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A CHARACTERISATION OF DUPLICATION SELF VERTEX SWITCHING IN GRAPHS

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Abstract: A vertex $v \in V(G)$ is said to be a *self vertex switching* of G if G is isomorphic to G^v, where G^v is the graph obtained from G by deleting all edges of G incident to v and adding all edges incident to v which are not in G. A vertex v is called a *duplication self vertex switching* of a graph G if the resultant graph obtained after duplication of v has v as a self vertex switching. In this paper, we give some properties of duplication self vertex switching.

AMS Subject Classification: 05C60

Key Words: switching, self vertex switching, duplication self vertex switching, D(vG), $dss_1(G)$

1. Introduction

For a finite undirected graph $G(V, E)$ with $|V| = p$ and a set $\sigma \subseteq V$, Seidel [5] defined the *switching* of G by σ as the graph $G^\sigma(V, E')$, which is obtained from G by removing all edges between σ and its compliment $V - \sigma$ and

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adding as edges all non edges between σ and $V - \sigma$. When $\sigma = \{v\} \subseteq V$, the corresponding switching $G^{\{v\}}$ is called a *vertex switching* and is denoted by G^v [6]. Switching is an equivalence relation and the associated equivalence classes are called *switching classes*. For a survey of switching classes of graphs we refer to Seidel [5]. A subset σ of $V(G)$ to be a *self switching* of G if $G \cong G^\sigma$. The set of all self switchings of G with cardinality k is denoted by $SS_k(G)$ and its cardinality by $ss_k(G)$. If $k = 1$, then we call the corresponding self switching as *self vertex switching* [8]. For any vertex $v \in V(G)$, the *open neighbourhood* $N(V)$ of v is the set of all vertices adjacent to v. That is $N(V) = \{u \in V(G)/\ uv \in E(G)\}$. The *closed neighbourhood* of v is defined by $N[v] = N(v) \cup \{v\}$. Two vertices u and v in G are said to be *interchange similar* if there is an automorphism α of G such that $\alpha(u) = v$ and $\alpha(v) = u$ [4]. In [7], a characterization is given for a cut vertex in G to be a self vertex switching where G is a connected graph such that any two self vertex switchings if exists , are interchange similar. The existance of graphs with given number of self vertex switchings were discussed in [1]. The trees [3] and unicyclic graphs [2] are characterized for self vertex switchings.

In this paper, we give some properties of duplication self vertex switching and we prove that if G has an automorphism which maps elements of $N(v)$ onto $[N(v)]^c$, then v is a duplication self vertex switching of G. We also prove that if v is a duplication self vertex switching of G, then there is an automorphism on G which maps elements of $N(v)$ onto elements of $[N(v)]^c$. We consider simple graphs only.

We consider the following results which are required in the subsequent section.

Theorem 1.1. (see [7]) *If v is a self vertex switching of a graph G of order p, then p is odd and $d_G(v) = (p-1)/2$.*

Theorem 1.2. (see [8]) *Let G be a graph in which any two self vertex switchings, if any, are interchange similar. If v is a self vertex switching of G, then $G - v$ has an automorphism which maps elements of $N(v)$ onto $[N(v)]^c$.*

2. Definitions and Properties of Duplication Self Vertex Switching

Definition 2.1. (see [9]) Duplication of a vertex v of graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G'.

The new graph obtained after duplication of a vertex v is denoted as $D(vG)$. The number of duplication self vertex switching is denoted by $dss_1(G)$.

Example 2.2. Consider the path P_4. The duplication of each vertex of P_4 are given in Figure 2.1.

