International Journal of Pure and Applied Mathematics
Volume 118 No. 2 2018, 397-406
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v118i2.22
ijpam.eu

DERIVATIONS OF SOME CLASSES OF ZINBIEL ALGEBRAS

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Abstract: In this study we deal with derivation of finite low dimensional Zinbiel algebras. We provide by some simple properties of the derivation algebras. Description of the derivation algebras with their dimensions for complex Zinbiel algebras of dimensions two,three and four are given and summarized in tabular form.

AMS Subject Classification: 16D70
Key Words: Lie algebra, Zinbiel Algebra, Derivation

1. Introduction

In 1965, Bloh [3] introduced Leibniz algebras which he termed D-algebra as a generalization of Lie algebras and in 1993, French Mathematician J. L. Loday [9] initiated the study of structural properties of Leibniz algebras which is now the usage found in most literatures. Other generalizations of Lie algebras besides Leibniz algebras are Malcev algebras, Lie superalgebras, binary Lie algebras, e.t.c., which have all emerged after decades of unflinching effort of different researchers across the globe.

In 1995, Loday [10] while studying cup product for Leibniz cohomology, introduced Zinbiel algebras and noticed that the operad of Zinbiel algebras is dual to the operad of Leibniz algebras in the sense Koszul duality [9]. After its introduction several studies have been conducted on Zinbiel algebras.Adashev

Received: 2017-10-19
Revised: 2017-11-07
Published: March 23, 2018

et. al. [1] gave classification of nul-filiform and filiform Zinbiel algebras and extended classification of complex Zinbiel algebras of dimension ≤ 3 to dimension 4. Adashev's classification is a follow up from Dzhumadil'daev's classification of nilpotent Zinbiel algebras up to dimension 3 [4].

Jacobson's hypothesis [6] that non degenerate derivation of an arbitrary Lie algebras is nilpotent spurred interest in the theory of derivation of Lie algebras. Rakhimov and Al-Hossain [14] studied the derivations of some Classes of finite dimensional Leibniz Algebra.
In this paper, we study the derivation algebras of low dimensional Zinbiel algebras. The outline of this paper will cover the following: section 1 is an introduction. Section 2 employed a technique for finding derivations from previous literatures on derivation of finite dimensional Lie and Leibniz algebras. In section 3 this technique is used to come up with derivations of low dimensional Zinbiel algebras.

2. Preliminaries

This section contains main definitions used and some results obtained for Leibniz algebras.

Definition 1. A Lie algebra L over a field K is an algebra satisfying the following conditions:

$$[x,x] = 0, \forall\, x \in L \tag{1}$$

$$\Big[[x,y],z\Big] + \Big[[y,z],x\Big] + \Big[[z,x],y\Big] = 0, \;\forall\; x,y,z \in L. \tag{2}$$

Definition 2. A Zinbiel algebra is a vector space A together with a bilinear map $\circ : A \times A \to A$ such that

$$(x \circ y) \circ z = x \circ (y \circ z) + x(z \circ y)$$

Under the conditions of a new product presented as $x \circ y + y \circ x$.
it is possible to obtain a structure of commutative associative algebra on A.The definition of a sequence for a provided Zinbiel algebra A can be given in the following way:

$$A^1 = A, A^{k+1} = A \circ A^k, \qquad k \geq 1.$$

Definition 3. A derivation of Zinbiel algebra A is a linear transformation $d : A \to A$ satisfying

$$d(x \circ y) = d(x) \circ y + x \circ d(y)$$

for all $x, y \in A$.

The set of all derivations of a Zinbiel algebra A we denote by $Der(A)$. The $Der(A)$ is an associative algebra with respect the composition operation $\circ$ and it is a Lie algebra with respect to the bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$.

Definition 4. A Zinbiel algebra A is called characteristically nilpotent if $Der(A)$ is nilpotent.

The study characteristically nilpotent algebras is important in connection with the observations made by Jacobson in [6] and further developments of this concept in [2], [5], [7], [8], [11], [13]. Let $x \in A$. Define L_a and R_a to be the elements of $Hom_k(A)$ determined by $L_a(x) = a \circ x \quad and \quad R_a(x) = x \circ a \quad for \quad all \quad x \in A$.
The following propositions can be deduced easily.

