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SUBSTITUTABLE INVENTORY MODEL WITH POSTPONED DEMANDS

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Abstract: This article presents a framework to study about a two-commodity continuous review inventory system with postponed demands. The demand for one commodity will be satisfied by an item of the other commodity if the inventory level of one commodity becomes empty, that is, the two commodities are substitutable. We fix W_i $(i = 1, 2)$ and $w_i\,(1 \leq w_i \leq W_i)$, to be the maximum storage capacity and the reorder level respectively for the ith commodity, and $Q_i\,(= W_i - w_i > w_i + 1)$ denote the ordering quantity for the ith commodity when both the inventory levels are less than or equal to their respective reorder levels. We assume that the primary customer for the ith commodity is of unit size and their time points form a Poisson process. Pool is the place with infinite capacity for those primary customers who find both the commodities unavailable. First come first serve rule is followed for the pooled customers. The service for the pooled customers is met only if at least one commodity is greater than or equal to $w_i + 1\,(i = 1, 2)$. Under steady state case, we obtain the joint probability distribution of the number of customers in the pool and inventory levels. Ultimately, the total expected cost function and the system performance measures are derived.

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1. Introduction

The inventory model with postponed demands have been the subject matter for many researchers in the past. The problem under discussion differs from classical inventory model with lead times. In the case of continuous review inventory system, the items demanded by the customers are delivered instantly from the stock if it is available and any demand that takes place during stock-out period were either lost (lost sales case) or satisfied only after the ordered items are materialised (backlogging case). After replenishment, the backlogged customers are selected for service in the latter case. But in some real life situations we may note that the same service rule will not be followed, the backlogged demand have to wait even after the replenishment i.e., postponed demand case. But the lead time plays a vital role in both the cases.

In our work, we analyse a two commodity substitutable inventory system with postponed demands. The inventory model with postponed demand was originally examined by Berman et al. [5]. They investigated an inventory model with deterministic service times and any arrivals during stock out period are placed in the queue. A review of inventory system under joint replenishment is initiated by Goyal and Satir [8]. Kalpakam et al. [10] who studied a coordinated multi-item inventory model under a joint replenishment policy. Krishnamoorthy et al. [12] have analysed a two commodity inventory problem without lead time, using theoretical results of direct Markov renewal process. A single server queueing system with a postponed work was investigated by Deepak et al. [6]. They introduced a model with a pool of infinite capacity which is alternative to finite capacity queues in which overflow customers are lost to the system forever.

Several articles have appeared in dealing with inventory model with postponed demand case. Sivakumar and Arivarignan [18] modelled an inventory system of perishable items with postponed demands. They assumed that the arrival of demands are according to a MAP and the lead time has phase type distribution. A perishable inventory model with finite pool and negative customers was analyzed by Manuel et al. [15]. An inventory model of perishable commodities with infinite pool and two types of customers: positive and negative, was analyzed by Sivakumar and Arivarignan [20]. In their work, they assumed that arrivals of regular and negative customers have independent MAPs. Recently, Jeyaraman et al. [9] considered a perishable inventory model with server vacations and postponed demands.

The work on inventory system with postponed demands by Krishnamoorthy and Islam [13] is extended in this article. Unlike single commodity, the two commodity inventory model has more complexities in the reordering procedures. In general, we need separate reorder levels to extend the (s, S) ordering policy to two commodity inventory model and both the commodities are independent of each other. This policy will hike the total cost of ordering. In this paper we consider a two commodity inventory system with postponed demands and coordinated reorder policy wherein both the commodities are substitutable i.e., the demand occur during the stock-out period of first commodity will be satisfied by an item of the second commodity and vice versa. Moreover products such as laptops, mobile phones, etc., may have different features and the customer would be satisfied by one item when the other is not available. There are so many advantages in coordinated reorder policy. The procurement of both the commodities are made from the same supply and items have to be supplied by the same transport facility. Hence, this reordering policy minimises the total cost when compare with independent reordering policy. In the last two decades, many researchers have been more interested in working on two-commodity inventory system. We refer to [1, 2, 3, 4, 7, 11, 17, 19, 21, 22] for a detailed study in two-commodity inventory models with or without substitutable items.

The rest of the paper is systematized as follows. A detailed description of the model is explained in section 2. In section 3, we carry out the analysis part of the problem. The measures of system performance are computed in the last section.

