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ASYMPTOTIC ESTIMATES OF SOME POSITIVE INTEGRALS OUTSIDE AN EXCEPTIONAL SETS

O.B. Skaskiv[1], O.Yu. Tarnovecka[2], D.Yu. Zikrach[3] §

[1,2,3]Ivan Franko National University of Lviv
1, Universytetska St., Lviv, 79000, UKRAINE

Abstract: Consider the functions represented by integrals of the form

$$F(x) = \int_0^{+\infty} g(t)f(tx + \beta(t)\tau(x))d\nu(t).$$

We found conditions for this functions under which Borel's type relation $\ln F(x) \le (1 + o(1)) \ln \mu(x, F)$ holds as $x \to +\infty$ outside some set E of finite Lebesgue measure.

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1. Introduction

Let $\mathcal{I}(\nu, f, \tau, \beta)$ be a class of a functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ represented by integrals of the form

$$F(x) = \int_0^{+\infty} g(t)f(tx + \beta(t)\tau(x))d\nu(t)$$

for each $x \in \mathbb{R}_+$. Here ν is a locally finite measure on $\mathbb{R}_+$, g positive ν-measurable function, f positive increase to $+\infty$ on $[0; +\infty)$ function such that $f(0) = 1$ and $\ln f(x)$ is convex on the interval $[0; +\infty)$. Functions $\beta(t)$ and $\tau(x)$ are nonnegative on $[0; +\infty)$ such that $f_x(t) := f(tx + \beta(t)\tau(x))$ is ν-measurable

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§Correspondence author

function for every fixed $x \geq 0$. We put $\mathcal{I}(\nu, f) \equiv \mathcal{I}(\nu, f, 0, 0)$ with $\beta(t) \equiv 0$, and $\mathcal{I}(\nu) \equiv \mathcal{I}(\nu, f)$ with $f(x) \equiv e^x$, i.e. we obtain the classes of functions represented by integrals of the form

$$F(x) = \int_0^{+\infty} g(t) f(tx) d\nu(t) \text{ and } F(x) = \int_0^{+\infty} g(t) e^{tx} d\nu(t),$$

respectively. By supp ν we denote the support of the measure ν. Note, that in the case of supp $\nu = \{\lambda_n : n \geq 0\}$, where (λ_n) is a sequence of positive numbers, the classes $\mathcal{I}(\nu, f)$ and $\mathcal{I}(\nu)$ are generalizations of the classes of functions represented by series

$$F(x) = \sum_{n=0}^{+\infty} a_n f(x\lambda_n) \text{ and } F(x) = \sum_{n=0}^{+\infty} a_n e^{x\lambda_n},$$

respectively.

For $F \in \mathcal{I}(\nu, f, \tau, \beta)$ and $x \geq 0$ we denote

$$\mu(x, F) = \sup\{g(t) f(tx + \beta(t)\tau(x)) : t \in \text{supp } \nu\}.$$

In the papers [1, 2] we find the following theorem.

Theorem A ([1, 2]). *In order that for every function* $F \in \mathcal{I}(\nu)$

$$\ln F(x) \leq (1 + o(1)) \ln \mu(x, F) \tag{1}$$

holds as $x \to +\infty$ *outside some set* E *of finite Lebesgue measure (meas* $E < +\infty$*), necessary and sufficient that*

$$(\exists t_0 > 0): \quad \int_{t_0}^{+\infty} \frac{d \ln \nu_0(t)}{t} < +\infty, \tag{2}$$

where $\nu_0(t) := \nu((0; t])$.

For the multiple Laplace–Stieltjes integrals we find similarly theorems in [6, 7]. For entire Dirichlet series of the form

$$F(x) = \sum_{n=0}^{+\infty} a_n e^{x\lambda_n}, \quad 0 = \lambda_0 < \lambda_n \uparrow +\infty \ (1 \leq n \uparrow +\infty),$$

analogous Theorem was proved in [8]. In this case, conditions (2) can be replaced by

$$\sum_{n=1}^{+\infty} \frac{1}{n\lambda_n} < +\infty. \tag{3}$$

Some analogs of classical Wiman's inequality was obtained for the class $I(\nu)$ in [3]. Many authors ([4, 5]), showed their interest in studying the Laplace–Stieltjes integrals in a classical sense. In the present article, we prove some improvements of Theorem A for the class $\mathcal{I}(\nu, f, \tau, \beta)$ and obtain some corollaries for the Taylor–Dirichlet type series.

2. Main Results

Theorem 1. *If function $\tau(x)$ is positive differentiable on $[x_0, +\infty)$ such that $\tau'(x) \geq 1$ $(x \geq x_0)$ and condition (2) holds with*

$$\nu_0(t) = \nu\big(\{u \geq 0\colon \ln f(u + \beta(u)) \leq t\}\big),$$

then for every function $F \in \mathcal{I}(\nu, f, \tau, \beta)$ there exists a set E of finite Lebesgue measure such that asymptotic relation (1) holds as $x \to +\infty$ $(x \notin E)$.

