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SIMPLE GRAPHOIDAL COVERS ON DIGRAPHS

Venkat Narayanan[1 §], Suresh Suseela[2], Kala[3]

[1]Department of Mathematics
St. Joseph's College of Engineering
Chennai, INDIA

[2]Department of Mathematics
St. John's College
Tirunelveli, INDIA

[3]Department of Mathematics
M.S. University
Tirunelveli, INDIA

Abstract: Let D be a digraph. A graphoidal cover of D is a set ψ of (not necessarily open) paths in D, such that every path in ψ has at least length 1, every vertex of D is an internal vertex of at most one path in ψ and every arc of D is in exactly one path in ψ. The minimum cardinality of a graphoidal cover of D is called the graphoidal covering number of D and is denoted by η. If every two paths in ψ have at most one common vertex, then it is called simple graphoidal cover of D. The minimum cardinality of a simple graphoidal cover of D is called simple graphoidal covering number of D and is denoted by η_s. In this paper, the simple graphoidal covering number on digraphs is determined. The simple graphoidal covering number on Product of two digraphs is also determined.

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1. Introduction

All the graphs considered in this paper are directed, weakly connected, finite,

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§Correspondence author

loop less and without multiple arcs. By a digraph D=(V,E),means a finite set V(the elements which are called vertices) together with a set E of ordered pairs of elements of V (these pairs are called arcs). The order and size of the D are denoted by p and q respectively. Two vertices are said to be adjacent if they are connected by an arc. For theoretical terminology of graph Harary [6] is referred. All the graphs considered in this paper are assumed to be connected and non-trivial. If $P = (v_1, v_2, v_3, ...v_n)$ be a path or cycle in a graph D, the vertices $v_2, v_3, v_4, ...v_{n-1}$ are called internal vertices of P and v_1, v_n are called external vertices of P. Two paths P and Q are said to be internally disjoint if no vertex of D is an internal vertex of both P and Q. The concept of graphoidal cover was introduced by Acharya and Sampath Kumar [1][2]. Arumugam and Pakkiam [2] [3] determined the graphoidal covering number of digraphs. Further Arumugam and Suresh Suseela [4] determined the acyclic graphoidal covering number on digraphs. The concept of simple graphoidal cover was introduced by Arumugam and Shahul Hamid [5]. This paper determines simple graphoidal covering number on some family of digraphs and on product of two digraphs.

2. Main Results

Definition 1. [3] A graphoidal cover of D is a set ψ of (not necessarily open) paths in D satisfying the following conditions.

(i) Every path in ψ has at least length 1.

(ii) Every vertex of D is an internal vertex of at most one path in ψ.

(iii) Every arc of D is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of D is called the graphoidal covering number of D and is denoted by η.

Definition 2. [5] A simple graphoidal cover of a graph D is a graphoidal cover ψ of D such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of D is called simple graphoidal covering number of D and is denoted by η_s.

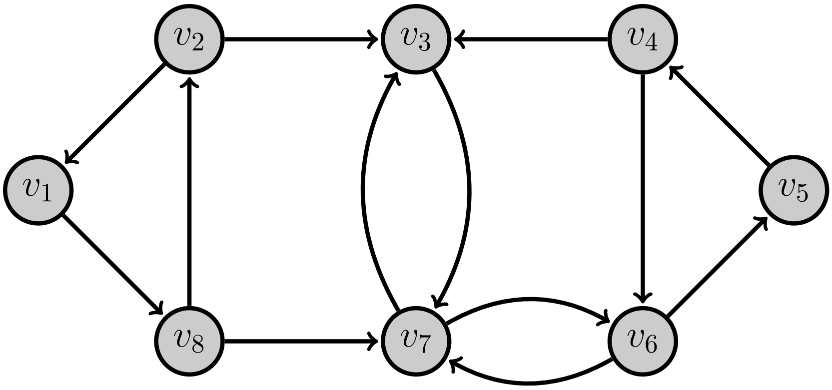

Figure 1

Example 2.1. Consider the digraph D given in Figure 1, The collection paths of D are

$$P_1 = (v_2v_1v_8v_2),$$
$$P_2 = (v_3v_7v_3),$$
$$P_3 = (v_7v_6v_7),$$
$$P_4 = (v_6v_5v_4v_6),$$
$$P_5 = (v_4v_3),$$
$$P_6 = (v_2v_3),$$
$$P_7 = (v_8v_7).$$

Then $\psi = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$ is a minimum simple graphoidal cover of D and $\eta_s(D) = 7$.

Definition 3. [5] Let ψ be a collection of internally disjoint paths in D. A vertex v of D is said to be an interior vertex of ψ, if it is an internal vertex of some path in ψ. Any vertex which is not an interior vertex is said to be an exterior vertex of ψ.

