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# A DELAYED SIS EPIDEMIC MODEL WITH NON-LINEAR INCIDENCE RATE

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**Abstract:** This paper makes a modest attempt to study an SIS epidemic model with time delay corresponding to the infectious period and non-linear incidence rate, where the growth of susceptible individuals is governed by the logistic equation. The local stability of the disease-free equilibrium and the endemic equilibrium with and without delay has also been analyzed. Conditions for the existence of Hopf bifurcation of the endemic equilibrium were established from applying time delay as a bifurcation parameter. Further, the analytical results supported by numerical simulations.

**AMS Subject Classification:** 92B, 92D, 34D, 70K
**Key Words:** delayed SIS epidemic model, non-linear incidence rate, basic reproduction number, Hopf bifurcation

## 1. Introduction

A grave threat posed to the welfare of society is the prevalence of diseases. One

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major culprit is urbanization which while making life easy for everyone has also created and nurtured fertile conditions for the onset of diseases [5]. Many of these diseases have the potential to explode into epidemics leading to decimation of a huge chunk of population. This makes it very necessary to generate a mathematical model that can predict the contour of an epidemic and recommend ways for diminishing its impact. The mathematical tools emerge as a saviour to not only explore the possibility of gauging the nature and spread of an epidemic but also come up with viable means to tackle it [3].
As a follow up to the arrival of epidemic, several models were conceived to account for the nature and dissemination of such diseases and following the success of one such popular model - Kermack-McKendrick model [1], several more appeared to answer some of the questions pertaining to the onset of such diseases. SIR model is one such model that relies heavily on the assumption of there being several intermediate conditions following the outbreak of a disease. The individuals concerned are first labelled susceptible (S) and then infected (I) and then removed (R) following recovery. The end result is either complete recuperation or death from the condition. The SIS model on the other hand [7, 8], supposes the return to previous condition of a victim after initial recovery. Thus this model makes allowance for an individual becoming vulnerable to the disease, reporting infection after coming in contact with those already infected and then recovering temporarily only to become affected again. These models have fired popular imagination since they have richly contributed to the study of STD, such as gonorrhoea and AIDS [6]. An important guiding factor in the development of these models is obviously the formulation of the incidence rate as a determinant in mathematical interpretation. The bilinear and standard incidence rates have frequently figured in classical epidemic models [4, 5, 9, 10, 11]. Non linear incidence rates of the form $\beta S^p I^q$ have been investigated by Liu et al. [9]. A very general form of nonlinear incidence rate was explored in brief by Derrick and van den Driessche [12].

Experience informs one that where infectious diseases are concerned, the infected individuals do not immediately infect another individual and that a sufficient amount of time needs to be factored in before any confirmed case of infection is made. This goes by the name of latent period ($\tau$) and is mathematically interpreted as the delay seen in the infection to take effect. The delay thus caused will destabilize the equilibrium in evidence until then and Hopf bifurcation ensures the availability of periodic solutions to such a model [2, 7].

In this paper, we consider a delayed SIS epidemic model with a nonlinear incidence rate as follows:

$$\begin{cases} \frac{dS}{dt} = rS(1-\frac{S}{k}) - dS - \beta S^2 I(t-\tau) + \gamma I \\ \\ \frac{dI}{dt} = \beta S^2 I(t-\tau) - dI - \gamma I \end{cases} \tag{1}$$

Where $S(t), I(t)$ refer to the number of individuals who are prone to contracting the disease and those who have already contracted the disease respectively. $r$ marks the rate of growth of the population vulnerable to infection, $k$ represents the ability to carry the infection of the community that is unaffected, $d$ is a reference to the death rate of the population concerned and $\gamma$ shows the rate at which individuals recover from infection while $\beta$ is the coefficient of infection.

The paper is organized as follows; the II section obtains the disease free equilibrium and endemic equilibrium in terms of local stability when there is delay and when there is not. The conditions satisfying Hopf bifurcation with respect to equilibrium of the endemic kind is established, applying the criterion of time delay as one of the parameters involved in bifurcation. Section III is devoted to supplying numerical simulation based on the models developed for the purpose. The concluding section discusses the findings arrived at based on the equations.

