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CERTAIN PROPERTIES OF THE ESSENTIAL SPECTRA IN BANACH SPACES

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Abstract: For an arbitrary Banach space X, let $\mathcal{L}(X)$ be the set of all bounded linear operators on X. We apply closed range theorem to determine some duality properties of the essential spectrum on $\mathcal{L}(X)$. Specifically, we introduce different parts of the essential spectrum and establish their duality relations. We also consider certain algebraic properties of the essential spectra in Banach spaces.

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1. Introduction

Let $\mathcal{L}(X)$ be the space of bounded linear operators on a Banach space X. Also, let X^* denotes the dual space of X. For $T \in \mathcal{L}(X)$, the spectrum of T, denoted by $\sigma(T)$, is defined as the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is not invertible. The spectrum $\sigma(T)$ is nonempty, closed and compact subset of the complex plane $\mathbb{C}$, see for instance [10, 11]. The point spectrum of T, $\sigma_p(T)$, is the set of eigenvalues of T, while the approximate point spectrum of T, $\sigma_{ap}(T)$ consists of all $\lambda \in \mathbb{C}$ for which there exists a sequence $(x_n)_n$ of unit vectors such that $(\lambda I - T)x_n \to 0$ as $n \to \infty$. For $T : X \to Y$ where both X and Y are Banach spaces, we define the nullspace (kernel) of T by $N(T) = \{x \in D(T) : Tx = 0\}$

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and the range of T by $R(T) = \{Tx : x \in D(T)\}$, where $D(T)$ denotes the domain of T. The adjoint of T will be denoted by T^*. For an arbitrary subset M of a Banach space X, let $M^\perp := \{\phi \in X^* : \phi(x) = 0 \text{ for all } x \in M\}$ denote the annihilator of M in X^*. The kernels and ranges of a closed operator and its adjoint are intimately related via annihilators [3]. For instance, it is well known that for a densely defined closed operator $T : X \to X$, it can be easily verified that $N(T^*) = R(T)^\perp$ and $R(T^*) = N(T)^\perp$.

An operator $T \in \mathcal{L}(X, Y)$ is said to be Fredholm if the dimension of $N(T)$ and the codimension of $R(T)$ are finite, that is, $\dim N(T) < \infty$ and the codim $R(T) < \infty$. Note that codim $R(T) = \dim Y/R(T)$ and the index of T, denoted by ind(T), is given by ind$(T) = \dim N(T) - \operatorname{codim} R(T)$. The essential spectrum of an operator $T \in \mathcal{L}(X)$, $\sigma_e(T)$ is therefore the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is not Fredholm. For a good theory of spectra and essential spectra, we refer to [3, 4, 5, 7, 8, 9] and references therein. If $T \in \mathcal{L}(X)$, the essential spectral radius $r_e(T)$ is defined as $r_e(T) = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$. In this paper, we give some results on the essential spectrum of bounded linear operators acting on a Banach space. More precisely, we give some algebraic and duality properties of the essential spectra. The following results which are readily available in literature (see [[3], Theorem A.1.10, Proposition 1.2.3, Theorem A.1.8] respectively) will be used in the sequel and for completion, we give their statements:

Theorem 1 (Closed Range Theorem)**.** *Let X and Y be Banach spaces and $T : D(T) \to Y$ be a densely defined and closed linear operator from X into Y. Then the following assertions are equivalent:*
(i). $R(T)$ is closed in Y
(ii). $R(T^*)$ is closed in X^*
(iii). $R(T^*) = N(T)^\perp$
(iv). $T : D(T) \to R(T)$ is open.
(v). $T^* : D(T^*) \to R(T^*)$ is open.
(vi). $R(T) = N(T^*)^\perp$

Theorem 2. *For every operator $T \in \mathcal{L}(X)$ on a Banach space X and for every $\lambda \in \mathbb{C}$ the following assertions are equivalent:*
(i). $\lambda I - T$ is bounded below, i.e. $\exists c > 0$: $\|(\lambda I - T)x\| \geq c\|x\|$.
(ii). $N(\lambda I - T) = \{0\}$ and $R(\lambda I - T)$ closed.
(iii). $\lambda \notin \sigma_{ap}(T)$.
Moreover, $\sigma_{ap}(T)$ is a closed subset of $\sigma(T)$ that contains the boundary $\partial\sigma(T)$. In particular, $\sigma_{ap}(T)$ is nonempty whenever X is non-trivial.