Theorem 4. *[5] For any simple graphoidal cover ψ of a (p,q) of graph D, let t_ψ denote the number of exterior vertices of ψ and let t= min t_ψ, where the minimum is taken overall simple graphoidal covers ψ of D, Then $\eta_s(D)$=q-p+t.*

Corollary 5. *[5] For any digraph D, $\eta_s(D) \geq q - p$. Moreover, the following are equivalent* *(i)* $\eta_s(D) = q - p$.

(ii) There exists a simple graphoidal cover of D without exterior vertices.

(iii) There exists a set P of internally disjoint and edge disjoint paths without exterior vertices such that any two paths in P have at most one vertex in common.

Theorem 6. *For any graph G, there exists an orientation D of G such that* $\eta_s(D) = \eta_s(G)$

Proof. Let ψ be the minimum graphoidal cover of G. By assigning the directions to edge of G in such a way that every path becomes directed path. Let D be the orientation of G. Collection of all directed paths corresponding to the path ψ is a simple grapoidal cover of G and hence $\eta_s(D) \leq \eta_s(G)$. Further deleting all directions from the edges of G, and the simple graphoidal cover of G is obtained and hence $\eta_s(G) \leq \eta_s(D)$. Thus $\eta_s(D) = \eta_s(G)$ □

Theorem 7. *Let D be a digraph, then* $\eta_s(D) = |E(D)|$ *if and only if id(v)=0 or od(v)=0,* $\forall v \in V(D)$.

Proof. Suppose id(v)=0 or od(v)=0, $\forall v \in V(D)$. Then for any simple graphoidal cover of D, no vertex is an internal vertex of any path in ψ. Hence t $=$ min $t_\psi = p$ and $\eta_s(D) = q - p + p = q$. Conversely, let D be a digraph with $\eta_s(D) = |E(D)|$. If there exists a vertex $v \in V(D)$ such that id(v) ≥ 1 and od(v) ≥ 1. Let v_1, v_2 be two vertices of D such that v_1v and $vv_2 \in$ E(D). Now there are two cases:

Case (i) if $v_1 = v_2$, Let C= (v_1vv_1). Then $\psi = \{C\} \cup \{E(D) - E(C)\}$ is a simple graphoidal cover of D and $|\psi|$ =1 which is a contradiction to $|E(D)|$=2.

Case(ii) if $v_1 \neq v_2$, Let P= (v_1vv_2). Then $\psi = \{P\} \cup \{E(D) - E(P)\}$ is a simple graphoidal cover of D and $|\psi|$ =1 which is a contradiction to $|E(D)|$=2. Hence id(v)=0 or od(v)=0, $\forall v \in V(D)$. □

Theorem 8. *Let D be a complete symmetric digraph, then* $\eta_s(D) = q - p$ *if and only if p=3*

Proof. Let D be a complete symmetric graph with p=3. Then the collection of paths of D are given by $P = \{(v_1, v_2, v_1), (v_2, v_3, v_2), (v_3, v_1, v_3)\}$ is a simple graphoidal cover of D in which all vertices made are internal. By corollary 5 , $\eta_s(D) = q - p$. Conversely, If there exists a complete symmetric graph with

$\eta_s(D) = q - p$. It is necessary to prove that p=3. when it is considered as $p \neq 3$, then there are two cases

Case (i) if $p < 3$, Let v_1, v_2 be the two vertices of D so that $v_1 v_2$ and $v_2 v_1 \in$ E(D). Let P = $(v_1 v_2 v_1)$. Then $\psi = \{P\} \cup \{E(D) - E(P)\}$ is a simple graphoidal cover of D in which v_1 is not internal. Therefore $\eta_s(D) = q - p + 1$ is contradiction.

Case (ii) if $p > 3$, Let $v_1, v_2, ...v_n$ be the vertices in D. Let P= $(v_1 v_2 .....v_n)$. Since D is a complete symmetric graph, there exists $v_1 v_n$ and $v_n v_1 \in$ E(D). Clearly the arcs $v_1 v_n$ and $v_n v_1 \in E(D) - \{P\}$. Then $\psi = \{P\} \cup \{E(D) - E(P)\}$ is not a simple graphoidal cover of D, since any two paths in ψ have more than one vertex in common which contradicts to corollary 5. Hence p=3. □

Theorem 9. *Let D be a isograph, then*

$$\eta_s(D) = \begin{cases} q - p + 1 & \text{if k=1 or when } k \geq 2 \text{ and p=2k+1} \\ q - p & \text{if } k \geq 2 \text{ and p} > \text{2k+1} \end{cases}$$

