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BI-IDEALS IN Γ-SO-RINGS

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Abstract: A Γ-so-ring is a structure possessing a natural partial ordering, an infinitary partial addition and a ternary multiplication, subject to a set of axioms. The partial functions under disjoint-domain sums and functional composition is a Γ-so-ring. In this paper we introduce the notions of partial bi-ideal and bi-ideal of Γ-so-ring and we obtain the characteristics of them.

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1. Introduction

Partially defined infinitary operations occur in the contexts ranging from integration theory to programming language semantics. The general cardinal algebras studied by Tarski in 1949, Housdorff topological commutative groups studied by Bourbaki in 1966, Σ-structures studied by Higgs in 1980, sum or-

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dered partial monoids & sum ordered partial semirings studied by Arbib, Manes, Benson[3, 6] and Streenstrup[17] are some of the algebraic structures of the above type.

M. Murali Krishna Rao[9] in 1995 introduced the notion of a Γ-semiring as a generalization of semirings and Γ-rings, and extended many fundamental results of semirings and Γ-rings to Γ-semirings. In [10, 11] we introduced the notion of Γ-so-ring, obtained a necessary and sufficient condition for the quotient R/θ to be a Γ/σ-so-ring, where (θ, σ) is a congruence relation on (R, Γ) and (ϕ, ρ)-representation of Γ-so-rings. In [12, 13, 14, 15, 16], we introduced the notions of an ideal, a prime ideal, a semiprime ideal and an irreducible ideal in a Γ-so-ring R and obtained many characteristics of them in R. As a continuation, in this paper we introduce the notions of partial bi-ideal and bi-ideal of Γ-so-ring and we generalize the results of Kaushik and Moin Khan [5] to Γ-so-rings.

2. Preliminaries

In this section we collect important definitions and results from [6, 10, 12, 13, 14, 15, 17].

Definition 2.1. [6] A partial monoid is a pair (M, Σ) where M is a nonempty set and Σ is a partial addition defined on some, but not necessarily all families $(x_i : i \in I)$ in M subject to the following axioms:

(i) Unary Sum Axiom. If $(x_i : i \in I)$ is a one element family in M and $I = \{j\}$, then $\Sigma(x_i : i \in I)$ is defined and equals x_j.

(ii) Partition-Associativity Axiom. If $(x_i : i \in I)$ is a family in M and $(I_j : j \in J)$ is a partition of I, then $(x_i : i \in I)$ is summable if and only if $(x_i : i \in I_j)$ is summable for every j in J, $(\Sigma(x_i : i \in I_j) : j \in J)$ is summable, and $\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j) : j \in J)$.

Definition 2.2. [10] Let (R, Σ) and (Γ, Σ') be two partial monoids. Then R is said to be a partial Γ-semiring if there exists a mapping $R \times \Gamma \times R \longrightarrow R$ (images to be denoted by $x\gamma y$ for $x, y \in R$ and $\gamma \in \Gamma$) satisfying the following axioms:

(i) $x\gamma(y\mu z) = (x\gamma y)\mu z$,

(ii) a family $(x_i : i \in I)$ is summable in R implies $(x\gamma x_i : i \in I)$ is summable in R and $x\gamma[\Sigma(x_i : i \in I)] = \Sigma(x\gamma x_i : i \in I)$,

(iii) a family $(x_i : i \in I)$ is summable in R implies $(x_i\gamma x : i \in I)$ is summable in R and $[\Sigma(x_i : i \in I)]\gamma x = \Sigma(x_i\gamma x : i \in I)$,

(iv) a family $(\gamma_i : i \in I)$ is summable in Γ implies $(x\gamma_i y : i \in I)$ is summable in R and $x[\Sigma'(\gamma_i : i \in I)]y = \Sigma(x\gamma_i y : i \in I)$ for all $x, y, z, (x_i : i \in I)$ in R and

$\mu, \gamma, (\gamma_i : i \in I)$ in Γ.

Definition 2.3. [10] A nonempty subset A of a partial Γ-semiring R is said to be *partial* Γ*-subsemiring* if

(i) A is a partial submonoid of R, and

(ii) $A\Gamma A \subseteq A$.

