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RELATION BETWEEN MCPHERSON NUMBER AND BLOTTING NUMBER OF GRAPHS

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Abstract: The concept of the *McPherson number* denoted by $\Upsilon(G)$, of a simple, undirected, finite, connected graph G on n vertices was introduced in 2015. The initial notion resulted in a grah G which recursively evolved into a mixed complete graph. Hence, a graph having both edges and arcs resulted. Recently the notion of the *blotting number* denoted, $\mathfrak{b}(G)$ of a graph G was introduced. This prompted the thinking that the *McPherson recursion* could rather result in edges instead of arcs, and it was found that $\mathfrak{b}(G) = \Upsilon(\overline{G})$.

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1. Introduction

For general notation and concepts in graphs and digraphs see [1, 2]. We will write that a graph G has order $\nu(G) = n \geq 1$ and size $\varepsilon(G) = p \geq 0$ with minimum and maximum degree $\delta(G)$ and $\Delta(G)$, respectively. The degree of a vertex $v \in V(G)$ is denoted $d_G(v)$ or when the context is clear, simply as $d(v)$. Unless mentioned otherwise all graphs G are simple, undirected, finite, connected graphs. An edgeless graph on n vertices is called a n-null graph denoted, $\mathfrak{N}_n$.

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The concept of the *McPherson number*[1] denoted by $\Upsilon(G)$, of a graph G on n vertices was introduced in [3]. Further results were published in [4]. The recursive concept, called the *McPherson recursion*, is a series of *vertex explosions* such that on the first iteration a vertex $v \in V(G)$ explodes to arc (directed edges) to all vertices $u \in V(G)$ for which the edge $vu \notin E(G)$, to obtain the mixed graph G_1. Now G_1 is considered on the second iteration and a vertex $w \in V(G_1) = V(G)$ may explode to arc to all vertices $z \in V(G_1)$ if edge $wz \notin E(G)$ and arc (w, z) or $(z, w) \notin E(G_1)$. The *McPherson number* of a simple connected graph G is the minimum number of iterative vertex explosions say ℓ, to obtain the mixed graph G_ℓ such that the underlying graph $G^*_\ell \simeq K_n$.

It is easy to see that the total number of arcs created is $\varepsilon(K_n) - \varepsilon(G)$. It is equally easy to see that $\Upsilon(K_n) = 0$ and $\Upsilon(K_n - uv)_{uv \in E(K_n)} = 1$. It is not that easy to see that the sequence of vertex explosions does not generally obey the commutative law.

An application of the McPherson number is determining the minimum number and the location of the corresponding vertices (points) in a graph (network) to set up bulk logistical depots from which operations are managed to establish edges (links) to ensure complete connectivity (direct functionality) between all pairs of vertices (points). This approach assists in minimising various management, logistical and other operational requirements.

Recently, Susanth et. al [5] introduced the notion of the *blotting number* of a graph G denoted by $\mathfrak{b}(G)$. *Effacement* of a vertex $v \in V(G)$ of a graph G means the removal of all edges incident with vertex v. The blotting number $\mathfrak{b}(G)$ is the minimum number of vertices of G to be effaced so that the resultant graph is a n-null graph. We call any such minimum set of vertices a *blotting set* of G and denote such set, $\mathfrak{B} = \{v_1, v_2, v_3, \dots, v_\mathfrak{b}\}$ or for brevity a $\mathfrak{B}$-set. Note that $\mathfrak{b}(\mathfrak{N}_n) = 0$.

This prompted the thinking that the *McPherson recursion* could rather result in edges instead of arcs. *Facement* of a vertex $v \in V(G)$, $d(v) < n - 1$ of graph G means the addition of edges such that the closed neighbourhood of v is, $N[v] = V(G)$. With facement of vertices in mind the McPherson number $\Upsilon(G)$ is the minimum number of vertices to be faced so that the resultant graph is a complete graph, K_n. We call any such minimum set of vertices a *McPherson set* of G and denote such set, $\mathfrak{M} = \{v_1, v_2, v_3, \dots, v_\Upsilon\}$ or for brevity a $\mathfrak{M}$-set. Note that $\Upsilon(K_n) = 0$.

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[1]Whilst listening to an amasing djembe drumming group from Ghana on Friday, 10 October 2014, celebrating the 15^{th} anniversary of Klitsgras Drumming Circle, the concept of McPherson numbers struck Kokkie's mind. It reminds us there are mathematics in music and vice versa.