Proof. Let D be a isograph (i.e) id(v)= od (v)=k, now there are three cases:

Case (i) When k=1, The collection paths of D are given by

$P = (v_1, v_2, v_3......, v_n, v_1)$
$Q = E(D) - P.$

Let $\psi = \{P \cup Q\}$ is a simple graphoidal cover of D in which v_1 is not an internal. Therefore $\eta_s(D) \leq q - p + 1$. Now, let ψ be any simple graphoidal cover of G. Now there are two cases:

Case (i) When $p < 4$, If ψ contains only paths, then no vertices can be made internal. If ψ contains atleast one cycle, then minimum one vertex can not be made internal. Therefore $t_\psi \geq 1$. Hence $\eta_s(D) \geq q - p + 1$. Thus $\eta_s(D) = q - p + 1$.

Case (ii) When $p \geq 4$, If ψ contains only paths, then minimum three verticies can not be made internal. If ψ contains atleast one cycle, then minimum one vertex can not be made internal. Therefore $t_\psi \geq 1$. Hence $\eta_s(D) \geq q - p + 1$. Thus $\eta_s(D) = q - p + 1$.

Case (ii) When $k \geq 2$ and p=2k+1. Let $C_1 = \{v_1, v_2, v_3....., v_{i-1}, v_i, v_{i+1}....,$ $v_1\}$ be a cycle. Choose a vertex $v_i \in C_1$ so that each v_i is adjacent only with v_{i-1} and v_{i+1} whereas not adjacent to other vertices in C_1. Let $C_2 =$ $\{v_i, w_1, w_2....., w_{i-1},\ w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in$

C_2 so that each w_i is adjacent only with w_{i-1} and w_{i+1} whereas not adjacent to any other vertices in C_2. Continuing like this a collection of cycles $\{C_1, C_2, C_3......C_n\}$ are derived. Let $\psi = \{\{C_1, C_2, C_3....C_n\} \cup \{E(D) - \{C_1, C_2, C_3....C_n\}\}$ is a simple graphoidal cover in which only one vertex can not be made internal. Therefore $\eta_s(D) \leq q - p + 1$. Now, let ψ be any simple graphoidal cover of G. If ψ contains only paths, then no vertices can be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $t_\psi \geq 1$. Hence $\eta_s(D) \geq q - p + 1$. Thus $\eta_s(D) = q - p + 1$.

Case(iii) When $k \geq 2$ and $p > 2k+1$. Let $C_1 = \{v_1, v_2, v_3....., v_{i-1}, v_i, v_{i+1}...., v_1\}$ be a cycle. Choose a vertex $v_i \in C_1$ so that each v_i is only adjacent with v_{i-1} and v_{i+1} whereas not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, w_1, w_2....., w_{i-1}, w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in C_2$ such that each w_i is only adjacent with w_{i-1} and w_{i+1} and not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3......C_n\}$ are derived which makes p-1 vertices internal. Let $x_i \in C_i$ and $y_j \in C_j$ be two vertices so that both vertices incident with non internal vertex v_k. Let the arcs be $x_i v_k, v_k y_j \in E(D)$. Now there are two sub cases:

Subcase (i) if $x_i = y_j$, then a cycle can be found and so $C_{n+1} = \{x_i, v_k, x_i\}$. Then $\psi = \{C_1, C_2, C_3......C_n, C_{n+1}\} \cup \{E(D) - \{C_1, C_2, C_3....C_n, C_{n+1}\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$.

Subcase (ii) if $x_i \neq y_j$, then a path can be found and $P = \{x_i, v_k, y_j\}$. Then $\psi = \{C_1, C_2, C_3...C_n\ , P\} \cup \{E(D) - \{C_1, C_2, C_3......C_n, P\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$. □

Theorem 10. *For any directed Tree T, $\eta_s(T) = n - 1$, where n is the number of vertices v such that id(v)=0 or od (v)=0.*

Proof. We prove the result by induction by on q, where q is the number of arc of T.

When q =1, then n=2 and $\eta_s(T) = 1$. Suppose the result is true for any directed tree with q-1 arcs. Let T be any directed tree with $q > 1$ arcs. Let n be the number of vertices v of T such that od(v)=0 (or) id(v)=0. Let u be the end vertex of T. It is assumed that id(u)=0 and od(u)=1. Let (uv) be the unique arc incident with u.

Case(i) $od(v) = 0$. Clearly $T - \{u\}$ be a directed tree with q-1 arcs and n-1 vertices w, with id(w)=0 or od(w)=0. Hence $T - \{u\}$ has a simple graphoidal cover of ψ such that $|\psi| = n-2$. Now $\psi_1 = \{\psi \cup \{uv\}\}$ is a minimum simple graphoidal cover of T with $|\psi_1| = n-1$.