Definition 2.4. [17] The *sum ordering* $\leq$ on a partial monoid (M, Σ) is the binary relation such that $x \leq y$ if and only if there exists a h in M such that $y = x + h$ for $x, y \in M$.

Definition 2.5. [17] A *sum-ordered partial monoid* or *so-monoid*, in short, is a partial monoid in which the sum ordering is a partial ordering.

Definition 2.6. [10] A partial Γ-semiring R is said be a sum-ordered partial Γ-semiring (in short Γ-so-ring) if the partial monoids R and Γ are so-monoids.

Definition 2.7. [10] A nonempty subset A of a Γ-so-ring R is said to be Γ*-subso-ring* if

(i) A is a partial subso-monoid of R, and

(ii) $A\Gamma A \subseteq A$.

Definition 2.8. [12] Let R be a partial Γ-semiring, A be a nonempty subset of R and Ω be a nonempty subset of Γ. Then the pair (A, Ω) of (R, Γ) is said to be a left (right) partial Γ-ideal of R if it satisfies the following:

(i) $(x_i : i \in I)$ is a summable family in R and $x_i \in A\ \forall i \in I$ implies $\Sigma_i x_i \in A$,

(ii) $(\alpha_i : i \in I)$ is a summable family in Γ and $\alpha_i \in \Omega\ \forall i \in I$ implies $\Sigma_i \alpha_i \in \Omega$, and

(iii) for all $x \in R,\ y \in A$ and $\alpha \in \Omega$, $x\alpha y \in A$ $(y\alpha x \in A)$.

If (A, Ω) is both left and right partial Γ-ideal of a partial Γ-semiring R, then (A, Ω) is called a *partial* Γ*-ideal* of R. If $\Omega = \Gamma$, then A is called a *partial ideal* of R.

Definition 2.9. [12] Let R be a Γ-so-ring, A be a nonempty subset of R and Ω be a nonempty subset of Γ. Then the pair (A, Ω) is said to be a left (right) Γ-ideal of R if it satisfies the following:

(i) (A, Ω) is a left (right) partial Γ-ideal of R,

(ii) $x \in R$ and $y \in A$ such that $x \leq y$ implies $x \in A$, and

(iii) $\alpha \in \Gamma$ and $\beta \in \Omega$ such that $\alpha \leq \beta$ implies $\alpha \in \Omega$.

If (A, Ω) is both left and right Γ-ideal of a Γ-so-ring R, then (A, Ω) is called a Γ*-ideal* of R. If $\Omega = \Gamma$, then A is called an *ideal* of (R, Γ).

Definition 2.10. [12] Let R be a Γ-so-ring. If A, B are subsets of R and Γ_1 is a subset of Γ, define $A\Gamma_1 B$ as the set $\{x \in R \mid \exists\ a_i \in A, \gamma_i \in \Gamma_1, b_i \in B, \Sigma_i a_i\gamma_i b_i$ exists and $x \leq \Sigma_i a_i\gamma_i b_i\}$.

If $A = \{a\}$ then we also denote $A\Gamma_1 B$ by $a\Gamma_1 B$. If $B = \{b\}$ then we also denote $A\Gamma_1 B$ by $A\Gamma_1 b$. Similarly if $A = \{a\}$ and $B = \{b\}$, we denote $A\Gamma_1 B$ by $a\Gamma_1 b$ and thus $a\Gamma_1 b = \{x \in R \mid x \leq a\gamma b$ for some $\gamma \in \Gamma_1\}$.

Theorem 2.11. *[12] Let R be a complete Γ-so-ring. If A and B are ideals of R then $A\Gamma B$ is an ideal of R. Moreover, $A\Gamma B \subset A \bigcap B$.*

3. Bi-Ideals

We introduce the notions of a partial bi-ideal in partial Γ-semiring and a bi-ideal in Γ-so-ring as follows:

Definition 3.1. Let R be a partial Γ-semiring. A partial Γ-subsemiring B of R is said to be a *partial bi-ideal* of (R, Γ) if and only if $B\Gamma R\Gamma B \subseteq B$.

Definition 3.2. Let R be a Γ-so-ring. A Γ-subso-ring B of R is said to be a *bi-ideal* of (R, Γ) if and only if $B\Gamma R\Gamma B \subseteq B$.