## 2. Equilibrium Points and Their Stability

System (1.1) always has a disease-free equilibrium $E_0\left(\frac{k(r-d)}{r}, 0\right)$, which exist when $r > d$. To find the endemic equilibrium, set

$$\begin{aligned} rS\left(1-\frac{S}{k}\right) - dS - \beta S^2 I + \gamma I = 0 \\ \beta S^2 I - (d+\gamma)I = 0 \end{aligned} \tag{2}$$

Define the basic reproduction number as follows

$$R_0 = \frac{k(r-d)}{r}\sqrt{\frac{\beta}{d+\gamma}} \tag{3}$$

From system (1.1) it follows that
(i) if $r > d$ and $R_0 < 1$, then there is no positive equilibrium;
(ii) if $R_0 > 1$, then there is a unique positive equilibrium $E^*(S^*, I^*)$ of the

system (1.1), called the "endemic equilibrium", given by

$$S^* = \sqrt{\frac{d+\gamma}{\beta}}, \quad I^* = \frac{r(d+\gamma)(R_0-1)}{dk\beta}$$

### 2.1. Stability of the Disease-Free Equilibrium

**Theorem 1.** *The disease-free equilibrium* $E_0\left(\frac{k(r-d)}{r}, 0\right)$ *of system (1.1) is locally asymptotically stable when* $r > d$ *and* $R_0 < 1$ *and unstable if* $R_0 > 1$

*Proof.* The Jacobian matrix of the system (1.1) at $E_0$ is given by

$$J(E_0) = \begin{bmatrix} r-d-\frac{2r}{k}S^* & -\beta S^{*2}e^{-\lambda\tau}+\gamma \\ 0 & \beta S^{*2}e^{-\lambda\tau}-(d+\gamma) \end{bmatrix} \tag{4}$$

The characteristic equation is

$$(\lambda+r-d)\left(\lambda - \frac{\beta k^2}{r^2}(r-d)^2e^{-\lambda\tau}+d+\gamma\right) = 0 \tag{5}$$

When $\tau = 0$, (2.4) has two roots $\lambda_1 = -(r-d) < 0$, and $\lambda_2 = \dfrac{\beta k^2}{r^2}(r-d)^2-(d+\gamma)$

Hence $E_0$ is locally asymptotically stable when $r > d$ and $R_0 < 1$ and unstable when $R_0 > 1$. □

### 2.2. Stability of the Endemic Equilibrium and Hopf Bifurcation

The Jacobian matrix of system (1.1) at endemic equilibrium $E^*$ is given by

$$J(E^*) = \begin{bmatrix} r-d-\frac{2r}{k}S^*-2\beta S^*I^* & -\beta S^{*2}e^{-\lambda\tau}+\gamma \\ 2\beta S^*I^* & \beta S^{*2}e^{-\lambda\tau}-(d+\gamma) \end{bmatrix} \tag{6}$$

The characterstic equation of (2.5) is given by

$$\lambda^2 + p_1\lambda + p_2 + e^{-\lambda\tau}(q_1\lambda + q_2) = 0 \tag{7}$$

where

$$p_1 = \frac{2r}{k}S^* + 2\beta S^*I^* + d + \gamma - (r-d), \quad q_1 = -\beta S^{*2}$$

$$p_2 = \frac{2r}{k}(d+\gamma)S^* + 2\beta dS^*I^* - (r-d)(d+\gamma), \quad q_2 = -\frac{2r\beta}{k}S^{*3} + (r-d)\beta S^{*2}$$

**Case 1:** For $\tau = 0$, (2.6) becomes

$$\lambda^2 + (p_1+q_1)\lambda + (p_2+q_2) = 0 \tag{8}$$

Now

$$p_1 + q_1 = \frac{2r}{k}S^* + 2\beta S^*I^* + d + \gamma - (r-d) - \beta S^{*2} = \frac{1}{S^*}\left(2\beta S^{*2}I^* + dI^*\right) > 0,$$

$$\begin{aligned} p_2+q_2 = \frac{2r}{k}(d+\gamma)S^* + 2\beta S^*I^* - (r-d)(d+\gamma) - \frac{2r}{k}(d+\gamma)S^* + (r-d)\beta S^{*2} \\ = 2\beta dS^*I^* > 0 \end{aligned}$$

By Routh-Hurwitz criteria, all roots of (2.7) are real and negative or complex conjugate with negative real part.
Hence the system (1.1) without delay is locally asymptotically stable when $R_0 > 1$.

**Case 2:** If $\tau > 0$, suppose that there is a positive $\tau_0$ such that equation (2.6) has pair of purely imaginary roots $\pm i\omega$, $\omega > 0$.