Case(ii) $id(v) = 1, od(v) \geq 1$. In this case $T-\{u\}$ is a directed tree with q-1 arcs and n vertices w, with id(w)=0 and od(w)=0. Hence $T-\{u\}$ is a simple graphoidal cover with $|\psi| = n-1$. Since v has indegree 0 in $T-\{u\}$, v is not an internal vertex of any path in ψ. Now, Let $P = \{v, v_1, v_2, v_3....., v_k\}$ be a path in ψ. Let $P_1 = \{u, v, v_1, v_2, v_3....., v_k\}$. Then $\psi_1 = \{\{\psi - \{P\}\} \cup P_1\}$ is a simple graphoidal cover of T and $|\psi_1| = n-1$.

Case(iii) $id(v) > 1, od(v) \geq 1$. Then $T-\{u\}$is a directed tree with q-1 arcs and n-1 vertices w, with id(w)=0 or od(w)=0. As in case (1), we can obtain a graphoidal cover ψ_1 of T with $|\psi_1| = n-1$. Hence $\eta_s(T) \leq n-1$. Now for any simple graphoidal cover of ψ of T, $t_\psi \geq n$ and hence t= min $t_\psi \geq n$. Hence $\eta_s(T) = q-p+t \geq (p-1)-p+n = n-1$. Thus $\eta_s(T) = n-1$. □

Corollary 11. *For any rooted tree T, $\eta_s(T) = n$, where n is the number of vertices v of T such that id(v)=1 and od(v)=0.*

Theorem 12. *Let D be a directed graph with $id(v), od(v) \geq 2$, for each $v \in V(D)$. Then $\eta_s(D) = q-p$.*

Proof. Let $P_1 = \{v_1, v_2, v_3..., v_{i-1}, v_i, v_{i+1}....v_n\}$ be a path(cycle). Choose a vertex $v_i \in P_1$ so that each v_i is only adjacent with v_{i-1} and v_{i+1}, whereas not adjacent to any other vertices in P_1. Let $P_2 = \{v_i, w_1, w_2.., w_{i-1}, w_i, w_{i+1}...., w_n\}$ be another path(cycle), choose a vertex $w_i \in P_2$ so that each w_i is only adjacent with w_{i-1} and w_{i+1}, whereas not adjacent to other vertex in C_2. Continuing like this, a collection of paths(cycles) $\{P_1, P_2, P_3......P_n\}$ are derived. Let $\psi = \{P_1, P_2, P_3....P_n\} \cup \{E(D) - \{P_1, P_2, P_3......P_n\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q-p$. □

Remark 2.1. The above result is not true for all complete symmetric graphs with $p > 3$. For example, consider the complete symmetric graph with p=4 in figure 2. Clearly $id(v_i), od(v_i) \geq 2$. But $\eta_s(D) \neq q-p$, by theorem 2.4.

Theorem 13. *Let D be a tournament with p vertices ($p \geq 6$), then $\eta_s(D) = q-p$ holds only*

(i) if $id(v_i), od(v_i) \geq 2$, for each v_i.

(ii) if exactly one vertex (id(v)=1 and od(v)= p-2) or (id(v)= p-2 and od(v)=1) and remain $id(v_i), od(v_i) \geq 2$, for each v_i.

(iii) *if exactly one vertex (id(v)=1 and od(v)= p-2) and (id(v)=p-2 and od(v)=1) and remain* $id(v_i), od(v_i) \geq 3$*, for each* v_i*.*

(iv) *if p=8 and exactly one vertex (id(v)=1 and od(v)=p-2) and (id(v)=p-2 and od(v)=1) and atmost one vertex (id(v)=2 and od(v)=p-3) and remain* $id(v_i), od(v_i) \geq 2$*, for each* v_i*.*

(v) *if p* $\geq$ *10 and exactly one vertex (id(v)=1 and od(v)=p-2) and (id(v)=p-2 and od(v)=1) and atmost two vertices (id(v)=2 and od(v)=p-3) and remain* $id(v_i), od(v_i) \geq 2$*, for each* v_i*.*

Proof. Let D be a tournament of order p ($p \geq 6$), Then there are five cases to prove this result.