1.1. Notations:

$$\delta_{ij} \; : \; \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{\delta_{ij}} \; : \; 1 - \delta_{ij}$$

$$k \in V_i^j \; : \; k = i, i+1, \ldots j$$

$X_1(t)$: Number of demands in the pool at time t

$X_2(t)$: Inventory level of the first commodity at time t

$X_3(t)$: Inventory level of the second commodity at time t

2. Formulation of the Model

We examine a two-commodity substitutable inventory system with postponed demands in the Quasi Birth and Death (QBD) process model. The maximum storage range for the ith commodity is W_i(i=1,2). The customers arrive according to the Poisson processes with parameters given by, α_1 for the first commodity and α_2 for the second commodity. The reorder level for the ith commodity is fixed and is denoted as w_i. The reordering amount for the ith commodity is $Q_i\,(= W_i - w_i > w_i + 1)$ items and this is when both the inventory levels are less than or equal to their corresponding reorder level w_i. The lead-time is negative exponentially distributed with rate β. Consider both the commodities to be substitutable. To be more clear, if the stock level of one of the commodities hits zero, then the other commodity is used to meet the customer needs. There is a place pool of infinite capacity, where the arriving primary customers are sent to when the inventory level of both the commodities are zero. When the stock level is either $X_2(t) \geq w_1 + 1$ or $X_3(t) \geq w_2 + 1$ or both(i.e., $X_2(t) \geq w_1 + 1$ and $X_3(t) \geq w_2 + 1$), then the need of the outside customers and pooled customers can be met. Whereas on the other hand the first commodity is $X_2(t) \leq w_1$ and the second commodity is $X_3(t) \leq w_2$, only the outside customers will be encountered while the pooled customers will have to wait until the replenishment. We assume that the demanding customer takes in the offer of postponement with an independent Bernoulli trial with probability $p, 0 \leq p < 1$, when the inventory level is zero. The customers also has freedom to reject the offer with probability $q = 1 - p$ and it is considered to be lost. The postponed customers are retained in a pool of infinite capacity. Once the replenishment is completed, the customers in the pool are selected with exponentially distributed time lag with $\lambda > 0$.

3. Mathematical Analysis

Let $X_1(t), X_2(t)$ and $X_3(t)$ denote the number of customers in the pool, inventory level of first commodity and inventory level of the second commodity respectively. From the assumptions it can be shown that the three dimensional homogeneous continuous-time stochastic process $L(t) = \{X_1(t), X_2(t), X_3(t)\}$ with discrete state space $E = \{(i_1, i_2, i_3) : i_1 \geq 0, 0 \leq i_2 \leq W_1, 0 \leq i_3 \leq W_2\}$ is a Markov process.

The infinitesimal generator matrix Θ of the Markov chain L can be conveniently expressed in a block partitioned matrix with entries

$$\Theta = \begin{pmatrix} \Delta_3 & \Delta_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \Delta_2 & \Delta_1 & \Delta_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \Delta_2 & \Delta_1 & \Delta_0 & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