Proposition 5. *Let A be an algebra. Then $Der(A)$ is lie algebra for the bracket $[f, g] = f \circ g - g \circ f$*

Proposition 6. *Let $d \in Hom_k(A)$. Then the following conditions are equivalent :*

1. $d \in der(A)$
2. $[d, L_x] = L_{d(x)}$ *for* $x \in A$
3. $[d, R_y] = R_{d(y)}$ *for* $y \in A$

Lemma 7. *The sets $R(A) = \{R_x | x \in A\}$, $L(A) = \{L_x | x \in A\}$ are subalgebras of the Zinbiel algebra $Der(A)$.*

Proof. The proof is derived from these identities

$$R_{x \circ y} = R_y R_x, \quad L_{x \circ y} = L_x L_y.$$

□

Next section deal with applications of the algorithm to low-dimensional complex Zinbiel algebras. Note that there is no one-dimensional Zinbiel algebra except for abelian.

3. An algorithm for finding derivations

Provided that $\{e_1, e_2, \cdots, e_n\}$ are the basis of A as n-dimensional complex Zinbiel algebra, then the $e_i e_j$ components with $i, j = 1, 2, \cdots, n$ can be referred to as the A structure constants on the basis $\{e_1, e_2, \cdots, e_n\}$. if

$$e_i \circ e_j = \sum_{k=1}^{n} \gamma_{ij}^{k} e_k$$

then

$$\{\gamma_{ij}^{k}, i, j, k, \leq n\}$$

is denoted the set of structure constants of A. over the field of complex numbers $\mathbb{C}$. It is important to discuss the Zinbiel algebras derivations. So, in matrix, $d = (d_{ij})_{i,j=1,2,\cdots,n}$ with the basis as $\{e_1, e_2, \cdots, e_n\}$. If the structure constants $\{\gamma_{ij}^{k}\}$ are given we get the system in the form presented below:

$$\sum_{k=1}^{n} \gamma_{ij}^{k} d_{tk} = \sum_{k=1}^{n} (d_{ki} \gamma_{kj}^{t} + d_{kj} \gamma_{ik}^{t}) \tag{3}$$

for $1 \leq i, j, t \leq n$.

Example 8. Let A be the two dimensional algebras with composition law $e_1 \circ e_1 = e_2$ on a basis $\{e_1, e_2\}$. We show that the system d_1, d_2 of derivations given by $d_1(e_1) = e_1, d_1(e_2) = 2e_2, d_2(e_1) = e_2$ from a basis of the algebra $Der(A)$.

It is easy to verify that d_1 and d_2 are derivations of A and they are linearly independent.

Let $d \in Der(A)$

$$der(e_k) = \begin{cases} \alpha_1 e_1 + \alpha_2 e_2 & for \quad k = 1 \\ \beta_1 e_1 + \beta_2 e_2 & for \quad k = 2 \end{cases}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are scalars.
Applying d to the equality $e_1 \circ e_1 = e_2$, we find

$$\begin{aligned} d(e_1 \circ e_1) &= d(e_1) \circ e_1 + e_1 \circ d(e_1) \\ d(e_2) &= (\alpha_1 e_1 + \alpha_2 e_2) \circ e_1 + e_1 \circ (\alpha_1 e_1 + \alpha_2 e_2) \\ \beta e_1 + \beta_2 e_2 &= \alpha_1 e_2 + \alpha_1 e_2 \end{aligned}$$

which imply $\beta_1 = 0 \quad and \quad \beta_2 = 2\alpha_1$. Thus , if $x = \lambda_1 e_1 + \lambda_2 e_2$ then

$$\begin{aligned} d(x) = d(\lambda_1 e_1 + \lambda_2 e_2) &= \lambda_1 d(e_1) + \lambda_2 d(e_2) \\ &= \lambda_1(\alpha_1 e_1 + \alpha_2 e_2) + \lambda_2(2\alpha_1 e_2) \\ &= \alpha_1(\lambda_1 e_1 + 2\lambda_2 e_2) + \alpha_2(\lambda_1 e_2) \\ &= \alpha_1 d_1(x) + \alpha_2 d_2(x). \end{aligned}$$

This approach can be applied to be able to find the complex Zinbiel algebras derivations in dimension 2, 3, and 4. Besides, we apply the classification result from [12].