Proof of Theorem 1. Let for fixed $x > 0$

$$G := G_x = \Big\{t > 0\colon (\ln f(u))'\Big|_{u=xt+\tau(x)\beta(t)} \cdot (t + \tau'(x)\beta(t)) \leq 2g'(x)\Big\},$$

where $g(x) := \ln F(x)$. First we prove that from the condition $\tau'(x) \geq 0$ $(x \geq x_0)$ follows

$$F(x) \leq 2\int_G a(t)f(xt + \tau(x)\beta(t))\nu(dt) \tag{4}$$

for all $x \geq x_0$. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}_+\setminus G} a(t)f(xt + \tau(x)\beta(t))d\nu(t)\\
\leq &\int_{\mathbb{R}_+\setminus G} a(t)f'(xt + \tau(x)\beta(t))\Big((\ln f(u))'\Big|_{u=xt+\tau(x)\beta(t)}\Big)^{-1} d\nu(t)\\
\leq &\frac{1}{2g'(x)}\int_{\mathbb{R}_+\setminus G} a(t)f'(xt + \tau(x)\beta(t))(t + \tau'(x)\beta(t))d\nu(t)\\
\leq &\frac{1}{2g'(x)}\int_{\mathbb{R}_+} a(t)f'(xt + \tau(x)\beta(t))(t + \tau'(x)\beta(t))d\nu(t) = \frac{F(x)}{2}.
\end{aligned}$$

Therefore,

$$F(x) = \int_G a(t)f(xt + \tau(x)\beta(t))\nu(dt)$$

$$+\int_{\mathbb{R}_+\setminus G} a(t)f(xt+\tau(x)\beta(t))d\nu(t)\le$$
$$\le\int_G a(t)f(xt+\tau(x)\beta(t))\nu(dt)+\frac{F(x)}{2},$$

and inequality (4) follows.

Without loss of generality we may assume that $\tau(x)\ge 1$ $(x\ge x_0)$ and $x_0\ge 1$, because from the condition $\tau'(x)\ge 1$ $(x\ge x_0)$ follows that $\varliminf\limits_{x\to+\infty}\tau(x)/x\ge 1$.

The convexity of the function $\ln f(x)$ implies

$$(\ln f(u))'\ge\frac{\ln f(u)}{u}\ge\frac{\ln f(u_1)-\ln f(0)}{u_1-0}$$

for all $u\ge u_1>0$. Therefore,

$$(\ln f(u))'\Big|_{u=xt+\tau(x)\beta(t)}\ge\frac{\ln f(u)}{u}\Big|_{u=xt+\tau(x)\beta(t)}\ge\frac{\ln f(u)}{u}\Big|_{u=t+\beta(t)}\tag{5}$$

for all $x\ge x_0, t>0$. So,

$$G=G_x\subset\Big\{t>0\colon\ \frac{\ln f(u)}{u}\Big|_{u=t+\beta(t)}\cdot(t+\beta(t))\le 2g'(x)\Big\}$$
$$=\Big\{t>0\colon\ \ln f(t+\beta(t))\le 2g'(x)\Big\}:=G_1\quad(x\ge x_0,\ t>0).$$

Hence and from inequality (4) we obtain for $x\ge x_0$

$$F(x)\le 2\int_{G_1}a(t)f(xt+\tau(x)\beta(t))\nu(dt)\le\mu(x,F)\cdot\nu_0(2g'(x)).\tag{6}$$

We remark that condition (2) is equivalent to

$$\int_{t_0}^{+\infty}\frac{\ln\nu_0(t)}{t^2}dt<+\infty.$$

Hence (see [1]) we have that there exist a continuous function $\psi\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $\psi(t)\uparrow+\infty$ $(t_0\le t\to+\infty)$ and

$$\int_0^{+\infty}\frac{dt}{\psi(t)}<+\infty,$$
$$\ln\nu_0(t)=o(\psi^{-1}(t))\ (t\to+\infty).\tag{7}$$

For the function $\psi_0(t) = \psi(t)/2$, we define a set

$$E := \{x > x_0 \colon g'(x) \geq \psi_0(g(x))\}.$$

The set E is of finite Lebesgue measure. Indeed, we have

$$meas(E \cap [x_0, +\infty)) = \int_E dx \leq \int_E \frac{g'(x)}{\psi_0(g(x))} dx \tag{8}$$

$$\leq \int_0^{+\infty} \frac{dt}{\psi_0(t)} < +\infty. \tag{9}$$

Finally, we deduce from inequality (6) using second relation of (7)

$$\begin{aligned} \ln F(x) &\leq \ln 2 + \ln \mu(x, F) + \ln \nu_0(2g'(x)) \\ &\leq \ln 2 + \ln \mu(x, F) + \ln \nu_0(\psi(g(x))) \\ &= \ln \mu(x, F) + o(g(x)), \end{aligned}$$

as $x \to +\infty$ $(x \notin E)$. This implies relation (1).