The sub matrices are given by

$$[\Delta_0]_{i_2 j_2} = \begin{cases} F & j_2 = i_2, \quad i_2 = 0 \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[F]_{i_3 j_3} = \begin{cases} \alpha_1 + \alpha_2, & j_3 = i_3, \quad i_3 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$[\Delta_2]_{i_2 j_2} = \begin{cases} C_0 & j_2 = i_2, \quad i_2 = 0 \\ C_1 & j_2 = i_2, \quad i_2 \in V_1^{w_1} \\ C_2 & j_2 = i_2 - 1, \quad i_2 \in V_1^{w_1} \\ C_3 & j_2 = i_2, \quad i_2 \in V_{w_1+1}^{W_1} \\ C_4 & j_2 = i_2 - 1, \quad i_2 \in V_{w_1+1}^{W_1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[C_0]_{i_3 j_3} = \begin{cases} \lambda, & j_3 = i_3 - 1, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[C_1]_{i_3 j_3} = \begin{cases} q\lambda, & j_3 = i_3 - 1, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[C_2]_{i_3 j_3} = \begin{cases} p\lambda, & j_3 = i_3, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[C_3]_{i_3 j_3} = \begin{cases} q\lambda, & j_3 = i_3 - 1, \quad i_3 \in V_1^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[C_4]_{i_3 j_3} = \begin{cases} \lambda, & j_3 = i_3, \quad i_3 = 0 \\ p\lambda, & j_3 = i_3, \quad i_3 \in V_1^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[\Delta_3]_{i_2 j_2} = \begin{cases} B_0 & j_2 = i_2 - 1, \quad i_2 \in V_1^{W_1} \\ G_4 & j_2 = i_2 + Q_1, \quad i_2 \in V_0^{w_1} \\ B_1 & j_2 = i_2, \quad i_2 = 0 \\ B_2 & j_2 = i_2, \quad i_2 \in V_1^{w_1} \\ B_3 & j_2 = i_2, \quad i_2 \in V_{w_1+1}^{W_1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[B_0]_{i_3 j_3} = \begin{cases} \alpha_1+\alpha_2, & j_3 = i_3, \quad i_3 = 0 \\ \alpha_1, & j_3 = i_3, \quad i_3 \in V_1^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[G_4]_{i_3 j_3} = \begin{cases} \beta, & j_3 = i_3 + Q_2, \quad i_3 \in V_0^{w_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[B_1]_{i_3 j_3} = \begin{cases} \alpha_1+\alpha_2, & j_3 = i_3 - 1, \quad i_3 \in V_1^{W_2} \\ -(\alpha_1+\alpha_2+\beta), & j_3 = i_3, \quad i_3 \in V_0^{w_2} \\ -(\alpha_1+\alpha_2), & j_3 = i_3, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[B_2]_{i_3 j_3} = \begin{cases} \alpha_2, & j_3 = i_3 - 1, \quad i_3 \in V_1^{W_2} \\ -(\alpha_1+\alpha_2+\beta), & j_3 = i_3, \quad i_3 \in V_0^{w_2} \\ -(\alpha_1+\alpha_2), & j_3 = i_3, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[B_3]_{i_3 j_3} = \begin{cases} \alpha_2, & j_3 = i_3 - 1, \quad i_3 \in V_1^{W_2} \\ -(\alpha_1+\alpha_2), & j_3 = i_3, \quad i_3 \in V_0^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[\Delta_1]_{i_2 j_2} = \begin{cases} B_0 & j_2 = i_2 - 1, \quad i_2 \in V_0^{W_1} \\ G_4 & j_2 = i_2 + Q_1, \quad i_2 \in V_0^{w_1} \\ D_1 & j_2 = i_2, \quad i_2 = 0 \\ D_2 & j_2 = i_2, \quad i_2 \in V_1^{w_1} \\ D_3 & j_2 = i_2, \quad i_2 \in V_{w_1+1}^{W_1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[D_1]_{i_3 j_3} = \begin{cases} \alpha_1+\alpha_2, & j_3 = i_3 - 1, \quad i_3 \in V_1^{W_2} \\ -(\alpha_1+\alpha_2+\beta), & j_3 = i_3, \quad i_3 \in V_0^{w_2} \\ -(\alpha_1+\alpha_2+\lambda), & j_3 = i_3, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[D_2]_{i_3 j_3} = \begin{cases} \alpha_2, & j_3 = i_3 - 1, \quad i_3 \in V_1^{W_2} \\ -(\alpha_1+\alpha_2+\beta), & j_3 = i_3, \quad i_3 \in V_0^{w_2} \\ -(\alpha_1+\alpha_2+\lambda), & j_3 = i_3, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[D_3]_{i_3 j_3} = \begin{cases} \alpha_2, & j_3 = i_3 - 1, \quad i_3 \in V_1^{W_2} \\ -(\alpha_1+\alpha_2+\lambda), & j_3 = i_3, \quad i_3 \in V_0^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

It may be noted that the above matrices Δ_0, Δ_1, Δ_2 and Δ_3 are square matrices of order $(W_1+1)(W_2+1)$ and the other matrices are square matrices of order (W_2+1).

3.1. Steady State Analysis

Let $\Delta = \Delta_0 + \Delta_1 + \Delta_2$. It can be easily seen that Δ is given by

$$[\Delta]_{i_2 j_2} = \begin{cases} G_1 & j_2 = i_2 - 1, \quad i_2 \in V_{w_1+1}^{W_1} \\ G_2 & j_2 = i_2, \quad i_2 \in V_{w_1+1}^{W_1} \\ G_3 & j_2 = i_2, \quad i_2 = 0 \\ G_4 & j_2 = i_2 + Q_1, \quad i_2 \in V_0^{w_1} \\ G_5 & j_2 = i_2, \quad i_2 \in V_1^{w_1} \\ G_6 & j_2 = i_2 - 1, \quad i_2 \in V_1^{w_1} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$$[G_1]_{i_3 j_3} = \begin{cases} \alpha_1 + \alpha_2 + \lambda, & j_3 = i_3, \quad i_3 = 0 \\ \alpha_1 + p\lambda, & j_3 = i_3, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[G_2]_{i_3 j_3} = \begin{cases} \alpha_2 + p\lambda, & j_3 = i_3 - 1, \quad i_3 \in V_1^{W_2} \\ -(\alpha_1 + \alpha_2 + \lambda), & j_3 = i_3, \quad i_3 \in V_0^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[G_3]_{i_3 j_3} = \begin{cases} \alpha_1 + \alpha_2, & j_3 = i_3 - 1, \quad i_3 \in V_1^{w_2} \\ \alpha_1 + \alpha_2 + \lambda, & j_3 = i_3 - 1, \quad i_3 \in V_{w_2+1}^{W_2} \\ -(\alpha_1 + \alpha_2 + \beta), & j_3 = i_3, \quad i_3 \in V_1^{w_2} \\ -(\alpha_1 + \alpha_2 + \lambda), & j_3 = i_3, \quad i_3 \in V_{w_2+1}^{W_2} \\ -\beta, & j_3 = i_3, \quad i_3 = 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$[G_5]_{i_3 j_3} = \begin{cases} \alpha_2, & j_3 = i_3 - 1, \quad i_3 \in V_1^{w_2} \\ \alpha_2 + q\lambda, & j_3 = i_3 - 1, \quad i_3 \in V_{w_2+1}^{W_2} \\ -(\alpha_1 + \alpha_2 + \beta), & j_3 = i_3, \quad i_3 \in V_0^{w_2} \\ -(\alpha_1 + \alpha_2 + \lambda), & j_3 = i_3, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