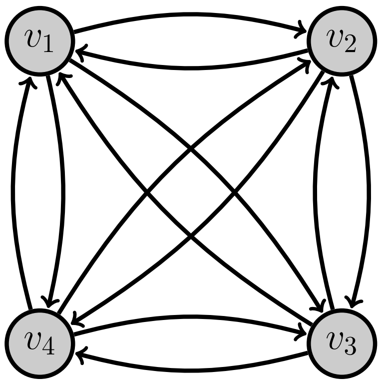

Figure 2

Case (i) Let $C_1 = \{v_1, v_2, v_3....., v_{i-1}, v_i, v_{i+1}...., v_1\}$ be a cycle. Choose a vertex $v_i \in C_1$ such that each v_i is only adjacent with v_{i-1} and v_{i+1}, and not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, w_1, w_2....., w_{i-1}, w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in C_2$ so that each w_i is only adjacent with w_{i-1} and w_{i+1}, and not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3....C_n\}$ are derived. Let $\psi = \{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$.

Case(ii) Let v_k be a vertex in V(D) such that $id(v_k) = 1$ and $od(v_k) = p-2$ Or $id(v_k) = p-2$ and $od(v_k) = 1$, then $C_1 = \{v_1, v_2, v_3....., v_{i-1}, v_i, v_{i+1}, ...v_k, ...,$

$v_1\}$ be a cycle. Choose a vertex $v_i \in C_1, (v_i \neq v_k)$ so that each v_i is only adjacent with v_{i-1} and v_{i+1}, while not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, w_1, w_2....., w_{i-1}, w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in C_2, (w_i \neq v_k)$ such that each w_i is only adjacent with w_{i-1} and w_{i+1}, and not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3....C_n\}$ are derived. Let $\psi = \{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$.

Case(iii) Let v_k, v_l be a vertex in V(D) such that $id(v_k) = 1$,$od(v_k) = p-2$, $id(v_l) = p-2$ and $od(v_l) = 1$, then $C_1 = \{v_1, v_2, v_3....., v_{i-1}, v_i, v_{i+1}, ..., v_l, v_k, ...,$ $v_1\}$ be a cycle. Choose a vertex $v_i \in C_1, (v_i \neq v_k$ and $v_l)$ so that each v_i is only adjacent with v_{i-1} and v_{i+1}, while not adjacent to other vertices in C_1. Let $C_2 = \{v_i, w_1, w_2..., w_{i-1}, w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in C_2, (w_i \neq v_k$ and $v_l)$ so that each w_i is only adjacent with w_{i-1} and w_{i+1}, while not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3......C_n\}$ are derived. Let $\psi = \{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$.

Case(iv) Let D be a digraph with p=8, v_l, v_m be a vertex in V(D) such that $id(v_l) = 1$, $od(v_l) = p - 2$, $id(v_m) = p - 2$ and $od(v_m) = 1$. Let K denote the number of vertices having $id(v) = 2$ and $od(v) = p-3$. Now there are two cases for K = 0 and 1.

Subcase (i) if K =0, (i.e.) no vertex is of degree $id(v) = 2$ and $od(v) = p - 3$. Then each vertex other than v_l, v_m has $id(v_i), od(v_i) \geq 2$. Let $C_1 = \{v_1, v_2, v_3....., v_{i-1}, v_i, v_{i+1}, ..., v_m, v_l, .., v_1\}$ be a cycle. Choose a vertex $v_i \in C_1, (v_i \neq v_m$ and $v_l)$ so that each v_i is only adjacent with v_{i-1} and v_{i+1}, while not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, , w_1, w_2....., w_{i-1}, w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in C_2, (w_i \neq v_m, v_l\ )$ so that each w_i is only adjacent with w_{i-1} and w_{i+1}, while not adjacent to any other vertices in C_2. Continuing like this, collection of cycles $\{C_1, C_2, C_3......C_n\}$ are obtained. Let $\psi = \{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$.

Subcase (ii) if K =1, Let v_n be the vertex in V such that $id(v_n) = 2$ and $od(v_n) = p - 3$. Then $C_1 = \{v_1, v_2, v_3..., v_{i-1}, v_i, v_{i+1}, .., v_m, v_l, .., v_1\}$ be a cycle. Choose a vertex $v_i \in C_1, (v_i \neq v_m$ and $v_l)$ so that each v_i is only adjacent with v_{i-1} and v_{i+1}, while not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, v_n, w_1, w_2....., w_{i-1}, w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in C_2, (w_i \neq v_m, v_n$ and $v_l\ )$ such that each w_i is only adjacent with w_{i-1} and w_{i+1}, while not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3......C_n\}$ are obtained. Let $\psi = \{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$.

Case(v) Let D be a digraph with p ($p \geq 10$), and v_l, v_m be a vertex in V such that $id(v_l) = 1$,$od(v_l) = p - 2$, $id(v_m) = p - 2$ and $od(v_m) = 1$ and Let K denote the number of vertices having $id(v) = 2$ and $od(v) = p - 3$. Now,there are three cases for K = 0,1 and 2.