2. Important Results of McPherson Numbers

We recall important results found in [3]. Note that the terminology *faced, facement*, were not mentioned in [3].

Lemma 2.1. *[3] For the McPherson number of a graph G on n vertices, we have $\Upsilon(G) \leq \varepsilon(K_n) - \varepsilon(G)$.*

Lemma 2.2. *[3] The McPherson number of a graph G on n vertices is obtained through the McPherson recursion. Let any vertex with degree equal to $\delta(G)$ be faced on the first iteration to obtain the graph G_1. Then let any vertex of G_1 with degree equal to $\delta(G_1)$ be faced on the second iteration to obtain G_2 and, so on. If after exactly ℓ vertex facements, $G_\ell \simeq K_n$ then, $\Upsilon(G) = \ell$.*

Lemma 2.3. *[3] Consider two graphs G and H both on n vertices with $G \neq H$. If after minimum say, t recursive facements of G, we have $G_t \simeq H$, then $\Upsilon(G) = \Upsilon(H) + t$.*

Lemma 2.4. *[3] For any n-null graph, $n \in \mathbb{N}$ we have that:*

(a) The vertex facement sequence obeys the commutative law,

(b) The McPherson number is given by $\Upsilon(\mathfrak{N}_n) = n - 1$.

From [5] we recall the next result.

Theorem 2.5. *[5] For complete graph K_n, $\mathfrak{b}(K_n) = n - 1$.*

Lemma 2.4(b) read together with Theorem 2.5 brings our main result.

Theorem 2.6. *For a graph G we have: $\mathfrak{b}(G) = \Upsilon(\overline{G})$.*

Proof. Consider a blotting set $\mathfrak{B} = \{v_1, v_2, v_3, \ldots, v_{\mathfrak{b}}\}$ of graph G. Clearly, from the definition of the complement of graph G, the corresponding facement of $v_i \in \mathfrak{B}$ after effacement results in $\overline{G}$. Therefore, $\Upsilon(\overline{G}) \leq \mathfrak{b}(G)$. Assume $\Upsilon(\overline{G}) < \mathfrak{b}(G)$. Consider a McPherson set $\mathfrak{M} = \{v_1, v_2, v_3, \ldots, v_{\Upsilon}\}$ of graph $\overline{G}$. Clearly, from the definition of the complement of graph $\overline{G}$, the corresponding effacement of $v_i \in \mathfrak{M}$ after facement results in $\overline{(\overline{G})} = G$. Therefore, $\Upsilon(G) \leq \mathfrak{b}(\overline{G})$ alternatively, $\Upsilon(\overline{G}) \geq \mathfrak{b}(G)$. But, the inequalities are contradictory hence, $\mathfrak{b}(G) = \Upsilon(\overline{G})$. □

Theorem 2.6 makes it possible to find the equivalent results in terms of McPherson number of the complement graph of all results found in [5]. Similarly, all results found in [3, 4] can be formulated in terms of the blotting number of the corresponding complement graph.

3. Two Special Results

Our next result characterises a graph G for which the McPherson set $\mathfrak{M}$ is unique. It follows immediately that the corresponding blotting set $\mathfrak{B}$ of the complement graph $\overline{G}$ is unique.

Theorem 3.1. *The McPherson set $\mathfrak{M}$ of a graph G is unique if and only if, resulting from the McPherson recursions each graph, $G = G_0$, G_1, G_2,..., $G_{\Upsilon-1}$ has a unique vertex v_i, $0 \leq i \leq \Upsilon - 1$ for which $d(v_i) = \min\{d(v_j) : v_j \in V(G_i)\}$.*

Proof. Clearly if $G = G_0$ has two or more vertices say, $v_0, v_1, v_2, \ldots, v_t$, $t \leq n$ for which $d(v_0) = d(v_1) = d(v_2) = \cdots = d(v_t) = \min\{d(v_j) : v_j \in V(G_0)\}$ the choice for the first iteration of the McPherson recursion is not unique. Else it is, and the first iteration allows for the maximum addition of edges to obtain G_1. Through immediate induction, the result that if, $d(v_0) = \min\{v_j : v_j \in V(G_0)\}$ and unique, $d(v_1) = \min\{v_j : v_j \in V(G_1)\}$ and unique, $\cdots, d(v_{\Upsilon-1}) = \min\{v_j : v_j \in V(G_{\Upsilon-1})\}$ and unique, then the set $\{v_0, v_1, v_2, \ldots, v_{\Upsilon-1}\}$ is the unique $\mathfrak{M}$-set of G.