$$[G_6]_{i_3 j_3} = \begin{cases} \alpha_1 + \alpha_2, & j_3 = i_3, \quad i_3 = 0 \\ \alpha_1, & j_3 = i_3, \quad i_3 \in V_1^{w_2} \\ \alpha_1 + p\lambda, & j_3 = i_3, \quad i_3 \in V_{w_2+1}^{W_2} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly Δ is a square matrix of order $(W_1+1)(W_2+1)$ and the other matrices are square matrices of order (W_2+1).

Let $\mathbf{\Pi}$ be steady state probability vector of Δ. That is, $\mathbf{\Pi}$ satisfies

$\mathbf{\Pi}\Delta = \mathbf{0}$, $\mathbf{\Pi e} = 1$.

The vector $\mathbf{\Pi}$ can be represented by

$\mathbf{\Pi} = (\pi^{(0)}, \pi^{(1)}, ..., \pi^{(W_1)})$

where

$\pi^{(i_1)}=(\pi^{(i_1,0)}, \pi^{(i_1,1)}, ..., \pi^{(i_1,W_2)})$

Theorem 3.1. *The steady-state probability vector* Π *corresponding to the generator* Δ *is given by*

$$\mathbf{\Pi^{(i_1)}} = \mathbf{\Pi^{(0)}}\Omega_{i_1}, \quad i_1 = 0, 1, \ldots, W_1.$$

where

$$\Omega_{i_1} = \begin{cases} I, & i_1 = 0, \\ -G_3G_6^{-1}, & i_1 = 1, \\ (-1)^{i_1-1}\Omega_1(G_5G_6^{-1})^{(i_1-1)}, & i_1 = 2, \ldots, w_1, \\ (-1)^{w_1}\Omega_1(G_5G_6^{-1})^{(w_1-1)}(G_5G_1^{-1}), & i_1 = w_1+1, \\ (-1)^{i_1-1}\Omega_1(G_5G_6^{-1})^{(w_1-1)}(G_5G_1^{-1})(G_2G_1^{-1})^{(i_1-w_1-1)}, & i_1 = w_1+2, \ldots, Q_1, \\ \Omega_{i_1-1}(G_2G_1^{-1}) - (G_4G_1^{-1})\Omega_{i_1-Q_1-1}, & i_1 = Q_1+1, \ldots, W_1, \end{cases}$$

and $\mathbf{\Pi^{(0)}}$ *can be obtained by solving equation* $\mathbf{\Pi^{(W_1)}}G_2 + \mathbf{\Pi^{(w_1)}}G_4 = \mathbf{0}$ *and* $\mathbf{\Pi e} = 1$. *That is,* $\mathbf{\Pi^{(0)}}(\Omega_{W_1}G_2 + \Omega_{w_1G_4}) = 0$ *and* $\mathbf{\Pi^{(0)}}(I + \sum_{i_1=1}^{W_1}\Omega_{i_1})\mathbf{e}_{(W_2+1)} = 0.$

Proof : We have

$$\mathbf{\Pi}\Delta = \mathbf{0} \text{ and } \mathbf{\Pi e} = 1 \tag{3.1}$$

The first equation of the above yields the following set of equations:

$$\begin{aligned} \mathbf{\Pi^{(1)}}G_6 + \mathbf{\Pi^{(0)}}G_3 &= \mathbf{0}, \\ \mathbf{\Pi^{(i_1+1)}}G_6 + \mathbf{\Pi^{(i_1)}}G_3 &= \mathbf{0}, \quad i_1 = 0 \\ \mathbf{\Pi^{(i_1+1)}}G_6 + \mathbf{\Pi^{(i_1)}}G_5 &= \mathbf{0}, \quad i_1 = 1, .., w_1-1, \\ \mathbf{\Pi^{(i_1+1)}}G_1 + \mathbf{\Pi^{(i_1)}}G_5 &= \mathbf{0}, \quad i_1 = w_1, \\ \mathbf{\Pi^{(i_1+1)}}G_1 + \mathbf{\Pi^{(i_1)}}G_2 &= \mathbf{0}, \quad i_1 = w_1+1, .., Q_1-1, \\ \mathbf{\Pi^{(i_1+1)}}G_1 + \mathbf{\Pi^{(i_1)}}G_2 + \mathbf{\Pi^{(i_1-Q_1)}}G_4 &= \mathbf{0}, \quad i_1 = Q_1, , .., W_1-1, \\ \mathbf{\Pi^{(W_1)}}G_2 + \mathbf{\Pi^{(w_1)}}G_4 &= \mathbf{0}, \end{aligned}$$

Solving the above system of equations recursively and using the normalizing condition, we get the stated result. □

Next, we derive the condition under which the system is stable.