Theorem 3 (Annihilator Theorem)**.** *Let M be a closed linear subspace*

of a Banach space X. Then the restriction mapping from X^* onto M^* and the quotient mapping from X onto X/M induce the canonical identifications;
(i). $(X/M)^* \cong M^\perp$
(ii). $X^*/M^\perp \cong M^*$
in the sense of isometric norm isomorphisms between Banach spaces.

2. Some Algebraic Properties of the Essential Spectrum

Let X be a complex Banach space and consider the quotient space $C(X) = \mathcal{L}(X)/K(X)$ endowed with the usual vector space operations and the canonical quotient norm. Also let $q : \mathcal{L}(X) \to C(X)$ denote the corresponding quotient mapping so that $q(T) = T + K(X)$ for all $T \in L(X)$. Moreover, for $T^* \in \mathcal{L}(X^*)$, we define an equivalent map $\tilde{q}$ by $\tilde{q}(T^*) = T^* + K(X^*)$. The following facts can be easily verified:

(i) $C(X)$ is a Banach algebra with respect to the multiplication:

$$(T + K(X))(S + K(X)) = TS + K(X)$$

for all $T, S \in \mathcal{L}(X)$. $C(X)$ is called the Calkin algebra

(ii) Given an arbitrary $T \in \mathcal{L}(X)$, then T is a Fredholm operator on X if and only if $q(T)$ is invertible in the Calkin algebra $C(X)$

(iii) $\sigma_e(T) = \sigma(q(T))$

(iv) $\sigma_e(T)$ is compact and nonempty for each $T \in \mathcal{L}(X)$ provided that X is of infinite dimension.

Theorem 4. *Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Then $\sigma_e(T) = \sigma_e(T^*)$.*

Proof. It suffices to prove that for $\lambda \in \mathbb{C}$, $\lambda \notin \sigma_e(T)$ if and only if $\lambda \notin \sigma_e(T^*)$. Equivalently, for $\lambda \in \mathbb{C}$, $\lambda I - T$ is Fredholm if and only if $\lambda I - T^*$ is Fredholm. Thus it suffices to prove that $T : X \to Y$ is Fredholm if and only if $T^* : Y^* \to X^*$ is Fredholm. Recall that T is Fredholm if and only if dim $N(T)$, dim$Y/R(T) < \infty$. It is therefore enough to show that dim $N(T^*) =$ dim$Y/R(T)$ and dim $X^*/R(T^*) =$ dim$N(T)$. Now $N(T^*) = R(T)^\perp \cong (Y/R(T))^*$ and since $\dim(Y/R(T)) < \infty$, it follows that $\dim(Y/R(T))^* < \infty$. So dim$N(T^*) =$ dim $(Y/R(T))^* = \dim(Y/R(T)) < \infty$. By closed range theorem, $R(T^*)$ is closed and equals $N(T)^\perp$. Now $X^*/R(T^*) = X^*/N(T))^\perp \cong N(T)^*$. Since dim$N(T) < \infty$, we have dim$N(T)^* =$ dim $N(T)$ and so dim$(X^*/R(T^*) =$ dim $N(T) < \infty$. Conversely, if T^* is Fredholm then dim$N(T^*) < \infty$ and dim $X^*/R(T^*) < \infty$. It therefore suffices to show that dim $N(T) =$ dim$X^*/R(T^*)$ and dim $Y/R(T) =$ dim$N(T^*)$. Now $X^*/R(T^*) = X^*/N(T))^\perp \cong N(T)^*$ and since dim$(X^*/R(T^*)) < \infty$, it follows that dim$N(T)^* =$ dim $N(T) < \infty$. By the closed range theorem, $N(T^*) = R(T)^\perp \cong (Y/R(T))^*$. But dim $N(T^*) < \infty$, and so it follows that dim $(Y/R(T))^*) =$ dim $(Y/R(T)) < \infty$ as desired. □

An immediate consequence is the following

Corollary 5. *For* $S, T \in \mathcal{L}(X), \sigma_e(T+S) = \sigma_e((T+S)^*)$.

