quad (20)$$

$$\lim_{t \to \infty} e^{-\rho t} \lambda_2(t) = 0 \quad (21)$$

$$r_f(0) = r_f^0 \quad (22)$$

$$M(0) = M_0 \quad (23)$$

From equation (18), we get the optimal control for bank j:

$$r_j^* = \frac{-2a\lambda_1 + 2(\alpha + \theta)r_f + (\eta + 2\nu)M - 2r_{j-1}(\beta + \mu) - 2\lambda_2 s - 2w}{4(z - \kappa)} \quad (24)$$

Even without solving explicitly the equation (24), we can infer some important results:

Claim 4. *If $\beta + \mu > 0$ in (24), the relation between the interest rate spread chosen by bank 1 is inverse with the one chosen by bank 2. This implies that if a bank does not want to grant more credits there exists always other bank that will be able to grant credits with a lower interest rate level.*

Claim 5. *If $\eta + 2\nu > 0$, bank 1 will increased the interest rate spread when the macroeconomic environment is appropriate, and viceversa.*

Claim 6. *If the benchmark interest rate increases, then bank 1 has incentives to increase the spread interest rate by a proportion $2(\alpha + \theta)$*

Proposition 7. *If equation (24) satisfies the optimality conditions for a maximum, as well as the transversality conditions, then the Markovian strategy is a Nash Equilibrium for bank j.*

Proof. By assuming symmetry in the players, making $r_j = r_{j-1}$ and replacing in (24), we get:

$$r_j^* = \frac{(\eta + 2\nu)M(t) - 2\left(a\lambda_1(t) - (\alpha + \theta)r_f(t) + s\lambda_2(t) + w\right)}{2(\beta - 2\kappa + \mu + 2z)} \tag{25}$$

In order to have an explicit solution to equation (25), we have to solve the following system of ordinary differential equations[4], associated to optimality conditions:

$$\dot{r_f}(t) = a\left(d + r_f(t) - 2r_j\right) \tag{26}$$

$$\dot{M}(t) = \gamma M(t) - 2r_j s \tag{27}$$

$$\dot{\lambda_1}(t) = -a\lambda_1 - r_f(t) + r_j(-\alpha - \theta) + \rho\lambda_1(t) \tag{28}$$

$$\dot{\lambda_2}(t) = -\gamma\lambda_2 + r_j\left(-\frac{\eta}{2} - \nu\right) + \rho\lambda_2(t) \tag{29}$$

Once we have the solution for the system of differential equations, we substitute them into equation 25 and that is Feedback Nash equilibrium for bank 1. □

4.2. Feedback Nash Equilibrium via HJB

In this section, we assume that the players follow Feedback strategies, that is, the best response function also depends of the state variables. Feedback strategies and HJB equations are more suitable for analyzing banks behavior because the feature of subgame perfectness ([6]) in the sense that for every $t \in [0, T = \infty)$ the solution is a Nash Equilibrium. Also, the HJB equations are solved with the dynamic programming algorithm, which means that the solution is a Markov Control Process ([7]). A more detailed information about the development and applications of this kind of processes can be seen in [15].

[4]The solution of the system is available upon request.

The maximization problem for bank $j = \{1, 2\}$, is the same as in equation (7):

$$\underset{r_j \in U_j}{\text{Maximize}}\ \pi_j = \int_0^\infty e^{-\rho_j t} \left([L(\cdot) - D(\cdot) - C(\cdot)]r_j + I(\cdot)r_f\right) dt \tag{30}$$

subject to equations (1) and (2) with $r_f(0) = r_f^0 \geq 0$ and $M(0) = M_0$. The mathematical tool for getting the best response function for Bank j are the HJB equations. Thus, for bank j we have:

$$\rho_j V_j(r_f, M, t) - \frac{\partial V_j}{\partial t} = \max_{r_j \in U_j} \left\{ (L(\cdot) - D(\cdot) - C(\cdot))\, r_j + I(\cdot) + \frac{\partial V_j}{\partial r_f}(\dot{r_f}) + \frac{\partial V_j}{\partial M}(\dot{M}) \right\} \tag{31}$$

Maximization of the right hand-side gives the following rules for the feedback strategies of bank j:

$$r_j^* = \frac{2aV_2'(r_f) + 2(\alpha + \theta)r_f - 2sV_2'(M) + (\eta + 2\nu)M + 2\kappa r_{j-1} - 2w}{4(\beta + \mu + z)} \tag{32}$$

The game has infinite horizon, thus $\partial V_j(r_f, M, t)/\partial t = 0$, and the game is stationary. We now state the following proposition.