Theorem 3.2. *The stability condition of the system under study is given by*

$$\frac{\alpha_1+\alpha_2}{\lambda} < \frac{(1-\sum_{i_2=0}^{w_1}\sum_{i_3=0}^{w_2}\Pi^{(i_1,i_2)})}{\Pi^{(0,0)}} \tag{3.2}$$

Proof : From the well known result of Neuts [16] on the positive recurrence of Θ we have

$$\Pi\Delta_0\mathbf{e} < \Pi\Delta_2\mathbf{e}$$

and by exploiting the structure of the matrices Δ_0 and Δ_2, and Π the stated result follows. □

It can be seen from the structure of the rate matrix Θ and from the lemma 3.2, that the Markov process $\{(X_1(t), X_2(t), X_3(t)), t \geq 0\}$ with the state space E is regular. Hence the limiting probability distribution

$$\phi^{(i_1,i_2,i_3)} = \lim_{t\to\infty} Pr\left[X_1(t)=i_1, X_2(t)=i_2, X_3(t)=i_3 | X_1(0), X_2(0), X_3(0)\right],$$

exists and is independent of the initial state. That is, $\Phi = \left(\Phi^{(0)}, \Phi^{(1)}, \ldots,\right)$ satisfies

$$\Phi\Theta = \mathbf{0}, \quad \Phi\mathbf{e} = 1.$$

We partition the vector $\Phi^{(i_1)}$, for $i_1 = 0, 1, \ldots,$ as follows

$$\Phi^{(i_1)} = \left(\phi^{(i_1,0)}, \phi^{(i_1,1)}, \ldots, \phi^{(i_1,W_1)}\right),$$

where

$$\phi^{(i_1,i_2)} = \left(\phi^{(i_1,i_2,0)}, \phi^{(i_1,i_2,1)}, \ldots, \phi^{(i_1,i_2,W_2)}\right), \ i_2 = 0, 1, \ldots, W_1$$

Theorem 3.3. *If the stability condition given in (3.8), then the steady-state probability vector* $\mathbf{\Phi}$ *is given by*

$$\mathbf{\Phi^{(i_1)}} = \mathbf{\Phi^{(0)}} R^{i_1}, i_1 = 0, 1, \ldots,$$

where the matrix R satisfies the matrix quadratic equation

$$R^2\Delta_2 + R\Delta_1 + \Delta_0 = 0. \tag{3.3}$$

and the vector $\mathbf{\Phi^{(0)}}$ *is obtained by solving*

$$\mathbf{\Phi^{(0)}}(\Delta_3 + R\Delta_2) = 0$$

subject to the normalizing condition

$$\mathbf{\Phi}^{(\mathbf{0})}(I-R)^{-1}\mathbf{e}=1$$

.

Proof : The proof of this theorem follows from the well-known result on matrix-geometric methods [16]. □

3.2. Computation of R matrix

To analyze the QBD process, a very important matrix in evaluating the performance measures is the matrix R. It is known as the rate matrix of the Markov chain Θ, and it is the minimal non-negative solution of the matrix quadratic equation,

$$R^2\Delta_2 + R\Delta_1 + \Delta_0 = 0. \tag{3.4}$$

Since Δ_0 has a non-zero entry in its $(1,1)$ position only, the matrix R has only one row (first row) of non-zero entries as shown below:

$$R \;=\; \begin{pmatrix} r_{00} & r_{01} & \cdots & r_{0W_2} & r_{10} & \cdots & \cdots & \cdots & \cdots & r_{W_1W_2} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 & \vdots & \cdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Solving the equation (3.10) gives the following set of equations:

For $i_2 = 0$,

$$\begin{aligned} -r_{00}(\alpha_1+\alpha_2+\beta) + r_{01}(\alpha_1+\alpha_2) + r_{10}(\alpha_1+\alpha_2) + (\alpha_1+\alpha_2) &= 0, \\ -r_{0i_3}(\alpha_1+\alpha_2+\beta) + r_{0i_3+1}(\alpha_1+\alpha_2) + r_{i_2+1i_3}\alpha_2 &= 0,\ i_3 \in V_1^{w_2}, \\ r_{00}r_{0i_3+1}\lambda + r_{00}r_{i_2+1i_3}p\lambda - r_{0i_3}(\alpha_1+\alpha_2+\lambda) + r_{0i_3+1}(\alpha_1+\alpha_2) + r_{1i_3}\alpha_2 &= 0,\ i_3 \in V_{w_2+}^{W_2-} \\ r_{00}r_{1W_2}p\lambda - r_{0W_2}(\alpha_1+\alpha_2+\lambda) + r_{1W_2}\alpha_2 &= 0, \end{aligned}$$