Proof. Follows from the Theorem 4 above. □

Another consequence that relates the essential spectral radius of an operator and that of its adjoint is the following:

Corollary 6. *For* $S, T \in \mathcal{L}(X)$, *the following hold;*

(i) $r_e(T^*) = r_e(T)$

(ii) $r_e(T+S) = r_e((T+S)^*)$

Proof. (i) follows immediately from the Theorem 4 above as well as the definition of the essential spectral radius, while (ii) is a consequence of the Corollary 5. □

The next Theorem relates the essential spectrum of an operator multiplied by a scalar and the spectrum of that operator.

Theorem 7. *For* $T \in \mathcal{L}(X), \alpha \in \mathbb{R}$ *with* $\alpha \neq 0$, *we have* $\sigma_e(\alpha T) = \alpha\sigma_e(T)$. *Moreover* $r_e(\alpha T) = |\alpha| r_e(T)$.

Proof. Recall that $\lambda \in \sigma_e(\alpha T)$ if and only if $\lambda I - \alpha T$ is not Fredholm. Factoring out the scalar, we have $\alpha(\frac{\lambda}{\alpha}I - T)$ not Fredholm, and hence $\frac{\lambda}{\alpha}I - T$ not Fredholm. The latter statement is equivalent to $\frac{\lambda}{\alpha} \in \sigma_e(T)$, and hence $\lambda \in \alpha\sigma_e(T)$. Thus $\lambda \in \sigma_e(\alpha T)$ if and only if $\lambda \in \alpha\sigma_e(T)$. Therefore $\sigma_e(\alpha T) = \alpha\sigma_e(T)$ as claimed. By the definition of the essential spectra and the corresponding radius it immediately follows that $r_e(\alpha T) = |\alpha| r_e(T)$. □

Relating the spectra of two operators and that of their sum on Banach spaces is not obvious. In fact if A and B are two operators on a Banach space, then in general $\sigma(A), \sigma(B)$ and $\sigma(A+B)$ are not related. The question has been when can they be related? This question together with related ones were considered in [1, 2]. For the essential spectra analogue, Shappiro and Snow [6] considered such questions and obtained among others the following result;

Theorem 8. *Let $A, B \in \mathcal{L}(X)$ and suppose that A and B commute, then we have:*

(i) $\sigma_e(A+B) \subseteq \sigma_e(A) + \sigma_e(B)$.

Moreover, if B is Fredholm, then

(ii) $\sigma_e(AB) \subseteq \sigma_e(A)\sigma_e(B)$.

Now if we scale the operators A and B using scalars α and β, we obtain the following result,

Corollary 9. *Let A and B be defined as in Theorem 8 above. Then for $\alpha, \beta \in \mathbb{R}$, we have*

(i) $\sigma_e(\alpha A + \beta B) \subseteq \alpha\sigma_e(A) + \beta\sigma_e(B)$.

Moreover, if B is Fredholm, then

(ii) $\sigma_e\left((\alpha A)(\beta B)\right) \subseteq \alpha\beta\sigma_e(A)\sigma_e(B)$.

Proof. Using Theorems 7 and 8 above, we have $\sigma_e(\alpha A + \beta B) \subseteq \sigma_e(\alpha A) + \sigma_e(\beta B) = \alpha\sigma_e(A) + \beta\sigma_e(B)$, which proves (i). The proof of (ii) is similar. Indeed, $\sigma_e\left((\alpha A)(\beta B)\right) \subseteq \sigma_e(\alpha A)\sigma_e(\beta B) \subseteq \alpha\beta\sigma_e(A)\sigma_e(B)$. □

Another consequence giving the essential spectral radii relations is the following,

Corollary 10. *Let A and B be defined as in Theorem 8. Then for $\alpha, \beta \in \mathbb{R}$, we have*

(i) $r_e(\alpha A + \beta B) \leq |\alpha| r_e(A) + |\beta| r_e(B)$.