Subcase (i) if K =0, (i.e.) no vertex is of degree $id(v_n) = 2$ and $od(v_n) = p - 3$. Then each vertex other than v_l, v_m has $id(v_i), od(v_i) \geq 2$. Let $C_1 = \{v_1, v_2, v_3....., v_{i-1}, v_i, v_{i+1}, ..., v_m, v_l, ..., v_1\}$ be a cycle. Choose a vertex $v_i \in C_1, (v_i \neq v_m$ and $v_l)$ so that each v_i is only adjacent with v_{i-1} and v_{i+1}, while not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, v_n, w_1, w_2....., w_{i-1}, w_i, w_{i+1}....,$ $v_i\}$ be another cycle, choose a vertex $w_i \in C_2, (w_i \neq v_m, v_n$ and v_l) so that each w_i is only adjacent with w_{i-1} and w_{i+1}, while not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3......C_n\}$ are derived. Let $\psi = \{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$.

Subcase (ii) if K =1, then v_n be the vertex in V such that $id(v_n) = 2$ and $od(v_n) = p - 3$. Then $C_1 = \{v_1, v_2, v_3..., v_{i-1}, v_i, v_{i+1}, .., v_m, v_l, .., v_1\}$ be a cycle. Choose a vertex $v_i \in C_1, (v_i \neq v_m$ and $v_l)$ so that each v_i is only adjacent with v_{i-1} and v_{i+1}, while not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, v_n, w_1, w_2....., w_{i-1}, w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in C_2, (w_i \neq v_l, v_m$ and v_n) so that each w_i is only adjacent with w_{i-1} and w_{i+1}, while not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3......C_n\}$ are derived. Let $\psi = \{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$.

Subcase (iii) if K =2, then v_n, v_k be the two vertices in V such that $id(v_n) = id(v_k) = 2$ and $od(v_n) = od(v_k) = p - 3$. Then $C_1 = \{v_1, v_2, v_3....., v_{i-1}, v_i, v_{i+1}, ...,$ $v_m, v_l, ..., v_1\}$ be a cycle. Choose a vertex $v_i \in C_1, (v_i \neq v_m$ and $v_l)$ so that each v_i is only adjacent with v_{i-1} and v_{i+1}, while not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, v_n, v_k, w_1, w_2....., w_{i-1}, w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in C_2, (w_i \neq v_l, v_m, v_k$ and v_n) such that each w_i is only adjacent with w_{i-1} and w_{i+1}, while not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3......C_n\}$ are derived. Let $\psi = \{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n, C_{n+1}\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(D) = q - p$. □

Theorem 14. *Let D be a transitive tournament of order p, then* $\eta_s(D) = q$.

Proof. Since D be a transitive tournament, Then for any simple graphoidal cover ψ of D, no vertices can be made internal,since every path is an arc. Hence t= min $t_\psi = p$ and $\eta_s(D) = q - p + p = q$. □

Theorem 15. *Let G be a strong tournament with* $p \geq 3$*, then there exists an orientation D of G such that*

$$\eta_s(D) = \begin{cases} q - p + 1 & \textit{if p is odd } (p > 1) \\ q - p + 2 & \textit{if p=4} \\ q - p & \textit{otherwise} \end{cases}$$

Proof. Case (i) if p is odd ($p \geq 3$). Let $C_1 = \{v_1, v_2, v_3....., v_{i-1}, v_i, v_{i+1}...., v_1\}$ be a cycle. Choose a vertex $v_i \in C_1$ so that each v_i is only adjacent with v_{i-1} and v_{i+1} while not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, w_1, w_2..., w_{i-1}, w_i, w_{i+1}, ..., v_i\}$ be another cycle, choose a vertex $w_i \in C_2$ so that each w_i is only adjacent with w_{i-1} and w_{i+1} while not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3......C_n\}$ are derived. Let $\psi = \{\{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n\}\}$ in which only one vertex made not internal. Therefore $\eta_s(G) \leq q - p + 1$. Now, let ψ be any simple graphoidal cover of G. If ψ contains only paths, then no vertices can be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $t_\psi \geq 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

Case(ii) if p=4, then the collection of paths of G are given by

$P = (v_1 v_2 v_3 v_1)$
$Q = E(D) - \{P\}$

is a simple graphoidal cover in which two vertices v_1, v_4 can not be made internal.Therefore $\eta_s(G) \leq q-p+2$. Now, let ψ be any simple graphoidal cover of G. If ψ contains only paths, then no vertices can be made internal. If ψ contains atleast one cycle, then atleast two vertex can not be made internal. Therefore $t_\psi \geq 2$. Hence $\eta_s(G) \geq q - p + 2$. Thus $\eta_s(G) = q - p + 2$.