For $i_2 \in V_1^{w_1}$,

$$\begin{aligned}
-r_{i_2 0}(\alpha_1+\alpha_2+\beta)+r_{i_2 1}\alpha_2+r_{i_2+1 0}(\alpha_1+\alpha_2) &= 0,\\
-r_{i_2 i_3}(\alpha_1+\alpha_2+\beta)+r_{i_2 i_3+1}\alpha_2+r_{i_2+1 i_3}\alpha_1 &= 0,\ i_3\in V_1^{w_2},\\
r_{00}r_{i_2 i_3+1}q\lambda+r_{00}r_{i_2+1 i_3}p\lambda-r_{i_2 i_3}(\alpha_1+\alpha_2+\lambda)+r_{i_2 i_3+1}\alpha_2+r_{i_2+1 i_3}\alpha_1 &= 0,\ i_2\in V_{w_2+1}^{W_2-1}\\
r_{00}r_{i_2+1 W_2}p\lambda-r_{i_2 W_2}(\alpha_1+\alpha_2+\lambda)+r_{i_2+1 W_2}\alpha_1 &= 0,
\end{aligned}$$

For $i_2 \in V_{w_1+1}^{Q_1-1}$,

$$\begin{aligned}
r_{00}r_{i_2 1}q\lambda+r_{00}r_{i_2+1 0}\lambda-r_{i_2 0}(\alpha_1+\alpha_2+\lambda)+r_{i_2 1}\alpha_2+r_{i_2+1 0}(\alpha_1+\alpha_2) &= 0,\\
r_{00}r_{i_2 i_3+1}q\lambda+r_{00}r_{i_2+1 i_3}p\lambda-r_{i_2 i_3}(\alpha_1+\alpha_2+\lambda)+r_{i_2 i_3+1}\alpha_2+r_{i_2+1 i_3}\alpha_1 &= 0,\ i_3\in V_1^{w_2},\\
r_{00}r_{i_2 i_3+1}q\lambda+r_{00}r_{i_2+1 i_3}p\lambda-r_{i_2 i_3}(\alpha_1+\alpha_2+\lambda)+r_{i_2 i_3+1}\alpha_2+r_{i_2+1 i_3}\alpha_1 &= 0,\ i_3\in V_{w_2+1}^{W_2-1}\\
r_{00}r_{i_2+1 W_2}p\lambda-r_{i_2 W_2}(\alpha_1+\alpha_2+\lambda)+r_{i_2+1 W_2}\alpha_1 &= 0,
\end{aligned}$$

For $i_2 \in V_{Q_1+1}^{W_1-1}$,

$$\begin{aligned}
r_{00}r_{i_2 1}q\lambda+r_{00}r_{i_2+1 0}\lambda-r_{i_2 0}(\alpha_1+\alpha_2+\lambda)+r_{i_2 1}\alpha_2+r_{i_2+1 0}(\alpha_1+\alpha_2) &= 0,\\
r_{00}r_{i_2 i_3+1}q\lambda+r_{00}r_{i_2+1 i_3}p\lambda-r_{i_2 i_3}(\alpha_1+\alpha_2+\lambda)+r_{i_2 i_3+1}\alpha_2+r_{i_2+1 i_3}\alpha_1 &= 0,\ i_3\in V_1^{Q}\\
r_{00}r_{i_2 i_3+1}q\lambda+r_{00}r_{i_2+1 i_3}p\lambda-r_{i_2 i_3}(\alpha_1+\alpha_2+\lambda)+r_{i_2 i_3+1}\alpha_2+r_{i_2+1 i_3}\alpha_1 &+\\
r_{(i_2-Q_1,i_3-Q_2)}\beta &= 0,\ i_3\in V_{Q_2}^{W}\\
r_{00}r_{i_2+1 W_2}p\lambda-r_{i_2 W_2}(\alpha_1+\alpha_2+\lambda)+r_{i_2+1 W_2}\alpha_1+r_{(i_2-Q_1,W_2-Q_2)}\beta &= 0,
\end{aligned}$$

For $i_2 = W_1$,

$$\begin{aligned}
r_{00}r_{i_2 1}q\lambda-r_{i_2 0}(\alpha_1+\alpha_2+\lambda)+r_{i_2 1}\alpha_2 &= 0,\\
r_{00}r_{i_2 i_3+1}q\lambda-r_{i_2 i_3}(\alpha_1+\alpha_2+\lambda)+r_{i_2 i_3+1}\alpha_2 &= 0,\ i_3\in V_1^{Q_2-1},\\
r_{00}r_{i_2 i_3+1}q\lambda-r_{i_2 i_3}(\alpha_1+\alpha_2+\lambda)+r_{i_2 i_3+1}\alpha_2+r_{(i_2-Q_1,i_3-Q_2)}\beta &= 0,\ i_3\in V_{Q_2}^{W_2-1},\\
-r_{i_2 W_2}(\alpha_1+\alpha_2+\lambda)+r_{(i_2-Q_1,W_2-Q_2)}\beta &= 0,
\end{aligned}$$

From the above non-linear equations, we can obtain the computable explicit form of R using the Gauss-Seidel iterative process. Once R is determined, the stationary probabilities of queue length for the QBD process.