Proposition 8. *If there exists a value function $V_j(t, r_f, M)$ such that the feedback strategies $\phi(t, r_f, M) = r_j^* \quad \forall t \in [0, \infty)$, then r_j^* is a feedback Nash Equilibrium*

Proof. Given the structure of the game, we will assume the following candidate for the value function:

$$V = c_1 + c_2 r_f + \frac{1}{2} c_3 r_f^2 + c_4 M + \frac{c_5 M^2}{2} + c_6 M r_f; \tag{33}$$

and its partial derivatives are given by:

$$\frac{\partial V_2(\cdot)}{\partial r_f} = c_3 r_f + c_6 M + c_2 \tag{34}$$

$$\frac{\partial V_2(\cdot)}{\partial M} = c_6 r_f + c_5 M + c_4 \tag{35}$$

Next, we substitute the optimal control for bank 2, equation (32), the value function and its derivative into the RHS of equation (31) to obtain the value of the coefficients c_i with $i = \{1, \ldots, 6\}$. In order for us to get the value of the coefficients, we apply the method of undetermined coefficients (see [6]). After doing this, we got a system of algebraic equations. After solving it, we substitute the result in equation (32) to obtain the Feedback Nash Equilibrium. ☐

To obtain an analytically solution, we substitute equations (34) and (35) in equation (32):

$$r_j^* = \frac{2\left[a\left(c_3 r_f + c_6 M + c_2\right) - s\left(c_6 r_f + c_5 M + c_4\right)\right]}{4(\beta+\mu+z)} + \frac{2\left[(\alpha+\theta)r_f + \kappa r_{j-1} - w\right] + (\eta+2\nu)M}{4(\beta+\mu+z)} \quad (36)$$

5. Comparison of Scenarios

The goal of this section is to compare the control variables under the three different scenarios. As all the games are symmetric, we show the control variables for bank $j = \{1, 2\}$. From scenario 1, equation (15) states:

$$r_j^* = \frac{-2aV_j'(r_f) + 2(\alpha+\theta)r_f(t) + (\eta+2\nu)M + 2\kappa r_{j-1} - 2w}{4(\beta+\mu+z)}$$

From scenario 2 (24):

$$r_j^* = \frac{-2a\lambda_1 + 2(\alpha+\theta)r_f + (\eta+2\nu)M - 2r_{j-1}(\beta+\mu) - 2\lambda_2 s - 2w}{4(z-\kappa)}$$

From scenario 3 (36):

$$r_j^* = \frac{2aV_j'(r_f) + 2(\alpha+\theta)r_f - 2sV_j'(M) + (\eta+2\nu)M + 2\kappa r_{j-1} - 2w}{4(\beta+\mu+z)}$$

The control variable of scenario 1 and scenario 3 are similar; both were solved using HJB equations. The difference relies in the term $-2sV_j'(M)$ which measures the gain that a player can get from any position of the game. This term was obtained when the bank takes into account the macroeconomic environment before taking the decision of setting the level of spread interest rates. This term can change the control variable, thus scenario 2 captures more precisely the level of spread that each bank has to choose.

As long as $\kappa < 0$ claim 1 remains true, for scenarios 1 and 3, and for scenario 2 it is explicit the inverse relation between the control variables of both banks, when $N = 2$. Comparing scenario 2 with 3, when both take into account the macroeconomic environment, we can see that the optimal controls are quite similar. The difference is that one was solve via Maximum principle and the another using HJB equations. The adjoint equations and the value function play key roles in the solution of control problems with the Maximum principle and HJB equations, respectively. Moreover under certain conditions, for example, if the value function is smooth enough, then the adjoint equation and the value function are closely related ([16]).

6. A Stochastic Differential Game

In this section our interest relies in modeling a game where the state variables are driven by stochastic processes, in particular Ito-like processes. Thus, the ordinary differential equations of the previous sections become stochastic differential equations. In order to have a solvable stochastic differential game, we will consider only one state equation; the benchmark interest rate. In Financial Theory, interest rates model are key for most of the models. [12] and [14] state that the interest rates behave different in comparison with the prices of the stocks and have special features like: they are positive and bounded. One of the first models of interest rates was introduced by Vasicek ([12]), it is a one factor model that captures the mean reverting property of interest rates. The solution of the model is given by the process $(r_t)_{t\in\mathbb{R}_+}$ that solves the stochastic differential equation $dr_t = \beta(\alpha - r_t)dt + \sigma dB_t$ where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian Motion.