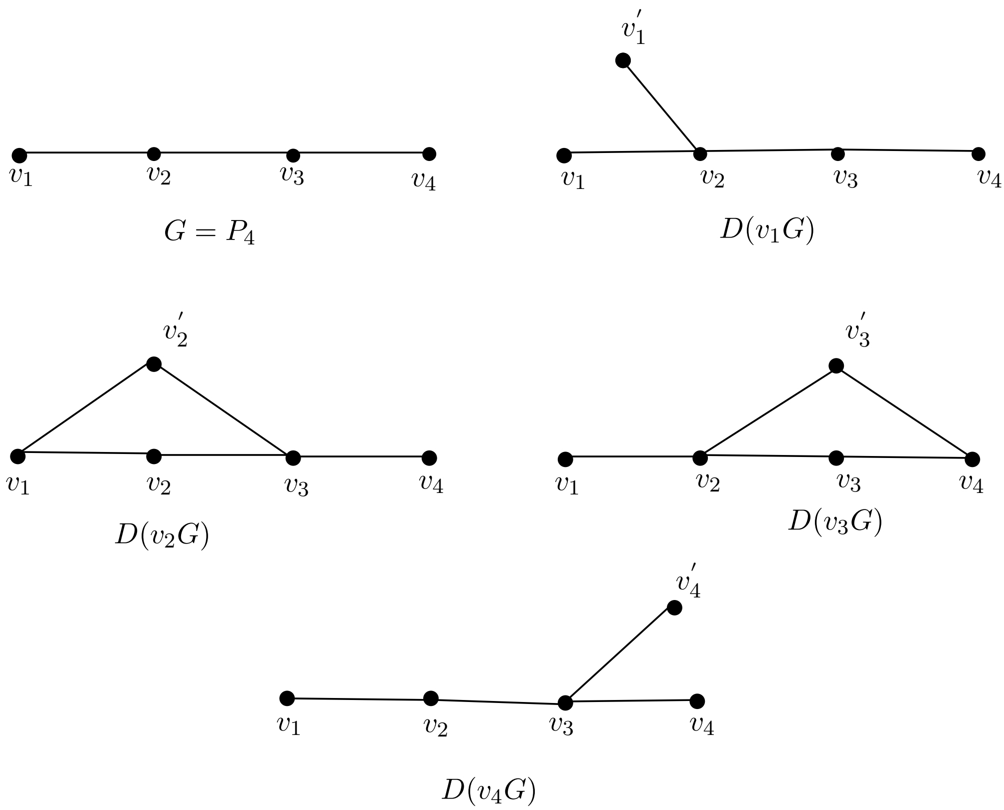

Fig.2.1

Remark 2.3. *Since graphs with odd number of vertices can have self vertex switchings, graphs with even number of vertices can have duplication self vertex switchings.*

The graph G given in Figure 2.2 has a duplication self vertex switching v.

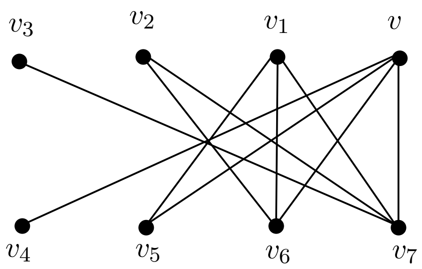

Fig.2.2. G

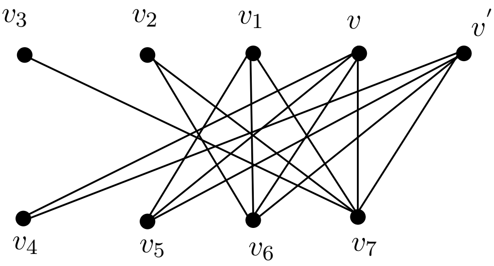

Fig.2.3. $D(vG)$

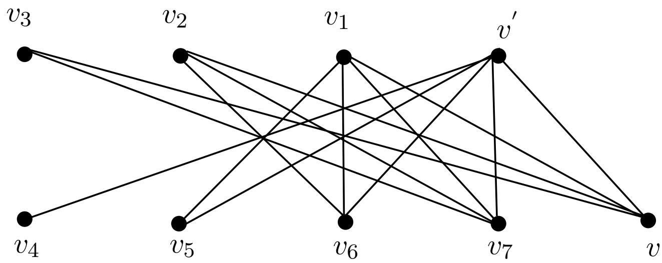

Fig.2.4. $D(vG)^v$

Result 2.4. *Let G be a (p, q) graph. Then $D(vG)$ is a $(p + 1,\ q + d_G(v))$ graph.*

Result 2.5. *For $v \in V(G)$, $d_G(v) = d_{D(vG)}(v)$.*

Theorem 2.6. *If v is a duplication self vertex switching of a graph G of order p, then p is even and $d_G(v) = p/2$.*

Proof. Let v be a duplication self vertex switching of a graph G of order p. Then v is a self vertex switching of the graph $D(vG)$ of order $p + 1$. By Theorem 2.3, $p+1$ is odd and $d_{D(vG)}(v) = (p+1-1)/2 = p/2$. Hence p is even and by Result 2.7, $d_G(v) = p/2$. Hence the theorem is proved. □

Theorem 2.7. *If v is a duplication self vertex switching of a graph G, then v and its duplication $v^{'}$ are interchange similar vertices. Further $v^{'}$ is also a self vertex switching of $D(vG)$.*

Proof. Let v be a duplication self vertex switching and let $v^{'}$ be a duplication of v. Then $D(vG) \cong D(vG)^v$, $N(v) = N(v^{'})$ and $[N[v]]^c - \{v^{'}\} = [N[v^{'}]]^c - \{v\}$. Define $f : V(D(vG)) \to V(D(vG))$ by $f(v) = v^{'}$, $f(v^{'}) = v$ and $f(u) = u$ for all $u \in V(G) - \{v\}$. Clearly f is an automorphism of $D(vG)$ and hence v and v' are interchange similar vertices in $D(vG)$. Hence v' is a self vertex switching of $D(vG)$ □