Put $\lambda = i\omega$ in (2.6), we get

$$-\omega^2 + p_1\omega i + p_2 + (q_1\omega i + q_2)(\cos\omega\tau - i\sin\omega\tau) = 0 \tag{9}$$

By separating the real and imaginary parts in (2.8), we get

$$\begin{cases} \omega^2 - p_2 = q_1\omega\sin\omega\tau + q_2\cos\omega\tau \\ \\ -p_1\omega = -q_2\sin\omega\tau + q_1\omega\cos\omega\tau \end{cases} \tag{10}$$

Squaring and adding both equations in (2.9), we obtain

$$\omega^4 + (p_1^2 - 2p_2 - q_1^2)\omega^2 + (p_2^2 - q_2^2) = 0 \tag{11}$$

Thus, if $(p_1^2 - 2p_2 - q_1^2) > 0$ and $(p_2^2 - q_2^2) > 0$, then there is no $\omega$ such that $i\omega$ is a root of (2.6). Thus the real parts of all roots of (2.6) are negative for all $\tau \geq 0$.

This shows that $E^*$ is asymptotically stable for all $\tau$ if the following conditions hold:

$$\begin{aligned} &i) \quad R_0 > 1 \\ &ii) \quad (p_1+q_1) > 0, (p_2+q_2) > 0 \\ &iii) \quad (p_1^2 - 2p_2 - q_1^2) > 0, (p_2^2 - q_2^2) > 0 \end{aligned}$$

If $(p_2^2 - q_2^2) < 0$ then there is a unique positive $\omega_0$ satisfying (2.10). That is, there is a single pair of purely imaginary roots $\pm i\omega_0$ to (2.6).
From (2.9) $\tau_n$ corresponding to $\omega_0$ can be obtained

$$\tau_n = \frac{1}{\omega_0} \arccos\left[\frac{(p_1q_1 - q_2)\omega_0^2 + p_2q_2}{q_1^2\omega_0^2 + q_2^2}\right] + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2..... \tag{12}$$

For $\tau = 0, E^*$ is stable, it remains stable for $\tau < \tau_0$ if $\left.\frac{dRe(\lambda)}{d\tau}\right|_{\lambda=i\omega_0} > 0$

Differentiating (2.6) with respect to $\tau$, we get

$$\frac{d\lambda}{d\tau}\left[2\lambda + p_1 - q_1e^{-\lambda\tau} + (q_1\lambda + q_2)\tau e^{-\lambda\tau}\right] = -\lambda(q_1\lambda + q_2)e^{-\lambda\tau} \tag{13}$$

$$\begin{aligned} \left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda + p_1 - q_1e^{-\lambda\tau} + (q_1\lambda + q_2)\tau e^{-\lambda\tau}}{-\lambda(q_1\lambda + q_2)e^{-\lambda\tau}} \\ \left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda + p_1}{-\lambda(q_1\lambda + q_2)e^{-\lambda\tau}} + \frac{q_1}{\lambda(q_1\lambda + q_2)} - \frac{\tau}{\lambda} \\ \left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda + p_1}{-\lambda(\lambda^2 + p_1\lambda + p_2)} + \frac{q_1}{\lambda(q_1\lambda + q_2)} - \frac{\tau}{\lambda} \end{aligned}$$

Thus,

$$\begin{aligned} &\left.\frac{dRe(\lambda)}{d\tau}\right|_{\lambda=i\omega_0} = Re\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=i\omega_0} \\ &= Re\left[\frac{2i\omega_0 + p_1}{-i\omega_0(-\omega_0^2 + p_1i\omega_0 + p_2)} + \frac{q_1}{i\omega_0(q_1i\omega_0 + q_2)} - \frac{\tau}{i\omega_0}\right] \\ &= Re\left[\frac{1}{\omega_0}\left(\frac{2i\omega_0 + p_1}{p_1\omega_0 + i(\omega_0^2 - p_2)} + \frac{q_1}{(-q_1\omega_0 + iq_2)} + \tau i\right)\right] \\ &= \frac{1}{\omega_0}\left(\frac{2\omega_0(\omega_0^2 - p_2) + p_1^2\omega_0}{p_1^2\omega_0 + (\omega_0^2 - p_2)^2} - \frac{q_1^2}{(q_1^2\omega_0^2 + q_2^2)}\right) \end{aligned}$$

$$= \frac{2\omega_0^2 + (p_1^2 - 2p_2 - q_1^2)}{(q_1^2\omega_0^2 + q_2^2)}$$

Under the condition $(p_1^2 - 2p_2 - q_1^2) > 0$, we have $\left.\frac{dRe(\lambda)}{dt}\right|_{\lambda=i\omega_0} > 0$.