6.1. The Duopoly Stochastic Differential Game

We assume now that dynamic system of the game is one-dimensional, and the the benchmark interest rate is driven by a stochastic process, a Vasicek like short-rate model.

$$dr_f = a[\theta(r_1 + r_2) - r_f]dt + \sigma(r_f, t)dW \tag{37}$$

where dW is a Brownian motion, a is the speed of adjustment, $\theta > 0$ is a proportion parameter and $\sigma(r_f, t)$ is the uncertainty associated with the interest rate; being this the risk factor. Assuming the same functional forms of previous sections, the game that both banks have to solve is the following:

$$\underset{r_j\in U_j}{\text{Maximize}}\ \Pi_j = \mathbf{E}\int_{t=0}^{\infty} e^{-\rho_j t}\left\{r_j[L(\cdot) - D(\cdot) - C(\cdot)] - I(\cdot)r_f\right\}dt \qquad j = \{1, 2\} \tag{38}$$

subject to the stochastic differential equation (37) and

$$r_f(0) = r_f^0 \geq 0 \tag{39}$$

$$\sigma(r_f, t) = \sigma^2 (r_f)^2 \tag{40}$$

$$r_j \in U_j \tag{41}$$

6.1.1. Stochastic Markovian Nash Equilibrium

The mathematical tool we will use for solving the game is the stochastic HJB equation, the interested reader can see [17] for formal proofs of this method and its applications.

Proposition 9. *The symmetric stochastic Markovian Nash equilibrium is given by:*

$$\Gamma^*(r_f, t) = -\frac{-r_f(t)\left(-ac_3 + \alpha + \theta\right) + ac_2 + w}{\beta - 2\kappa + \mu + 2z} \tag{42}$$

where we assume that $M = 0$, i.e., M is a constant so it can be zero, also a suitable economic interpretation is that we have steady macroeconomic conditions.

Proof. We start by writing the HJB equation for player $j = \{1, 2\}$

$$\rho V(r_f) = \max_{r_j \in U_j} \{ r_j [L(\cdot) - D(\cdot) - C(\cdot)] - I(\cdot) r_f + V'(r_f) \left[a(r_1 + r_2 - r_f) \right] + \frac{1}{2}\sigma^2 r_f(t)^2 V''(r_f) \} \tag{43}$$

The next steps are to state the first order conditions and to solve for the control variable:

$$r_j = \frac{-2aV'(r_f) + 2(\alpha + \theta) r_f(t) + (\eta + 2\nu)M - 2r_{j-1}(\beta + \mu) - 2w}{4(z - \kappa)} \tag{44}$$

We assume that both banks have the same space control, we look for a symmetric solution, thus $r_j = r_{j-1}$:

$$r_j = \frac{-2aV'(r_f) + 2(\alpha + \theta) r_f(t) + (\eta + 2\nu)M - 2w}{2(\beta - 2\kappa + \mu + 2z)} \tag{45}$$

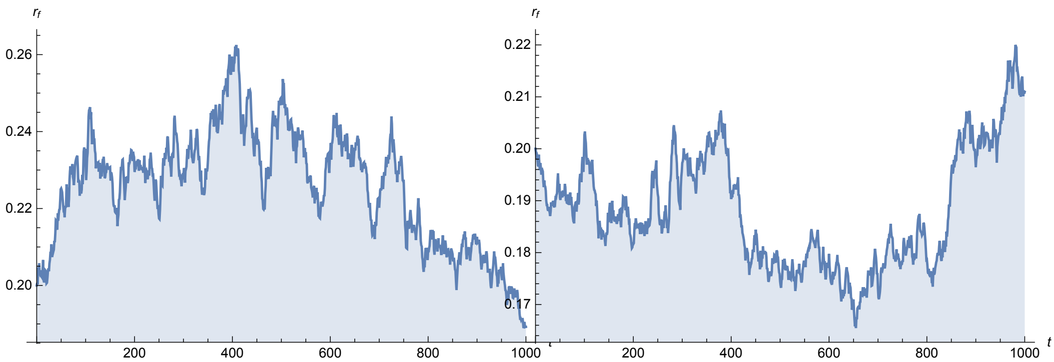
The game has a particular structure similar to a LQ system, therefore the value function has the form:

$$V(r_f) = c_1 + c_2 r_f + \frac{1}{2} c_3 (r_f)^2 \tag{46}$$

$$V'(r_f) = c_2 + c_3 r_f \tag{47}$$

After substituting equations (47) and (45) into the RHS, given by (43),then by usung the undetermined coefficients method, we obtain the values of $c_i \quad i = \{1, 2, 3\}$ $\square$

From proposition 9 we can see that the control variable is quit similar to one obtained in section 1, equation (15). The difference is that the optimal trajectory of r_f is driven by a diffusion process. If we assume a Ito process then we can simulate the optimal trajectories with the help of the software Wolfram Mathematica.