Observation 3.3. If B is a left/right/bothsided ideal of a Γ-so-ring R, then B is a bi-ideal of (R, Γ).

The following is an example of a Γ-so-ring in which a bi-ideal of (R, Γ) is not an ideal of (R, Γ).

Example 3.4. Let $\mathbb{Z}^-$ be the set of all negative integers. Let $R = \Gamma = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}^- \bigcup \{0\}\}$. Then R (Γ) is a so-monoid with finite support of usual matrix addition. Moreover, R is a Γ-so-ring with usual matrix multiplication. Take $B = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}^- \bigcup \{0\}\}$. Then B is a bi-ideal of (R, Γ) but not an ideal of (R, Γ). Since

$$\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -4 & -3 \end{pmatrix} = \begin{pmatrix} -12 & -10 \\ 0 & 0 \end{pmatrix} \notin B.$$

Theorem 3.5. *Let R be a Γ-so-ring. Then intersection of right and left ideals of R is a bi-ideal of R.*

Proof. Let A be a right ideal and B be a left ideals of R. Since

$$(A \bigcap B)\Gamma R\Gamma(A \bigcap B) \subseteq A\Gamma R\Gamma A \subseteq A\Gamma R$$

and A is a right ideal of R, we have $(A\bigcap B)\Gamma R\Gamma(A\bigcap B)\subseteq A$. Since

$$(A\bigcap B)\Gamma R\Gamma(A\bigcap B)\subseteq B\Gamma R\Gamma B\subseteq R\Gamma B$$

and B is a left ideal of R, we have $(A\bigcap B)\Gamma R\Gamma(A\bigcap B)\subseteq B$. Hence

$$(A\bigcap B)\Gamma R\Gamma(A\bigcap B)\subseteq A\bigcap B.$$

Therefore $A\bigcap B$ is a bi-ideal of R. □

Theorem 3.6. *Let R be a Γ-so-ring and $\{B_i \mid i\in I\}$ be a family of bi-ideals of (R,Γ). Then $B=\bigcap_{i\in I}B_i$ is also a bi-ideal of (R,Γ).*

Proof. Let $(x_j : j\in J)$ be a summable family in R and each $x_j\in B$ for $j\in J$. Then $\Sigma_j x_j$ exists in R and each $x_j\in B=\bigcap_{i\in I}B_i$ for $j\in J$. $\Rightarrow x_j\in B_i\ \forall\ i\in I$, $j\in J$. $\Rightarrow \Sigma_j x_j\in B_i\ \forall\ i\in I$. $\Rightarrow \Sigma_j x_j\in\bigcap_{i\in I}B_i=B$. Let $x\in B\Gamma B$. Then $x\le\Sigma_j b_j\alpha_j b'_j$ (if exists) where $b_j,b'_j\in B$ and $\alpha_j\in\Gamma$, $j\in J$. $\Rightarrow x\le\Sigma_j b_j\alpha_j b'_j$ where $b_j,b'_j\in B_i\ \forall\ i\in I$ and $\alpha_j\in\Gamma$, $j\in J$. $\Rightarrow x\in B_i\Gamma B_i\ \forall\ i\in I$. Since, for each $i\in I$, B_i is a bi-ideal of (R,Γ), $B_i\Gamma B_i\subseteq B_i\ \forall\ i\in I$. Thus $x\in B_i\ \forall\ i\in I$. $\Rightarrow x\in\bigcap_{i\in I}B_i=B$. Therefore B is a Γ-subso-ring of R.

To prove B is a bi-ideal of (R,Γ), let $x\in B\Gamma R\Gamma B$. Then $\exists\ b_i,c_i\in B$, $r_i\in R$ and $\alpha_i,\beta_i\in\Gamma$ such that $\Sigma_i b_i\alpha_i r_i\beta_i c_i$ exists in R and $x\le\Sigma_i b_i\alpha_i r_i\beta_i c_i$. $\Rightarrow b_i,c_i\in B_i\ \forall\ i\in I$, $r_i\in R$ and $\alpha_i,\beta_i\in\Gamma$ such that $x\le\Sigma_i b_i\alpha_i r_i\beta_i c_i$. $\Rightarrow x\in B_i\Gamma R\Gamma B_i\ \forall\ i\in I$. Since B_i is a bi-ideal of $(R,\Gamma)\ \forall\ i\in I$, $x\in B_i\ \forall\ i\in I$. $\Rightarrow x\in\bigcap_{i\in I}B_i=B$. Therefore $B\Gamma R\Gamma B\subseteq B$. Hence $B=\bigcap_{i\in I}B_i$ is a bi-ideal of (R,Γ). □