Theorem 2.8. *Let G be a graph and let v be any vertex of G. Then v is a duplication self vertex switching of G iff there is an automorphism on G which maps elements of $N(v)$ onto elements of $[N(v)]^c$.*

Proof. Let v be a duplication self vertex switching of G. Then v is a self vertex switching of $D(vG)$. By Theorem 2.9, v' is also a self vertex switching of $D(vG)$ and the vertices v and v' are interchange similar in $D(vG)$. By Theorem 2.4, $D(vG) - \{v\}$ has an automorphism which maps elements of $N(v)$ onto elements of $[N(v)]^c$. Since $G \cong D(vG) - \{v\}$, G has an automorphism which maps elements of $N(v)$ onto elements of $[N(v)]^c$.

Conversely, let f be an automorphism on G which maps elements of $N(v)$ onto $[N(v)]^c$. Let v' be a duplication of v. Then $G = D(vG) - \{v'\}$ and hence f is an automorphism on $D(vG) - \{v'\}$ which maps elements of $N(v)$ onto $[N(v)]^c$. Since $D(vG) - \{v'\} \cong D(vG) - \{v\}$, there exists an automorphism g on $D(vG) - \{v\}$ which maps elements of $N(v)$ onto $[N(v)]^c$. Define $f^* : V(D(vG)) \to V(D(vG)^v)$ such that $f^*(v) = v$ and $f^*(u) = g(u)$ for all $u \in D(vG) - \{v\}$. Clearly f^* is a bijection. To prove f^* is an isomorphism between $D(vG)$ and $D(vG)^v$, it is enough to check the adjacency property.

Let $u, w \in V(D(vG)) - \{v\}$.

1. v and u are adjacent in $D(vG)$

$$\begin{aligned}
&\Leftrightarrow u \in N(v) \text{ in } D(vG)\\
&\Leftrightarrow u \in N(v) \text{ in } D(vG) - \{v\}\\
&\Leftrightarrow g(u) \in [N(v)]^c \text{ in } D(vG) - \{v\}\\
&\Leftrightarrow g(u) \in [N(v)]^c \text{ in } D(vG)\\
&\Leftrightarrow v \text{ and } g(u) \text{ are non adjacent in } D(vG)\\
&\Leftrightarrow v \text{ and } g(u) \text{ are adjacent in } D(vG)^v\\
&\Leftrightarrow f^*(v) \text{ and } f^*(u) \text{ are adjacent in } D(vG)^v.
\end{aligned}$$

2. u and w are adjacent in $D(vG)$

$$\begin{aligned}
&\Leftrightarrow u \text{ and } w \text{ are adjacent in } D(vG) - \{v\}\\
&\Leftrightarrow g(u) \text{ and } g(w) \text{ are adjacent in } D(vG) - \{v\}\\
&\Leftrightarrow g(u) \text{ and } g(w) \text{ are adjacent in } D(vG)
\end{aligned}$$

$$\Leftrightarrow g(u) \text{ and } g(w) \text{ are adjacent in } D(vG)^v$$
$$\Leftrightarrow f^*(u) \text{ and } f^*(w) \text{ are adjacent in } D(vG)^v.$$

Hence v is a duplication self vertex switching of G and hence the theorem is proved. □

Theorem 2.9. *Let G be a graph of order p and let v be a duplication self vertex switching of G. Then for each vertex $u \in V(G)$ there exists a vertex $w(\neq u)$ in $V(G)$ such that $d(u) = d(w)$.*

Proof. Let v be a duplication self vertex switching of G. By Theorem 2.10, there exists an automorphism on G which maps elements of $N(v)$ onto elements of $[N(v)]^c$ and vice versa. This implies that for each vertex $u \in N(v)$, there exists a vertex $w(\neq u) \in [N(v)]^c$ such that $d_G(u) = d_G(w)$. That is for each vertex $u \in V(G)$, there exists a vertex $w(\neq u) \in V(G)$ such that $d_G(u) = d_G(w)$. □

Theorem 2.10. *If v is a duplication self vertex switching of a graph G of order p, then v is adjacent to a vertex of degree $p/2$ in G.*

Proof. Let v be a duplication self vertex switching of a graph G of order p. By Theorem 2.8, $d_G(v) = p/2$. Suppose v is non adjacent to any vertex of degree $p/2$ in G. Then by definition v is non adjacent to any vertex of degree $p/2$ in $D(vG)$. By Theorem 2.11, there exists a vertex u other than v in $V(G)$ such that $d_G(u) = p/2$ and $u \in [N(v)]^c$. By theorem 2.10., there is an automorphism on G which maps elements of $N(v)$ onto elements of $[N(v)]^c$. Hence for $u \in [N(v)]^c$, there is a $w \in N(v)$ such that $d_G(u) = d_G(w) = p/2$, which contradicts the assumption that v is non adjacent to any vertex of degree $p/2$ in G. Hence v is adjacent to a vertex of degree $p/2$ in G. □