(a) (b)

Figure 2: Optimal trajectories of r_f with arbitrary parameters

With an appropriate choice of the parameters $a, \alpha, \beta, \theta, \kappa, \mu, \sigma, w, z$, we can observe from Figures 2 that the optimal trajectories do not get negative values, even when it is similar to the Vasicek short-rate model.

7. Conclusions

In this paper we have modeled a non-cooperative game where two banks compete for granting credits and taking deposits. In the deterministic case, the system dynamics is given by two ordinary differential equations, while in the stochastic case, the system dynamics is given by a stochastic differential equations.

The main objective was to analyze the incentives that a bank has for granting credits for starting new projects. We found that in a duopoly game, the level of spread interest rate set by a bank depends on the state of the system, and the rate set by the competitors by a factor κ. The latter can take positive or negative values, when it is negative we call it a competitive factor, if it is positive is a collusion factor.

The analysis was made with different mathematical tools in order to compare them, we conclude that under certain assumption about the value function and adjoint equations the result can be similar. Both banks follow Markov decision processes, that was the motivation for modeling the game using stochastic differential equation, and the results were quite similar.

Finally, the model presented here can be extended in many ways, for example, instead of taking the factor κ as a constant it can be a function of time and the state dynamics. Also, we can add Poisson jumps and Markov regime switching to the stochastic differential equation of the short rate; see, for instance, [17] and [13].

References

[1] Brunnermeir, Markus K., Deciphring the Liquidity and Credit Crunch 2007-2008, *Journal of Economic Perspectives*, **81** (2009), 77-100.

[2] Docker, E. and Jorgense, S. and Van Long, N. and Sorger G., *Differential Games in Economics and Management Science*, Cambridge University Press, (2000).

[3] Engwerda, Jacob, *LQ Dynamic Optimization and Differential Games*, John Wiley and Sons, LTD, (2005).

[4] Freixas, Xavier and Rochet, Jean-Carlos, *Microeconomics of Banking*, MIT press, (2008).

[5] Gambacorta, Leonardo, How do Banks Set Interest Rates?, *National Bureau of Economic Research*,(2004).

[6] Haurie, Alain and Krawczyk, Jacek B. and Zaccour, Georges, *Games and Dynamic Games*, World Scientific, (2012).

[7] Hernández-Lerma, O. and Lasserre, J.B., *Discrete-Time Markov Control Processes: Basic Optimality Criteria*, Springer Science+Business Media New York, (1996).

[8] Hernández-del-Valle, A., C. I. Martínez-Garcia, and F. Venegas-Martínez, Sovereign Default in a Currency Area: A Monetary General Equilibrium Model, *International Journal of Pure and Applied Mathematics*, **Vol. 108, No. 2** (2016), 227-261.

[9] Hernández-del-Valle, A., C. I. Martínez-Garcia, and F. Venegas-Martínez, Equilibrium Characteristics of Sovereign Default in a Two Country Currency Area: A Two Player Dynamic Hierarchical Game, *Communications in Applied Analysis*, **Vol. 21, No. 2** (2017), 187-216.

[10] Mishkin, Frederic S., *The Economics of Money, Banking, and Financial Markets*, Addison-Wesley, (2004).

[11] Osborne, Martin J. and Rubinstein, Ariel, *A Course in Game Theory*, The MIT Press, (1994).

[12] Privault, Nicolas, *An Elementary Introduction to Stochastic Interest Rate Modelling*, World Scientific, (2012).

[13] Vallejo-Jiménez, B, and F. Venegas-Martínez, Optimal Consumption and Portfolio Rules When the Asset Price is Driven by a Time-Inhomogeneous Markov Modulated Fractional Brownian Motion with Multiple Poisson Jumps, *Economics Bulletin*, **Vol. 37, No. 1** (2017, 314-326.

[14] Venegas-Martínez, Francisco, *Riesgos Financieros y Económicos, Productos derivados y decisiones económicas bajo incertidumbre*, Cengage Learning, (2008).

[15] Venegas-Martínez, F. and Hernández-Lerma, Onésimo, Toma de decisiones de agentes racionales con procesos markvovianos: avances recientes en economía y finanzas, *El Trimestre Económico*, **81** (2012), 733-779.

[16] Yong, J. and Zhou, X.Y., *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag New York, (1999).

[17] Zhang, Cheng-ke and Zhu, Huai-nian and Zhou, Hai-ying and Bin, Ning, *Non-cooperative Stochastic Differential Game Theory of Generalized Markov Jump Linear Systems*, Springer, (2017).