(ii) $r_e((\alpha A)(\beta B)) \leq |\alpha\beta| r_e(A) r_e(B)$.

Proof. Follows immediately from the definition of the essential spectral radius and the Corollary 9 above. □

3. Parts of the Essential Spectrum and Duality

We begin this section by stating and proving the following Lemma,

Lemma 11. For $T \in \mathcal{L}(X)$, we have $\tilde{q}(T^*) = q(T)^*$ for all $T^* \in \mathcal{L}(X^*)$. In particular, $C(X)^* = C(X^*)$.

Proof. Clear from the fact that an operator is compact if and only if its adjoint is compact. Indeed, $q(T)^* = (T + K(X))^* = T^* + K(X)^* = T^* + K(X^*) = \tilde{q}(T^*)$. □

From Section 2 fact (iii), it is apparent that the essential spectrum $\sigma_e(T)$ is (by definition) the spectrum of the coset $T + K(X)$ in the Calkin algebra $C(X) = \mathcal{L}(X)/K(X)$ where $K(X)$ is the ideal of all compact operators on X. More precisely, $\sigma_e(T) = \sigma(q(T))$. Using this fact, we introduce the following parts of the essential spectrum on $\mathcal{L}(X)$:

Definition 12.

1. Essential approximate point spectrum $\sigma_{ap}^{ess}(T)$
   $\sigma_{ap}^{ess}(T) = \{\lambda \in \mathbb{C}$ there exists $x_n \subset X$ such that$(q(T) - \lambda I)x_n \to 0$ as $n \to \infty\}$.

2. Essential surjectivity spectrum $\sigma_{su}^{ess}(T)$
   $\sigma_{su}^{ess}(T) = \{\lambda \in \mathbb{C} : q(T) - \lambda I$ is not surjective $\}$.

3. Essential point spectrum $\sigma_p^{ess}(T)$
   $\sigma_p^{ess}(T) = \{\lambda \in \mathbb{C} : (q(T) - \lambda I)x = 0$ for some $x \neq 0, x \in X\}$. In other words $\sigma_p^{ess}(T)$ is the set of eigenvalues of $q(T)$.

4. Essential compression spectrum $\sigma_{com}^{ess}(T)$
   $\sigma_{com}^{ess}(T) = \{\lambda \in \mathbb{C} : R(q(T) - \lambda I)$ is not dense in $C(X)\}$.

From the above definition, it's apparent that $\sigma_{ap}^{ess}(T) = \sigma_{ap}(q(T))$, $\sigma_{su}^{ess}(T) = \sigma_{su}(q(T))$, $\sigma_p^{ess}(T) = \sigma_p(q(T))$ and $\sigma_{com}^{ess}(T) = \sigma_{com}(q(T))$, where σ_{ap}, σ_{su}, σ_p and σ_{com} are the usual approximate point spectrum, surjectivity spectrum, point spectrum and compression spectrum respectively. For a comprehensive theory on these spectra, we refer to [2, 3, 7].
There are some obvious relations between the various parts of the essential spectrum defined above. In particular, we give the following result,

Theorem 13. *Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$. Then*

(i) $\sigma_p^{ess}(T) \subseteq \sigma_{ap}^{ess}(T)$

(ii) $\sigma^{ess}_{com}(T) \subseteq \sigma^{ess}_{su}(T)$

(iii) $\sigma_e(T) = \sigma^{ess}_{p}(T) \cup \sigma^{ess}_{su}(T)$.
In particular, $\sigma_e(T) = \sigma^{ess}_{ap}(T) \cup \sigma^{ess}_{com}$.