The converse i.e. if $\mathfrak{M}$ is unique then $d(v_0) = \min\{v_j : v_j \in V(G_0)\}$ and unique, $d(v_1) = \min\{v_j : v_j \in V(G_1)\}$ and unique, $\cdots, d(v_{\Upsilon-1}) = \min\{v_j : v_j \in V(G_{\Upsilon-1})\}$ and unique, follows trivially. □

We recall that a domination set of a graph G is a minimum set of vertices X such that every $w \notin X$ is adjacent to at least one $v \in X$. In other words, $\bigcup\limits_{\forall v \in X,\ X\ a\ minimum} N[v] = V(G)$. Also, the domination number of G is denoted by $\gamma(G)$ where, $\gamma(G) = |X|$. We also recall that an acyclic graph has no induced cycle. An acyclic graph is also called a tree (or forest in the case G has disconnected components).

Theorem 3.2. *For an acyclic graph G of order $n \geq 2$, any $\mathfrak{B}$-set of G is a γ-set of G. Therefore, $\gamma(G) = \mathfrak{b}(G)$.*

Proof. We prove the result through mathematical induction. Let $T_n \in \{trees\ of\ order\ n \geq 2\}$. Since $T_2 = P_2$ with vertices say, v_1, v_2 both $\{v_1\}$, $\{v_2\}$ are both a $\mathfrak{B}$-set and a γ-set of T_2. Therefore, $\gamma(T_2) = \mathfrak{b}(T_2)$.

Since $T_3 = P_3 = v_1v_2v_3$ the set $\{v_2\}$ is the unique $\mathfrak{B}$-set and γ-set of T_3. Therefore, $\gamma(T_3) = \mathfrak{b}(T_3)$.

For T_4 we have, either the path $v_1v_2v_3v_4$ with the $\mathfrak{b}$-sets $\{v_1, v_3\}$, $\{v_2, v_4\}$ and $\{v_2, v_3\}$. Otherwise, T_4 is a star with central vertex say, u hence, $\{u\}$ is the unique $\mathfrak{b}$-set. Clearly in all cases a $\mathfrak{b}$-set of T_4 is also a γ-set. Therefore, in all cases $\gamma(T_4) = \mathfrak{b}(T_4)$.

Assume the result holds for any tree of order $n = k$ and consider any $T_k \in \{trees\ of\ order\ k \geq 2\}$. Now add a new pendant vertex u in any way to T_k to obtain a tree T_{k+1}.

Case-1: Attach u to any pendant vertex w of T_k. If $w \in \mathfrak{b}$-set of T_k it is also in γ-set of T_k and it admits to be both a $\mathfrak{b}$-set and γ-set of T_{k+1}. Therefore, $\gamma(T_k) = \mathfrak{b}(T_k) = \gamma(T_{k+1}) = \mathfrak{b}(T_{k+1})$. If $w \notin \mathfrak{b}$-set of T_k but w can be exchanged with some vertex in some $\mathfrak{b}$-set to get a new $\mathfrak{b}$-set, the result follows as before. Otherwise, the vertex u must be included in any $\mathfrak{b}$-set of T_k to obtain a $\mathfrak{b}$-set of T_{k+1}. Exactly the same resoning applies for the γ-set property. Hence, the result holds for a tree of order $n = k + 1$. Therefore, the result holds for all $n \in \mathbb{N}$, $n \geq 2$ if a new pendant vertex u is added to any pendant vertex of T_k.

Case-2: Attach u to any internal vertex of T_k. Similar reasoning to that in Case 1 proves the result. Hence, the result holds for a tree of order $n = k + 1$. Therefore, through induction it follows that for an acyclic graph G of order $n \geq 2$, any $\mathfrak{B}$-set of G is a γ-set of G hence, $\gamma(G) = \mathfrak{b}(G)$. □

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The author would like to dedicate the paper to the memories of his friend Col. (Retd.) Vic McPherson, who left this world on 20, April 2017.

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