Theorem 2.11. *In a non-regular graph G, the vertex non adjacent to any minimum degree vertex is not a duplication self vertex switching .*

Proof. Let w be a minimum degree vertex in G and let v be a vertex of G such that v is non adjacent to any minimum degree vertex of G. Then v is non adjacent to w in $D(vG)$ and hence w is a minimum degree vertex of $D(vG)$. This implies that $d_{D(vG)}(u) \geq \delta(D(vG)) + 2$ for all $u \in N(v)$. In $D(vG)^v$, the vertex w has degree $\delta(D(vG)) + 1$. This implies that $D(vG) \ncong D(vG)^v$ and hence v is not a duplication self vertex switching of G. □

Theorem 2.12. *In a non-regular graph G, the vertex non adjacent to any maximum degree vertex is not a duplication self vertex switching.*

Proof. Let w be a maximum degree vertex of G and let v be a vertex of G such that v is non adjacent to any maximum degree vertex of G. Then w is a maximum degree vertex of $D(vG)$ and w is non adjacent to v in $D(vG)$. This implies that $d_{D(vG)}(u) < \Delta(DvG))$ for all $u \in N(v)$ and $d(w) = \Delta(D(vG)) + 1$ in $D(vG)^v$. This implies that $D(vG) \ncong D(vG)^v$ and hence v is not a duplication self vertex switching of G. □

Definition 2.13. [8] *The set of positive integers, each is the degree of a vertex in a graph G, is denoted by $DS(G)$. The set of positive integers, each is the degree of a vertex which is adjacent to the vertex v in graph G, is denoted by $NDS(v)$. The set of positive integers, each is the degree of a vertex which is non adjacent to the vertex v in graph G, is denoted by $NDS(v)^c$. That is,*

$$DS(G) = \{n/n = d(u),\ u \in V(G)\},$$

$$NDS(v) = \{n/n = d(u),\ u \in N(v)\},$$

$$NDS(v)^c = \{n/n = d(u),\ u \in [N(v)]^c\}.$$

For $S \subseteq N$, let $S - 1 = \{s - 1/s \in S\}$.

Theorem 2.14. *If v is a duplication self vertex switching of a graph G, then $NDS(v) = NDS(v)^c$.*

Proof. Let v be a duplication self vertex switching of G. To prove the theorem, it is enough to prove that $NDS(v) \subseteq NDS(v)^c$ and $NDS(v)^c \subseteq NDS(v)$. Suppose that $NDS(v) \nsubseteq NDS(v)^c$. Then there exists an integer m such that $m \in NDS(v)$ but $m \notin NDS(v)^c$. This implies that there exists a vertex, say u, in G such that $u \in N(v)$ and $d(u) = m$ but $[N(v)]^c$ does not contain any vertex of degree m. Since v is a duplication self vertex switching of G, there is an automorphism on G which maps $u \in N(v)$ onto some $w \in [N(v)]^c$. This implies that $d(u) = d(w) = m$, which is a contradiction since $[N(v)]^c$ does not contain any vertex of degree m. Hence $NDS(v) \subseteq NDS(v)^c$. Similarly we can prove that $NDS(v)^c \subseteq NDS(v)$ and hence $NDS(v) = NDS(v)^c$. □

Theorem 2.15. *Let G be a graph of order $p \geq 3$. If G has a vertex of degree p-1, then $dss_1(G) = 0$.*

Proof. Suppose that $dss_1(G) \neq 0$. Then G has a duplication self vertex switching v and hence $d(v) = p/2$. Since order of G is p, the vertex which has degree $p - 1$ is adjacent to v in G. Consider a vertex u such that $d(u) = p - 1$ in G. Since v is a duplication self vertex switching of G, by Theorem 2.10, there is an automorphism on G, which maps elements of $N(v)$ onto elements of $[N(v)]^c$.

This implies that there exists a vertex w in $[N(v)]^c$ such that $d(w) = p - 1$. Since $p \geq 3$, $p/2 \neq p-1$ and hence $w \neq v$. This implies that w is non adjacent to v in G and $d(w) = p - 1$, which is a contradiction. Hence $dss_1(G) = 0$. □

Remark 2.16. *For $p = 2$, $p/2 = p - 1 = 1$. The graph $G = K_2$ has two vertices, each of degree $p - 1$. But $dss_1(G) = 2 \neq 0$.*

3. Conclusion

We have introduced duplication self vertex switching in graphs and proved few necessary conditions for a vertex to be a duplication self vertex switching. Also we gave a characterisation for a vertex to be a duplication self vertex switching.

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