The proof of Theorem 1 is completed. □

Similarly, we obtain such statement.

Theorem 2. *If function $\tau(x)$ is positive differentiable on $[x_0, +\infty)$ such that $\tau'(x) \geq 1$ $(x \geq x_0)$ and condition* (2) *holds with*

$$\nu_0(t) = \nu\big(\{u \geq 0 \colon \ln f(u) \leq t\}\big),$$

then for every function $F \in \mathcal{I}(\nu, f, \tau, \beta)$ there exists a set E of finite Lebesgue measure such that asymptotic relation (1) *holds as $x \to +\infty$ $(x \notin E)$.*

Proof of the Theorem 2. Because $\tau(x) \geq 0$ and $\tau'(x) \geq 0$ $(x \geq x_0)$, from the convexity of the function $\ln f(x)$ we get

$$(\ln f(u))' \Big|_{u = xt + \tau(x)\beta(t)} \geq \frac{\ln f(u)}{u} \Big|_{u = xt + \tau(x)\beta(t)} \geq \frac{\ln f(t)}{t} \tag{10}$$

for all $x \geq \max\{1, x_0\}, t > 0$ and thus

$$\begin{aligned} G = G_x \subset \Big\{t > 0 \colon \frac{\ln f(u)}{u}\Big|_{u=t} \cdot t \leq 2g'(x)\Big\} \\ = \Big\{t > 0 \colon \ln f(t) \leq 2g'(x)\Big\} := G_2 \quad (x \geq \max\{1, x_0\},\ t > 0). \end{aligned}$$

Now again condition (2) implies that conditions (7) hold. Thus as above we obtain that the set $E := \{x > x_0\colon g'(x) \geq \psi_0(g(x))\}$ has finite Lebesgue measure and from inequality (4) by second relation of (7) it follows

$$\begin{aligned}\ln F(x) \leq \ln 2 + \ln \mu(x, F) + \ln \nu(G_2) &= \ln 2 + \ln \mu(x, F) + \ln \nu_0(2g'(x)) \\ &\leq \ln 2 + \ln \mu(x, F) + \ln \nu_0(\psi(g(x))) \\ &= \ln \mu(x, F) + o(g(x))\end{aligned}$$

as $x \to +\infty$ $(x \notin E)$. The proof of Theorem 2 is completed. □

3. Corollaries

3.1. Laplace–Stieltjes type integrals. We consider the class $\mathcal{I}(\nu, f)$ of functions $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ represented by integrals of the form

$$F(x) = \int_0^{+\infty} g(t) f(tx) d\nu(t).$$

Corollary 3. *If condition* (2) *holds with* $\nu_0(t) = \nu\big(\{u \geq 0\colon \ln f(u) \leq t\}\big)$, *then for every function* $F \in \mathcal{I}(\nu, f)$ *there exists a set* E *of finite Lebesgue measure such that asymptotic relation* (1) *holds as* $x \to +\infty$ $(x \notin E)$.

The statement of Corollary 3 follows from Theorem 2 with $\beta(t) \equiv 0$. For $f(x) = e^x$ Corollary 3 implies immediately such assertion.

Corollary 4 (see Theorem A)**.** *If condition* (2) *holds with* $\nu_0(t) = \nu\big([0, t]\big)$, *then for every function* $F \in \mathcal{I}(\nu, f)$ *there exists a set* E *of finite Lebesgue measure such that asymptotic relation* (1) *holds as* $x \to +\infty$ $(x \notin E)$.

3.2. Taylor–Dirichlet type series. Let $\mathcal{TD}(\lambda, \beta, \tau)$ be the class of positive convergent for all $x \geq 0$ series of the form

$$F(x) = \sum_{n=0}^{+\infty} a_n e^{x\lambda_n + \tau(x)\beta_n}, \quad 0 = \lambda_0 < \lambda_n \uparrow +\infty \ (1 \leq n \uparrow +\infty).$$

Here $n_\lambda(t) = \sum_{\lambda_n \leq t} 1$ is a counting function of the sequence $\lambda = (\lambda_n)$, $\beta = (\beta_n)$ is a positive sequence, a differentiable function $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\tau'(x) \geq 0$ $(x \geq x_0)$. For $F \in \mathcal{TD}(\lambda, \beta, \tau)$ and $x \geq 0$ define

$$\mu_*(x, F) = \max\big\{a_n e^{x\lambda_n + \tau(x)\beta_n} : n \geq 0\big\}.$$

Theorem 2 implies such statement.