Case(iii) if p is even and $p \neq 4$, Let $C_1 = \{v_1, v_2, v_3..., v_{i-1}, v_i, v_{i+1}.., v_1\}$ be a cycle. Choose a vertex $v_i \in C_1$ so that each v_i is only adjacent with

v_{i-1} and v_{i+1}, while not adjacent to any other vertices in C_1. Let $C_2 = \{v_i, w_1, w_2....., w_{i-1}, w_i, w_{i+1}...., v_i\}$ be another cycle, choose a vertex $w_i \in C_2$ o that each w_i is only adjacent with w_{i-1} and w_{i+1}, while not adjacent to any other vertices in C_2. Continuing like this, a collection of cycles $\{C_1, C_2, C_3......C_n\}$ are obtained. Let $\psi = \{C_1, C_2, C_3......C_n\} \cup \{E(D) - \{C_1, C_2, C_3......C_n\}\}$ is a simple graphoidal cover of D in which all the vertices are made internal. By corollary 5, Therefore $\eta_s(G) = q - p$. □

3. Product of Graphs

In this section, the simple graphoidal covering number on product of graphs are determined. The product of two graphs G and H which can be defined as a graph whose vertex set is V(G) X V(H)= $\{(g, h) : g \in V(G)$ and $h \in V(H)\}$, while the edge set varies according to the nature of the product. Various kinds of graph products are defined in [7]. Here we discuss only Cartesian product, Strong product of two directed paths or directed cycles. The Cartesian Product of two graphs G and H denoted by G ∘ H has the edge set E(G ∘ H)= $\{((g, h), (g', h')) : g = g', (h, h') \in E(H)$ or $h = h', (g, g') \in E(G)\}$. The Strong Product of graphs G and H is denoted by G ⊗ H where E(G ⊗ H)=$\{((g, h), (g', h')) : g = g', (h, h') \in E(H)); (g, g') \in E(G), h = h'$ or $(g, g') \in E(G)$ and $(h, h') \in E(H)\}$.

For the positive integer m and n, the following notations are used

$P_m = (u_1, u_2......., u_m)$
$P_n = (u_1, u_2......., u_n)$
$C_m = (u_1, u_2......., u_m, u_1)$
$C_n = (u_1, u_2......., u_n, u_1)$

For all the product graphs G, the vertex set V(G)= $\{w_{ij} : 1 \leq i \leq m; 1 \leq j \leq n\}$ where $w_{i,j} = (u_i, v_j)$ is considered.

Theorem 16. *If D=* $P_m \circ P_n$*, where* $m, n \geq 2$ *then* $\eta_s(D) = q - p + 3$

Proof. if $m, n \geq 2$, the collection of path of D are

$P = (w_{11}, w_{12}, w_{13}.......w_{1n}, w_{2n}, w_{3n}, w_{4n}.....w_{mn})$
$Q = (w_{11}, w_{21}, w_{31}.......w_{m1}, w_{m2}, w_{m3}, w_{m4}.....w_{mn-1})$
$R_i = (w_{1i}, w_{2i}, w_{3i}.......w_{mi}) : i = 2, 3....n - 1$

$S = E(G) - \{P\} \cup \{Q\} \cup \{R_i\}$.

Let $\psi = \{P_i \cup Q \cup \ R_i \cup S\}$ is a simple graphoidal covering number of D in which the vertices $w_{11}, w_{m,n-1}$ and $w_{m,n}$ are not internal. Therefore $\eta_s(D) \leq q-p+3$. Now, let ψ be any simple graphoidal cover of D. If ψ contains only paths, then atleast three vertices can not be made internal. Since ψ contains cycles, no vertex can be made internal, Therefore $t_\psi \geq 3$. Hence $\eta_s(D) \geq q-p+3$. Thus $\eta_s(D) = q-p+3$. □

Theorem 17. *If D= $C_m \circ C_n$, where $m, n \geq 3$, then $\eta_s(D) = q-p+1$*

Proof. if $m, n \geq 3$, the collection of paths of D are

$P_i = (w_{i1}, w_{i2}, w_{i3}.......w_{in}, w_{i1}) : i = 1, 2...m$
$Q = (w_{11}, w_{21}, w_{31}.......w_{m1}, w_{11})$
$R = E(D) - \{P_i\} \cup \{Q\}$.