3.3. Computation of the vector $\mathbf{\Phi}^{(0)}$

Since R, Δ_0 and Δ_2 are special structured matrices, then the vector $\mathbf{\Phi}^{(0)}$ can be computed using the following set of non-linear equations:

For $i_2 = 0$,

$$\begin{aligned}
-\phi^{(0,0,0)}(\alpha_1+\alpha_2+\beta)+\phi^{(0,i_2,1)}(\alpha_1+\alpha_2)+\phi^{(0,i_2+1,0)}(\alpha_1+\alpha_2) &= 0,\\
-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2+\beta)+\phi^{(0,i_2,i_3+1)}(\alpha_1+\alpha_2)+\phi^{(0,i_2+1,i_3)}\alpha_2 &= 0,\ i_3\in V_1^{w_2},\\
\phi^{(0,0,0)}(r_{(i_2,i_3+1)}\lambda+r_{(i_2+1,i_3)}p\lambda)-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2)+&\\
\phi^{(0,i_2,i_3+1)}(\alpha_1+\alpha_2)+\phi^{(0,i_2+1,i_3)}\alpha_2 &= 0,\ i_3\in V_{w_2+1}^{W_2-1},\\
\phi^{(0,0,0)}r_{(i_2+1,i_3)}p\lambda-\phi^{(0,i_2,W_2)}(\alpha_1+\alpha_2)+\phi^{(0,i_2+1,W_2)}\alpha_2 &= 0,\ i_3=W_2,
\end{aligned}$$

For $i_2 \in V_1^{w_1}$,

$$\begin{aligned}
-\phi^{(0,i_2,0)}(\alpha_1+\alpha_2+\beta)+\phi^{(0,i_2,1)}\alpha_2+\phi^{(0,i_2+1,0)}(\alpha_1+\alpha_2) &= 0,\\
-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2+\beta)+\phi^{(0,i_2,i_3+1)}\alpha_2+\phi^{(0,i_2+1,i_3)}\alpha_1 &= 0,\ i_3\in V_1^{w_2},\\
\phi^{(0,0,0)}(r_{(i_2,i_3+1)}q\lambda+r_{(i_2+1,i_3)}p\lambda)-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2)+&\\
\phi^{(0,i_2,i_3+1)}\alpha_2+\phi^{(0,i_2+1,i_3)}\alpha_1 &= 0,\ i_3\in V_{w_2+1}^{W_2-1},\\
\phi^{(0,0,0)}r_{(i_2+1,i_3)}p\lambda-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2)+\phi^{(0,i_2+1,i_3)}\alpha_1 &= 0,\ i_3=W_2,
\end{aligned}$$

For $i_2 \in V_{w_1+1}^{Q_1-1}$,

$$\begin{aligned}
\phi^{(0,0,0)}(r_{(i_2,1)}q\lambda+r_{(i_2+1,0)}\lambda)-\phi^{(0,i_2,0)}(\alpha_1+\alpha_2)+\phi^{(0,i_2,1)}\alpha_2+\phi^{(0,i_2+1,0)}(\alpha_1+\alpha_2) &= 0,\\
\phi^{(0,0,0)}(r_{(i_2,i_3+1)}q\lambda+r_{(i_2+1,i_3)}p\lambda)-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2)+&\\
\phi^{(0,i_2,i_3+1)}\alpha_2+\phi^{(0,i_2+1,i_3)}\alpha_1 &= 0,\\
\phi^{(0,0,0)}r_{(i_2+1,i_3)}p\lambda-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2)+\phi^{(0,i_2+1,i_3)}\alpha_1 &= 0,
\end{aligned}$$

For $i_2 \in V_{Q_1}^{W_1-1}$,

$$\begin{aligned}
\phi^{(0,0,0)}(r_{(i_2,1)}q\lambda+r_{(i_2+1,0)}\lambda)-\phi^{(0,i_2,0)}(\alpha_1+\alpha_2)+\phi^{(0,i_2,1)}\alpha_2+\phi^{(0,i_2+1,0)}(\alpha_1+\alpha_2) &= 0\\
\phi^{(0,0,0)}(r_{(i_2,i_3+1)}q\lambda+r_{(i_2+1,i_3)}p\lambda)-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2)+\phi^{(0,i_2,i_3+1)}\alpha_2+\phi^{(0,i_2+1,i_3)}\alpha_1 &= 0\\
\phi^{(0,0,0)}(r_{(i_2,i_3+1)}q\lambda+r_{(i_2+1,i_3)}p\lambda)-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2)+\phi^{(0,i_2,i_3+1)}\alpha_2+&\\
\phi^{(0,i_2+1,i_3)}\alpha_1+\phi^{(0,i_2-Q_1,i_3-Q_2)}\beta &= 0\\
\phi^{(0,0,0)}r_{(i_2+1,i_3)}p\lambda-\phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2)+\phi^{(0,i_2+1,i_3)}\alpha_1+\phi^{(0,i_2-Q_1,i_3-Q_2)}\beta &= 0
\end{aligned}$$