Corollary 3.7. *Let R be a Γ-so-ring, I be an ideal and B be a bi-ideal of (R,Γ). Then $B\bigcap I$ is a bi-ideal of (R,Γ).*

Proof. Since every ideal is a bi-ideal of (R,Γ), by the Theorem 3.6, $B\bigcap I$ is a bi-ideal of (R,Γ). □

The following example illustrates that the set union of two bi-ideals in a Γ-so-ring need not be a bi-ideal.

Example 3.8. Let $R = \{0, a, b, c, d, e\}$. Define Σ on R as

$$\Sigma_i x_i = \begin{cases} x_j, & \text{if } x_i = 0\ \forall i \neq j, \text{ for some } j, \\ d, & \text{if } (x_j = a,\ x_k = b \text{ or } x_j = b,\ x_k = c \text{ for some } j,\ k) \\ & \text{and } x_i = 0\ \forall i \neq j, k, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Then R is a so-monoid.

Let $\Gamma = \{0', 1'\}$. Define Σ' on Γ as

$$\Sigma'_i \alpha_i = \begin{cases} 1',\ if\ \alpha_i = 0'\ \forall i \neq j\ for\ some\ j \\ undefined,\ otherwise. \end{cases}$$

Then Γ is a so-monoid.
Define a mapping $R \times \Gamma \times R \to R$ as follows:

$0'$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	0	0
d	0	0	0	0	0	0
e	0	0	0	0	0	0

$1'$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	0	d
e	0	a	b	c	d	e

Then R is a Γ-so-ring and $\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, a, b, c, d\}$, R are bi-ideals of (R, Γ). Now $\{0, a\} \cup \{0, b\} = \{0, a, b\}$ is not a bi-ideal of (R, Γ), since $a + b = d$ which is not in $\{0, a, b\}$.

Theorem 3.9. *Let R be a Γ-so-ring. Let B be a bi-ideal of (R, Γ) and A be a nonempty subset of R. Then the sets $B\Gamma A$ and $A\Gamma B$ are bi-ideals of (R, Γ).*

Proof. Note that $B\Gamma A = \{x \in R \mid \exists\ b_i \in B, \alpha_i \in \Gamma, a_i \in A, \Sigma_i b_i \alpha_i a_i$ exists and $x \leq \Sigma_i b_i \alpha_i a_i\}$. Since $0 \in B\Gamma A$, $B\Gamma A$ is a nonempty subset of R. Let $(x_i : i \in I)$ be a summable family in R and $x_i \in B\Gamma A$, $i \in I$. Then each $x_i \leq \Sigma_{i_j} b_{i_j} \alpha_{i_j} a_{i_j}$, where $b_{i_j} \in B$, $\alpha_{i_j} \in \Gamma$, $a_{i_j} \in A$, $j \in J$, $i \in I$. $\Rightarrow \Sigma_i x_i \leq \Sigma_i \Sigma_{i_j} b_{i_j} \alpha_{i_j} a_{i_j}$. $\Rightarrow \Sigma_i x_i \in B\Gamma A$. Let $x \in (B\Gamma A)\Gamma(B\Gamma A)$. Then $x \leq \Sigma_i b_i \alpha_i a_i \alpha'_i b'_i \alpha''_i a'_i$, where $b_i, b'_i \in B$, $\alpha_i, \alpha'_i, \alpha''_i \in \Gamma, a_i, a'_i \in A$, $i \in I$. Now $b_i, b'_i \in B$, $\alpha_i, \alpha'_i \in \Gamma, a_i \in A \subseteq R$, we have $b_i \alpha_i a_i \alpha'_i b'_i \in B\Gamma R\Gamma B$. Since B is a bi-ideal of (R, Γ), $B\Gamma R\Gamma B \subseteq B$ and so each $b_i \alpha_i a_i \alpha'_i b'_i \in B\ \forall\ i \in I$, so $x \in B\Gamma A$. Therefore $B\Gamma A$ is a Γ-subso-ring of R. Since B is a bi-ideal of