Let $\psi = \{P_i \cup Q \cup \ R\}$ is a simple graphoidal cover of D in which the vertex w_{11} is not internal. Therefore $\eta_s(D) \leq q-p+1$. Now, let ψ be any simple graphoidal cover of D. If ψ contains only paths, then atleast one vertex can not be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $t_\psi \geq 1$. Hence $\eta_s(D) \geq q-p+1$. Thus $\eta_s(D) = q-p+1$. □

Theorem 18. *If D= $P_m \circ C_n$, where $m \geq 2, n \geq 3$, then $\eta_s(D) = q-p+2$*

Proof. if $m \geq 2, n \geq 3$, the collection of paths of D are

$P_i = (w_{i1}, w_{i2}, w_{i3}.......w_{in}, w_{i1}) : i = 1, 2...m$
$Q = (w_{11}, w_{21}, w_{31}.......w_{m1})$
$R = E(D) - \{P_i\} \cup \{Q\}$.

Let $\psi = \{P_i \cup Q \cup \ R\}$ is a simple graphoidal cover of D in which the vertices w_{11}, w_{m1} are not internal. Therefore $\eta_s(D) \leq q-p+2$. Now, let ψ be any simple graphoidal cover of D. If ψ contains only paths, then atleast six vertices can not be made internal. If ψ contains atleast one cycle, then atleast two vertex can not be made internal. Therefore $t_\psi \geq 2$. Hence $\eta_s(D) \geq q-p+2$. Thus $\eta_s(D) = q-p+2$. □

Theorem 19. *If D= $P_m \otimes P_n$, where $m, n \geq 2$, then $\eta_s(D) = q-p+4$*

Proof. if $m \geq 2, n \geq 3$, the collection of paths of D are

$P_i = (w_{i1}, w_{i2}, w_{i3}.......w_{in}) : i = 1, 2...m$
$Q = (w_{11}, w_{21}, w_{31}.......w_{m1})$
$R = (w_{1n}, w_{2n}, w_{3n}.......w_{mn})$
$S = E(D) - \{P_i\} \cup \{Q\} \cup \{R\}.$

Let $\psi = \{P_i \cup Q \cup R \cup S\}$ is a simple graphoidal cover of D in which the vertices $w_{11}, w_{m1}, w_{1n}, w_{mn}$ are not internal. Therefore $\eta_s(D) \leq q - p + 4$. Now, let ψ be any simple graphoidal cover of D. If ψ contains only paths, then atleast four vertices can not be made internal. If ψ contains atleast one cycle, then atleast no vertex can not be made internal. Therefore $t_\psi \geq 4$. Hence $\eta_s(D) \geq q - p + 4$. Thus $\eta_s(D) = q - p + 4$. □

Theorem 20. *If D= $C_m \otimes C_n$, where $m, n \geq 3$, then $\eta_s(D) = q - p + 1$*

Proof. if $m, n \geq 3$, the collection of paths of D are

$P_i = (w_{1i}, w_{2i}, w_{3i}.......w_{mi}, w_{1i}) : i = 1, 2...n$
$Q = (w_{11}, w_{12}, w_{13}.......w_{1n}, w_{11})$
$R = E(D) - \{P_i\} \cup \{Q\}.$

Let $\psi = \{P_i \cup Q \cup R\}$ is a simple graphoidal cover of D in which the vertex w_{11} is not internal. Therefore $\eta_s(D) \leq q - p + 4$. Now, let ψ be any simple graphoidal cover of D. If ψ contains only paths, then no verticies can be made internal. If ψ contains atleast one cycle, then atleast one vertex can not be made internal. Therefore $t_\psi \geq 1$. Hence $\eta_s(D) \geq q - p + 1$. Thus $\eta_s(D) = q - p + 1$. □

Theorem 21. *If D= $P_m \otimes C_n$, where $m \geq 2, n \geq 3$, then $\eta_s(D) = q - p + 2$*

Proof. if $m \geq 2, n \geq 3$, the collection of paths of D are

$P_i = (w_{i1}, w_{i2}, w_{i3}.......w_{in} w_{i1}) : i = 1, 2...m$
$Q = (w_{11}, w_{21}, w_{31}.......w_{m1})$
$R = E(D) - \{P_i\} \cup \{Q\}.$

Let $\psi = \{P_i \cup Q \cup R\}$ is a simple graphoidal cover of D in which the vertices w_{11}, w_{m1} are not internal. Therefore $\eta_s(D) \leq q - p + 2$. Now, let ψ be any simple graphoidal cover of D. If ψ contains only paths, then six verticies can not be made internal. If ψ contains atleast one cycle, then atleast two vertex can not be made internal. Therefore $t_\psi \geq 2$. Hence $\eta_s(D) \geq q - p + 2$. Thus $\eta_s(D) = q - p + 2$. □

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