Proof. (i) is clear from the definition. For (ii), let $\lambda \in \sigma^{ess}_{com}(T)$. Then $\overline{R(q(T) - \lambda I)} \neq C(X)$, which implies that $q(T) - \lambda I$ is not surjective, as desired. Moreover,

$$\begin{aligned}
\sigma_e(T) &= \sigma(q(T)) = \{\lambda \in \mathbb{C} : q(T) - \lambda I \text{ is not invertible}\} \\
&= \{\lambda \in \mathbb{C} : q(T) - \lambda I \text{ is not bijective}\} \\
&= \{\lambda \in \mathbb{C} : q(T) - \lambda I \text{ is not surjective, or } q(T) - \lambda I \text{ is not injective}\} \\
&= \sigma^{ess}_{p}(T) \cup \sigma^{ess}_{su}(T),
\end{aligned}$$

which proves (iii). Finally, by the characterization of the approximate point spectrum given by Theorem 2, we have that $\lambda \in \sigma_{ap}(q(T))$ if and only if either $q(T) - \lambda I$ is not injective or $R(q(T) - \lambda I)$ is not closed. This completes the proof. □

Duality theory is known to provide connections between the various parts of the spectrum of an operator on X and the corresponding parts of the spectrum of the adjoint operator on the dual space X^*. In our next result we give an interesting relationship between the essential surjectivity spectrum of an operator and the essential approximate point spectrum of its adjoint on a Banach space, and vice versa.

Theorem 14. *Let X be an infinite dimensional Banach space and $T \in \mathcal{L}(X)$, then we have the following relations;*

(i) $\sigma^{ess}_{ap}(T) = \sigma^{ess}_{su}(T^*)$,

(ii) $\sigma^{ess}_{su}(T) = \sigma^{ess}_{ap}(T^*)$

where $\sigma^{ess}_{su}(T^*) = \sigma_{su}(\tilde{q}(T^*))$ *and* $\sigma_{ap}(T^*) = \sigma^{ess}_{ap}(\tilde{q}(T^*))$.

Proof. To prove (i), we shall show that $\mathbb{C} \setminus \sigma^{ess}_{ap}(T) = \mathbb{C} \setminus \sigma^{ess}_{su}(T^*)$. Using the characterization of the approximate point spectrum given by Theorem 2, it suffices to prove that $q(T)$ is bounded below if and only $\tilde{q}(T^*)$ is surjective. Since $q(T)$ is bounded below, it follows that $R(q(T))$ closed and $N(q(T)) = \{0\}$. By Closed range theorem $R(\tilde{q}(T^*)) = R(q(T)^*) = N(q(T))^{\perp} = \{0\}^{\perp} = C(X)$, and so $\tilde{q}(T^*)$ is surjective as desired. Conversely, let $R(\tilde{q}(T^*)) = C(X^*)$ and so $R(\tilde{q}(T^*)$ is closed. By Closed range theorem, $R(q(T))$ is closed and $N(q(T)) =$

$R(\tilde{q}(T^*))^\perp = (C(X^*))^\perp = \{0\}$. Therefore $q(T)$ is bounded below, and this completes the proof of (i).
Now, to prove (ii), we shall prove that $\mathbb{C} \setminus \sigma_{su}^{ess}(T) = \mathbb{C} \setminus \sigma_{ap}^{ess}(T^*)$. That is, for $\lambda \in \mathbb{C}$ we have; $\lambda \notin \sigma_{su}^{ess}(T)$ if and only if $\lambda \notin \sigma_{ap}^{ess}(T^*)$. Thus $\lambda I - q(T)$ surjective is equivalent to $\lambda I - \tilde{q}(T^*)$ bounded below. It therefore suffices to prove that $q(T)$ is surjective if and only if $q(T^*)$ bounded below. Now let $R(q(T)) = C(X)$, then by the Closed range theorem, $R(\tilde{q}(T^*))$ is closed and $N(\tilde{q}(T^*)) = R(q(T))^\perp = \{0\}$. Hence $\tilde{q}(T^*)$ is bounded below.
Conversely, if $\tilde{q}(T^*)$ is bounded below, then $R(\tilde{q}(T^*))$ is closed and $N(\tilde{q}(T^*)) = R(q(T))^\perp = \{0\}$. By the Closed range theorem $R(q(T))$ is closed and $R(q(T)) = N(\tilde{q}(T^*))^\perp = \{0\}^\perp = C(X)$. Therefore $q(T)$ is surjective, and this completes the proof. □

The following Theorem gives the relationship of the compression essential spectrum of an operator with the essential point spectrum of its adjoint on a Banach space and vice versa.