(R,Γ), $B\Gamma R\Gamma B \subseteq B$. Consider $B\Gamma(A\Gamma R)\Gamma B \subseteq B\Gamma R\Gamma B$ as $A\Gamma R \subseteq R$. So $B\Gamma A\Gamma R\Gamma B \subseteq B$. $\Rightarrow (B\Gamma A\Gamma R\Gamma B)\Gamma A \subseteq B\Gamma A$. $\Rightarrow (B\Gamma A)\Gamma R\Gamma(B\Gamma A) \subseteq B\Gamma A$. Hence $B\Gamma A$ is a bi-ideal of (R,Γ). Similarly we can prove that $A\Gamma B$ is a bi-ideal of (R,Γ). □

Theorem 3.10. *Let R be a Γ-so-ring and A, B be bi-ideals of (R,Γ). Then the set $A\Gamma B$ is a bi-ideal of (R,Γ).*

Proof. Since $A \subseteq R$ and B is a bi-ideal of (R,Γ), by the Theorem 3.9, $A\Gamma B$ is a bi-ideal of (R,Γ). □

Theorem 3.11. *If B is a bi-ideal of a Γ-so-ring R then $b\Gamma B\Gamma c$ is a bi-ideal of R for any b, $c \in R$.*

Proof. Consider $(b\Gamma B\Gamma c)\Gamma R\Gamma(b\Gamma B\Gamma C) \subseteq b\Gamma(B\Gamma R\Gamma B)\Gamma c \subseteq b\Gamma B\Gamma c$ (since B is a bi-ideal of R). Hence $b\Gamma B\Gamma c$ is a bi-ideal of R. □

Note that if A, B are ideals of a Γ-so-ring R, then $A\Gamma B \subseteq A\bigcap B$ as in [12]. However this result need not be true when A, B are bi-ideals. It is illustrated in the following example.

Example 3.12. Consider the Γ-so-ring R as in the Example 3.4. Take $A = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}^-\bigcup\{0\}\}$ and $B = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z}^-\bigcup\{0\}\}$. Then A, B are bi-ideals of (R,Γ) such that $A\bigcap B = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$ and $A\Gamma B = \{\begin{pmatrix} 0 & axb \\ 0 & 0 \end{pmatrix} \mid a,x,b \in \mathbb{Z}^-\bigcup\{0\}\}$. Hence $A\Gamma B \nsubseteq A\bigcap B$.

The following is an example of a Γ-so-ring R in which $A\bigcap B$ need not be contained in $A\Gamma B$, where A, B are bi-ideals of (R,Γ).

Example 3.13. Let $R = \{0,u,v,x,y,z\}$. Define Σ on R as

$$\Sigma_i x_i = \begin{cases} x_j, \ if \ x_i = 0 \ \forall i \neq j, \ for \ some \ j, \\ undefined, \ otherwise. \end{cases}$$

Then R is a so-monoid.
Let $\Gamma = \{0', 1'\}$. Define Σ' on Γ as

$$\Sigma'_i \alpha_i = \begin{cases} 1', \ if \ \alpha_i = 0' \ \forall i \neq j \ for \ some \ j \\ undefined, \ otherwise. \end{cases}$$

Then Γ is a so-monoid.
Define a mapping $R \times \Gamma \times R \to R$ as follows:

$0'$	0	u	v	x	y	z
0	0	0	0	0	0	0
u	0	0	0	0	0	0
v	0	0	0	0	0	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0
z	0	0	0	0	0	0

$1'$	0	u	v	x	y	z
0	0	0	0	0	0	0
u	0	u	0	0	0	u
v	0	0	v	0	0	v
x	0	0	0	0	0	x
y	0	0	0	0	0	y
z	0	u	v	x	y	z

Then R is a Γ-so-ring. Consider the bi-ideals $A = \{0, x, y\}$, $B = \{0, u, x\}$ of R. Then $A \bigcap B = \{0, x\}$ whereas $A\Gamma B = \{0\}$.