Therefore, the transversality condition holds and Hopf bifurcation occurs at $\tau = \tau_0$, $\omega = \omega_0$.

Summarizing the discussion above, we have the following conclusion.

**Theorem 2.** *Assume that $R_0 > 1$ then there is a positive $\tau_0$ such that the following results hold.*

*i) If $0 < \tau < \tau_0$ then system (1.1) has an endemic equilibrium $E^*$ which is locally asymptotically stable.*

*ii) If $\tau > \tau_0$ then system (1.1) undergoes Hopf bifurcation and a periodic orbit exists in the small neighbourhood of $E^*$*

## 3. Numerical Simulations

In this section, we substantiate as well as augment our analytical results through numerical simulations considering the following examples.

**Example 1:**
Consider the parameter values
$k = 0.2; r = 0.85; \beta = 0.8; d = 0.65; \gamma = 0.10;$
In this case, $R_0 = 0.0486 < 1$ then system (1.1) has a disease free quilibrium at (0.0471, 0) and is locally asymptotically stable (Figure 1).

**Example 2:**
Consider the parameter values
$k = 20; r = 0.85; \beta = 0.8; d = 0.65; \gamma = 0.10;$
In this case, $R_0 = 4.8602 > 1$ then system (1.1) has an endemic quilibrium at (0.9682, 0.2366) and is locally asymptotically stable (Figure 2 & Figure 3).

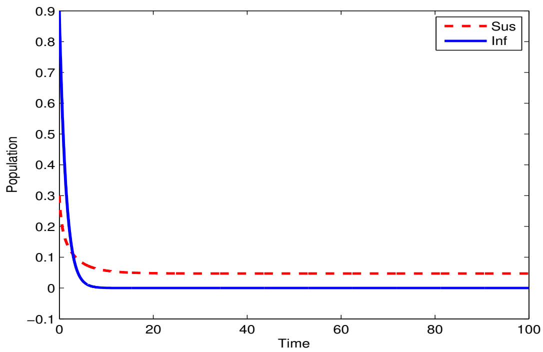


Figure 1

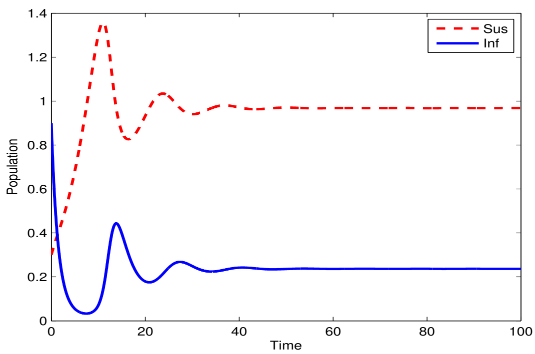


Figure 2

## 4. Concluding Remarks

This paper looked at a delayed SIS epidemic model having a nonlinear incidence rate. $R_0$, defined as basic reproduction number, is the agent governing the behavior of the model in question. The model was investigated keeping in mind its stability, both when there is delay and when there isn't. It was shown that disease free equilibrium is guaranteed only if $r > d$ and shows symptoms of being asymptotically stable in case of $R_0 < 1$. When $R_0 > 1$, endemic equilibrium is on the horizon and the model gathers stability which is asymptotic. Applying time delay as a parameter determining bifurcation, conditions ripe for the presence of Hopf bifurcation of endemic equilibrium were brought out. Local stability was shown to be a function of time delay $\tau$. By way of

Figure 3 represents phase-portraits of the system (1.1) at the endemic equilibrium

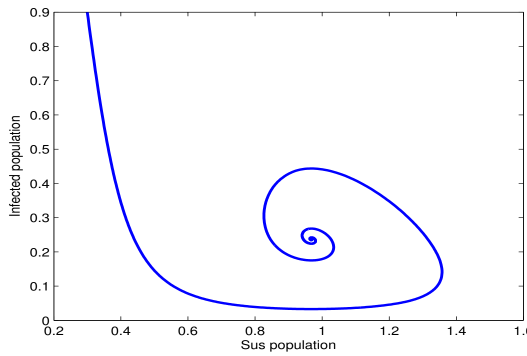


Figure 3

conclusion, numerical simulations were put in place to confirm results obtained theoretically.

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