Theorem 9. *Any two-dimensional Zinbiel algebra A is isomorphic to one of the following non-isomorphic Zinbiel algebras of*

$A_1 : \quad e_1 e_1 = e_2.$

Table 1: derivation of two-dimensional complex Zinbiel algebras

Isomorphism Class	Derivation	Dim
A_1	$\begin{pmatrix} d_{11} & 0 \\ d_{21} & 2d_{11} \end{pmatrix}$	2

Theorem 10. *Up to isomorphism, there exist three one parametic families and six explicit representatives of non Lie complex Zinbiel algebras of dimension three:*

$A_1 : \quad e_i e_j = 0$
$A_2 : \quad e_1 e_1 = e_3$
$A_3 : \quad e_1 e_1 = e_3, \quad e_2 e_2 = e_3$
$A_4 : \quad e_1 e_2 = \frac{1}{2} e_3, \quad e_2 e_1 = \frac{-1}{2} e_3$
$A_5 : \quad e_2 e_1 = e_3$
$A_6 : \quad e_1 e_1 = e_3, \quad e_1 e_2 = e_3, \quad e_2 e_2 = e_3$
$A_7 : \quad e_1 e_1 = e_2, \quad e_1 e_2 = e_3, \quad e_2 e_1 = e_3.$

Isomorphism Class	**Derivation**	**Dim**
A_1	$\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$	9
A_2	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & 2d_{11} \end{pmatrix}$	5
A_3	$\begin{pmatrix} d_{11} & d_{12} & 0 \\ -d_{12} & d_{11} & 0 \\ d_{31} & d_{32} & 2d_{11} \end{pmatrix}$	4
A_4	$\begin{pmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & d_{11}+d_{22} \end{pmatrix}$	6
A_5	$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ d_{31} & d_{32} & d_{11}+d_{22} \end{pmatrix}$	4
A_6	$\begin{pmatrix} d_{11} & -d_{21} & 0 \\ d_{21} & d_{11}+d_{21} & 0 \\ d_{31} & d_{32} & 2d_{11}+d_{21} \end{pmatrix}$	4
A_7	$\begin{pmatrix} d_{11} & 0 & 0 \\ d_{21} & 2d_{11} & 0 \\ d_{31} & 2d_{21} & 3d_{11} \end{pmatrix}$	3

Theorem 11. *The isomorphism class of four-dimensional complex nilpotent Zinbiel algebras are given by the following representatives*

$$\begin{array}{lllll}
A_1: & e_1e_1 = e_2, & e_1e_2 = e_3, & e_2e_1 = 2e_3, & e_1e_3 = e_4, \\
 & e_2e_2 = 3e_4, & e_3e_1 = 3e_4 & & \\
A_2: & e_1e_1 = e_3, & e_1e_2 = e_4, & e_1e_3 = e_4, & e_3e_1 = 2e_4 \\
A_3: & e_1e_1 = e_3, & e_1e_3 = e_4, & e_2e_2 = e_4, & e_3e_1 = 2e_4 \\
A_4: & e_1e_2 = e_3, & e_1e_3 = e_4, & e_2e_1 = -e_3 & \\
A_5: & e_1e_2 = e_3, & e_1e_3 = e_4, & e_2e_1 = -e_3, & e_2e_2 = e_4 \\
A_6: & e_1e_1 = e_4, & e_1e_2 = e_3, & e_2, e_1 = -e_3, & e_2e_2 = -2e_3 + e_4 \\
A_7: & e_1e_2 = e_3, & e_2e_1 = e_4, & e_2, e_2 = -e_3 &
\end{array}$$

$A_8: \quad e_1e_1 = e_3, \quad e_1e_2 = e_4, \quad e_2e_1 = -\alpha e_3, \quad e_2e_2 = -e_4 \quad \alpha \in \mathbb{C}$
$A_9: \quad e_1e_1 = e_4, \quad e_1e_2 = \alpha e_4, \quad e_2e_1 = -\alpha e_4, \quad e_2e_2 = e_4;$
$\quad e_3e_3 = e_4 \quad \alpha \in \mathbb{C}$
$A_{10}: \quad e_1e_2 = e_4, \quad e_1e_3 = e_4, \quad e_2e_1 = -e_4, \quad e_2e_2 = e_4,$
$\quad e_3e_1 = e_4$
$A_{11}: \quad e_1e_1 = e_4, \quad e_1e_2 = e_4, \quad e_2e_1 = -e_4, \quad e_3e_3 = e_4$
$A_{12}: \quad e_1e_2 = e_3, \quad e_2e_1 = e_4$
$A_{13}: \quad e_1e_2 = e_3, \quad e_2e_1 = -e_3, \quad e_2e_2 = e_4$
$A_{14}: \quad e_2e_1 = e_4, \quad e_2e_2 = e_3$
$A_{15}: \quad e_1e_2 = e_4, \quad e_2e_2 = e_3, \quad e_2e_1 = \frac{1+\alpha}{1-\alpha}e_4 \quad \alpha \in \mathbb{C}/\{1\}$
$A_{16}: \quad e_1e_2 = e_4, \quad e_2e_1 = -e_4, \quad e_3e_3 = e_4.$