Definition 3.14. Let R be a Γ-so-ring. Then we define the following:
(i) $(0 : R)_l = \{x \in R \mid x\Gamma R = 0\}$,
(ii) $(0 : R)_r = \{x \in R \mid R\Gamma x = 0\}$,
(iii) $(0 : R)_m = \{x \in R \mid R\Gamma x\Gamma R = 0\}$, and
(iv) $(0 : R)_b = \{x \in R \mid x\Gamma R\Gamma x = 0\}$.

Theorem 3.15. *Let R be a Γ-so-ring. Then the subsets $(0 : R)_l$, $(0 : R)_r$, $(0 : R)_m$ and $(0 : R)_b$ are bi-ideals of (R, Γ).*

Proof. We prove that $(0 : R)_l$ is a bi-ideal of (R, Γ). Since $0\Gamma R = 0$, $(0 : R)_l$ is a nonempty subset of R. Let $(x_i : i \in I)$ be a summable family in R and $x_i \in (0 : R)_l$, $i \in I$. Then $\Sigma_i x_i$ exists in R and $x_i\Gamma R = 0$, $i \in I$. We prove that $(\Sigma_i x_i)\Gamma R = 0$. Let $y \in (\Sigma_i x_i)\Gamma R$. Then $y \leq \Sigma_j(\Sigma_i x_i)\alpha_j r_j$ for $\alpha_j\Gamma$ and $r_j \in R$. $\Rightarrow y \leq \Sigma_j\Sigma_i(x_i\alpha_j r_j) = 0$ and so $y = 0$. Therefore $(\Sigma_i x_i)\Gamma R = 0$. Hence $\Sigma_i x_i \in (0 : R)_l$. Let $y \in (0 : R)_l\Gamma(0 : R)_l$. Then $y \leq \Sigma_i x_i\alpha_i y_i$, where $x_i, y_i \in (0 : R)_l$, $\alpha_i \in \Gamma$, $i \in I$. Since $x_i, y_i \in (0 : R)_l$, $x_i\Gamma R = 0$ and $y_i\Gamma R = 0$, $i \in I$. Now we prove that $y\Gamma R = 0$. Let $x \in y\Gamma R$. Then $x \leq \Sigma_j y\gamma_j s_j$ for some $\gamma_j \in \Gamma$, $s_j \in R$. $\Rightarrow x \leq \Sigma_j(\Sigma_i x_i\alpha_i y_i)\gamma_j s_j = \Sigma_j\Sigma_i x_i\alpha_i(y_i\gamma_j s_j) = 0$ and hence $y\Gamma R = 0$. $\Rightarrow y \in (0 : R)_l$. Thus $(0 : R)_l\Gamma(0 : R)_l \subseteq (0 : R)_l$. Therefore $(0 : R)_l$ is a Γ-subso-ring of R. Let $y \in (0 : R)_l\Gamma R\Gamma(0 : R)_l$. Then $y \leq \Sigma_i x_i\alpha_i r_i\beta_i y_i$, where $x_i, y_i \in (0 : R)_l$, $\alpha_i, \beta_i \in \Gamma$, $r_i \in R$, $i \in I$. Since $x_i, y_i \in (0 : R)_l$, $x_i\Gamma R = 0$ and $y_i\Gamma R = 0$, $i \in I$. Now we prove that $y\Gamma R = 0$. Let $x \in y\Gamma R$. Then $x \leq \Sigma_j y\gamma_j s_j$ for some $\gamma_j \in \Gamma$, $s_j \in R$. $\Rightarrow$ $x \leq \Sigma_j(\Sigma_i x_i\alpha_i r_i\beta_i y_i)\gamma_j s_j = \Sigma_j\Sigma_i x_i\alpha_i(r_i\beta_i y_i\gamma_j s_j) = 0$ and hence $y\Gamma R = 0$. $\Rightarrow$ $y \in (0 : R)_l$. Therefore $(0 : R)_l\Gamma R\Gamma(0 : R)_l \subseteq (0 : R)_l$. Hence $(0 : R)_l$ is a bi-ideal of (R, Γ). Similarly we can prove that $(0 : R)_r$, $(0 : R)_m$ and $(0 : R)_b$ are bi-ideals of (R, Γ). □

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