For $i_2 = W_1$,

$$\begin{aligned}
\phi^{(0,0,0)} r_{(i_2,1)} q\lambda - \phi^{(0,i_2,0)}(\alpha_1+\alpha_2) + \phi^{(0,i_2,1)}\alpha_2 &= 0,\\
\phi^{(0,0,0)} r_{(i_2,i_3+1)} q\lambda - \phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2) + \phi^{(0,i_2,i_3+1)}\alpha_2 &= 0,\ i_3 \in V_1^{Q_2-1},\\
\phi^{(0,0,0)} r_{(i_2,i_3+1)} q\lambda - \phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2) + &\\
\phi^{(0,i_2,i_3+1)}\alpha_2 + \phi^{(0,i_2-Q_1,i_3-Q_2)}\beta &= 0,\ i_3 \in V_{Q_2}^{W_2-1},\\
\phi^{(0,i_2-Q_1,i_3-Q_2)}\beta - \phi^{(0,i_2,i_3)}(\alpha_1+\alpha_2) &= 0,\ i_3 = W_2,
\end{aligned}$$

3.4. Computation of the vector $(I-R)^{-1}$

Due to the special structure of R matrix, the matrix $(I-R)^{-1}$ has the following structure:

$$(I-R)^{-1} = \begin{pmatrix}
\frac{1}{1-r_{00}} & \frac{r_{01}}{1-r_{00}} & \cdots & \frac{r_{0W_2}}{1-r_{00}} & \frac{r_{10}}{1-r_{00}} & \cdots & \cdots & \cdots & \cdots & \frac{r_{W_1W_2}}{1-r_{00}}\\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & 0 & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \cdots & 0 & \vdots & \cdots & \vdots & \vdots & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}$$

4. System Performance Measures

In this section we derive some performance measure of the system under consideration in the steady state.

4.1. Expected Inventory Level

Let η_{I_1} denote the expected inventory level of first commodity in the steady state. Then we have

$$\eta_{I_1} = \sum_{i_1=0}^{\infty}\sum_{i_2=1}^{W_1}\sum_{i_3=0}^{W_2} i_2\phi^{(i_1,i_2,i_3)}.$$

Let η_{I_2} denote the expected inventory level of second commodity in the steady state. Then we have

$$\eta_{I_2} = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{W_1}\sum_{i_3=1}^{W_2} i_3\phi^{(i_1,i_2,i_3)}.$$

4.2. Expected Reorder Rate

Let η_R denote the expected reorder rate in the steady state. Then we have

$$\begin{aligned}\eta_R = & \sum_{i_3=0}^{w_2}(\alpha_1+\delta_{i_30}\alpha_2)\phi^{(0,w_1+1,i_3)}+\sum_{i_2=0}^{w_1}(\alpha_2+\delta_{i_20}\alpha_1)\phi^{(0,i_2,w_2+1)}\\ &+\sum_{i_1=1}^{\infty}\sum_{i_3=0}^{w_2}(\alpha_1+p\lambda+\delta_{i_30}(\alpha_2+q\lambda))\phi^{(i_1,w_1+1,i_3)}\\ &+\sum_{i_1=1}^{\infty}\sum_{i_2=0}^{w_1}(\alpha_2+q\lambda+\delta_{i_20}(\alpha_1+p\lambda))\phi^{(i_1,i_2,w_2+1)}.\end{aligned}$$

4.3. Expected Number of Demands in the Pool

Let η_P denote the expected number of demands in the pool in the steady state. Then we have

$$\eta_P = \sum_{i_1=1}^{\infty}\sum_{i_2=0}^{W_1}\sum_{i_3=0}^{W_2} i_1\phi^{(i_1,i_2,i_3)}.$$

4.4. Expected Total Cost

The expected total cost per unit time in the steady-state for this model is defined to be

$$TC(W_1,W_2,w_1,w_2) = c_{h_1}\eta_{I_1}+c_{h_2}\eta_{I_2}+c_s\eta_R+c_r\eta_P,$$

where

c_s : Setup cost per order.
c_{h_1} : The inventory carrying cost of first commodity per unit item per unit time.
c_{h_2} : The inventory carrying cost of second commodity per unit item per unit time.
c_r : Back ordering cost of a demand in the orbit per unit time.

Conclusion

In this paper, we modeled an inventory system of substitutable item with postponed demands. The joint probability distribution of the number of pooled customers and inventory levels are obtained in the steady state case. Finally, various system performance measures are determined and the long-run total expected cost rate is derived.

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