Table 2: derivation of four-dimensional complex Zinbiel algebras

Isomorphism Class	**Derivation**	**Dim**
A_1	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & 2d_{11} & 0 & 0 \\ d_{31} & 3d_{21} & 3d_{11} & 0 \\ d41 & 4d_{31} & 6d_{21} & 4d_{11} \end{pmatrix}$	4
A_2	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & 2d_{11} & 0 & 0 \\ d_{31} & 0 & 2d_{11} & 0 \\ d_{41} & d_{42} & 3d_{31} + d_{21} & 3d_{11} \end{pmatrix}$	5
A_3	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & \frac{3}{2}d_{11} & 0 & 0 \\ d_{31} & 0 & 2d_{11} & 0 \\ d_{41} & d_{42} & 3d_{31} & 3d_{11} \end{pmatrix}$	4
A_4	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ 0 & 0 & d_{11} + d_{22} & 0 \\ d_{41} & d_{42} & 0 & 2d_{11} + d_{22} \end{pmatrix}$	5
A_5	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & 2d_{11} & 0 & 0 \\ 0 & -2d_{21} & 3d_{11} & 0 \\ d_{41} & d_{42} & -d_{21} & 4d_{11} \end{pmatrix}$	4

Isomorphism Class	Derivation	Dim
A_6	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{11} \end{pmatrix}$	5
A_7	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{11} \end{pmatrix}$	5
A_8	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{11} \end{pmatrix} (\alpha \neq 0)$	5
	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{11} \end{pmatrix} (\alpha = 0)$	5
A_9	$\begin{pmatrix} d_{11} & -d_{21} & 0 & 0 \\ d_{21} & d_{11} & 0 & 0 \\ 0 & 0 & d_{11} & 0 \\ d_{41} & d_{42} & d_{43} & 2d_{11} \end{pmatrix} (\alpha \neq 0)$	5
	$\begin{pmatrix} d_{11} & -d_{21} & -d_{31} & 0 \\ d_{21} & d_{11} & -d_{32} & 0 \\ d_{31} & d_{32} & d_{11} & 0 \\ d_{41} & d_{42} & d_{43} & 2d_{11} \end{pmatrix} (\alpha = 0)$	7
A_{10}	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & d_{11} & 0 & 0 \\ 0 & -d_{21} & d_{11} & 0 \\ d_{41} & d_{42} & d_{43} & 2d_{11} \end{pmatrix}$	5
A_{11}	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & d_{11} & 0 & 0 \\ 0 & 0 & d_{11} & 0 \\ d_{41} & d_{42} & d_{43} & 2d_{11} \end{pmatrix}$	5
A_{12}	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & d_{11}+d_{22} & 0 \\ d_{41} & d_{42} & 0 & d_{11}+d_{22} \end{pmatrix}$	6

Isomorphism Class	**Derivation**	**Dim**
A_{13}	$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & d_{11}+d_{22} & 0 \\ d_{41} & d_{42} & 0 & 2d_{22} \end{pmatrix}$	7
A_{14}	$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & 2d_{22} & 0 \\ d_{41} & d_{42} & d_{12} & d_{11}+d_{22} \end{pmatrix}$	7
A_{15}	$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & 2d_{22} & 0 \\ d_{41} & d_{42} & 2d_{12} & d_{11}+d_{22} \end{pmatrix} (\alpha \neq -1)$	7
	$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & 2d_{22} & 0 \\ d_{41} & d_{42} & d_{12} & d_{11}+d_{22} \end{pmatrix} (\alpha = -1)$	7
A_{16}	$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ d_{21} & 2d_{33}-d_{11} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ d_{41} & d_{42} & d_{43} & 2d_{33} \end{pmatrix}$	7

Corollary 12. *i The list of iso-morphism classes of Zinbiel algebras of* $2, 3,$ *and* 4 *dimensions contains no of characteristic nilpotent classes.*

ii The range of derivation algebras dimensions is between 2 *and* 9 *in this case.*

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