Theorem 15. *Let X be an infinite dimensional Banach space and let $T \in \mathcal{L}(X)$. Then we have;*

(i) $\sigma_{com}^{ess}(T) = \sigma_p^{ess}(T^*)$,

(ii) $\sigma_p^{ess}(T) \subseteq \sigma_{com}^{ess}(T^*)$

where $\sigma_p^{ess}(T^*) = \sigma_p(\tilde{q}(T^*))$ *and* $\sigma_{com}^{ess}(T^*) = \sigma_{com}(\tilde{q}(T^*))$.

Proof. We wish to prove that $\mathbb{C} \setminus \sigma_{com}^{ess}(T) = \mathbb{C} \setminus \sigma_p^{ess}(T^*)$. That is, for $\lambda \in \mathbb{C}$, we have $\lambda \notin \sigma_{com}^{ess}(T)$ if and only if $\lambda \notin \sigma_p^{ess}(T^*)$. This means $\lambda I - q(T)$ has a dense range, and this is equivalent to $\lambda I - \tilde{q}(T^*)$ being injective. So $\overline{R(\lambda I - q(T))} = C(X) \Leftrightarrow N(\lambda I - q(T^*)) = \{0\}$. It suffices to prove that $\overline{R(q(T))} = C(X)$ if and only if $N(\tilde{q}(T^*)) = \{0\}$. Since $N(\tilde{q}(T^*)) = R(q(T))^\perp$ and from the Hahn-Banach extension theorem, it follows that for each $M \subseteq Y$,

$$\overline{M} = Y \Leftrightarrow M^\perp = \{0\}. \tag{1}$$

Now from equation (1), $N(q(T^*)) = R(q(T))^\perp = \{0\}$ which completes the proof of (i). To prove (ii), we use the assertion (i) above but instead replace T with T^*. Then we have $\sigma_p^{ess}(T^*) \subseteq \sigma_p^{ess}(T^{**}) = \sigma_{com}^{ess}(T^*)$. □

Remark 16. If X is reflexive, that is, $X \cong X^{**}$; then $\sigma_p^{ess}(T) = \sigma_{com}^{ess}(T^*)$.

References

[1] M. Zima, A theorem on the spectral radius of the sum of two operators and its applications, *Bull. Austral. Math. Soc.*, **48** (1993), 427-434, **doi:** 10.1017/S0004972700015884.

[2] C. Davis, P. Rosenthal, Solving linear operator equations, *Can. J. Math.*, **26** (1974), 1384-1389, **doi:** 10.4153/CJM-1974-132-6.

[3] K. B. Laursen, M. M. Neumann, *Introduction to Local Spectral Theory*, London Math. Soc. Monographs, Oxford University press, New York, 2000.

[4] A. F. Ruston, *Fredholm Theory in Banach Spaces*, Cambridge University press, Cambridge, 1986, **doi:** 10.1017/CBO9780511569180.

[5] L. Fialkow, Essential spectra of elementary operators, *Trans. Amer. Math. Soc.*, **267** (1981), 157-174, **doi:** 10.1090/S0002-9947-1981-0621980-8.

[6] J. Shapiro, M. Snow, The Fredholm spectrum of the sum and product of two operators, *Trans. Amer. Math. Soc.*, **191** (1974), 387-393, **doi:** 10.1090/S0002-9947-1974-0454682-8.

[7] K. B. Laursen, *Essential spectra through local spectral theory.* Proc. Amer. Math. Soc. **125** (1997), 1425–1434.

[8] A. Dehici, *On some properties of spectra and essential spectra in Banach spaces.* Sarajevo Journal of Mathematics **11** (2015), 219–234.

[9] A. G. Aksoy, *The radius of the essential spectrum.* Journal of Mathematical Analysis and Applications **128** (1987), 101–107. http://dx.doi.org/10.1016/0022-247X(87)90217-4

[10] J. B. Conway, *A course in Functional Analysis.* Springer - Verlag, New York Inc., 1985. http://dx.doi.org/10.1007/978-1-4757-3828-5

[11] W. Rudin, *Functional Analysis.* McGraw - Hill, Inc., USA, 1973.