Corollary 5. *If for a sequence* $\lambda = (\lambda_n)$ *condition* (3) *holds, then for every function* $F \in \mathcal{TD}(\lambda, \beta, \tau)$ *there exists a set* E *of finite Lebesgue measure such that relation* $\ln F(x) = (1 + o(1)) \ln \mu_*(x, F)$ *holds as* $x \to +\infty$ $(x \notin E)$.

Indeed, condition (3) implies that condition (2) holds with

$$\nu_0(t) = n_\lambda(t).$$

We put $\nu\big((0, t]\big) = n_\lambda(t)$ and consider a function $\beta\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\beta(\lambda_n) = \beta_n$ $(n \geq 0)$. One has

$$\mu_*(x, F) \leq F(x) = \int_0^{+\infty} g(t) e^{xt+\tau(x)\beta} d\nu(t),$$

where $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ some function such that $g(\lambda_n) = a_n$. Then the Theorem 2 implies the statement of Corollary 5, because

$$\mu_*(x, F) = \max\big\{g(t)e^{xt+\tau(x)\beta}\colon x \in \operatorname{supp}\, \nu\big\} = \mu(x, F).$$

For a measurable set $E \subset \mathbb{C}$ we denote $\varkappa(E) := \displaystyle\int_E \frac{dxdy}{|z|}$, where $z = x + iy$. Let

$$\mathbb{C}_q := \{z = re^{i\theta} \in \mathbb{C}\colon\ r > 0,\ |\theta| < q\} \quad \text{and} \quad \mathbb{C}_+ := \mathbb{C}_{\pi/2}.$$

At $\tau(x) \equiv \ln x$ $(x \geq 1)$ from Corollary 5 by the proof of Theorem 1 we obtain the following corollary.

Corollary 6. *Let* $\{\beta_n\colon n \geq 0\} \subset \mathbb{Z}_+$. *If for a sequence* $\lambda = (\lambda_n)$ *condition* (3) *holds, then for every entire function of the form* $F(z) = \sum_{n=0}^{+\infty} a_n z^{\beta_n} e^{z\lambda_n}$ *there exists a set* $E \subset \mathbb{C}_+$, *such that*

$$\ln|F(re^{i\theta})| \leq (1 + o(1)) \ln \mu_1(r, \theta, F)$$

holds as $r \to +\infty$ $(re^{i\theta} \notin E)$ *for every* $\theta \in (-\pi/2, -\pi/2)$, *and* $\varkappa(E \cap \mathbb{C}_q) < +\infty$ *for every* $q, 0 < q < \pi/2$, *where* $\mu_1(r, \theta, F) = \max\big\{|a_n| r^{\beta_n} e^{r\lambda_n \cos\theta}\colon n \geq 0\big\}$.

Proof. Denote $\lambda_n^{(1)} := \lambda_n \cos\theta$ and $\nu_\theta(t) = n_{\lambda^{(1)}}(t)$ for $\theta \in (-\pi/2, -\pi/2)$. We remark that $\nu_\theta(t) = n_\lambda(t/\cos\theta) = \nu_0(t/\cos\theta)$. By using of Corollary 5 we have

$$\ln|F(re^{i\theta})| \leq g_\theta(r) := \ln \sum_{n=0}^{+\infty} |a_n| r^{\beta_n} e^{r\lambda_n^{(1)}} \leq (1 + o(1)) \ln \mu_1(r, \theta, F)$$

as $r \to +\infty$ $(r \notin E_\theta)$ for some set E_θ of finite Lebesgue measure such that (see (8))

$$meas(E_\theta \cap [0, +\infty)) = \int_{E_\theta} dx \leq \int_0^{+\infty} \frac{dt}{\psi_\theta(t)} := C_\theta < +\infty,$$

because $\tau'(x) = (\ln x)' > 0$ $(x > 0)$. Here $\psi_\theta(t) = \psi(t \cdot \cos\theta)$ and ψ is a function such that conditions (7) hold.

We put $E := \{z = re^{i\theta} \in \mathbb{C} \colon r \in E_\theta, |\theta| < \pi/2\}$. One has

$$C_\theta \le \frac{1}{\cos\theta} \int_0^{+\infty} \frac{dt}{\psi(t)} = \frac{C_0}{\cos\theta},$$

hence for every $q, 0 < q < \pi/2$ we have

$$\varkappa(E \cap \mathbb{C}_q) = \int_{E\cap\mathbb{C}_q} \frac{dxdy}{|z|} = \int_{-q}^{q} d\theta \int_{E_\theta} dr$$
$$\le C_0 \int_{-q}^{q} \frac{d\theta}{\cos\theta} \le \frac{2qC_0}{\cos q} < +\infty. \quad \square$$

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