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APPLICATIONS OF AN OPERATION APPROACHES ON *SC*-OPEN SETS

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Abstract: The paper introduces the concept of an operation on $SCO(X)$, the family of *sc*-open subsets of a topological space X. Using this operation, we define *sc*-γ-open sets, and study some of their related notions. Also, we introduce *sc*-γ-generalized closed sets and investigate their properties. Moreover, we introduce and study *sc*-γ-T_i spaces ($i \in \{0, \frac{1}{2}, 1, 2\}$) and *sc*-$(\gamma, \beta)$-continuous functions. Finally, some basic properties of functions with *sc*-β-closed graphs are obtained.

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Key Words: *sc*-γ-open sets, *sc*-γ-g.closed sets, *sc*-γ-T_i spaces ($i \in \{0, \frac{1}{2}, 1, 2\}$), *sc*-$(\gamma, \beta)$-continuous functions, *sc*-β-closed graphs

1. Introduction

The concepts of an operation on τ and α-closed graphs of functions is due to Kasahara [4]. After the work of Kasahara, Jankovic [3] defined the concept of

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operation-closures of α and investigated function with strongly closed graph. Ogata [8] defined and studied the concept of operation-open sets, and used it to investigate operation-separation axioms and operation-functions. Recently, Krishnan, Ganster and Balachandran [6] studied the notion of mapping γ on $SO(X)$, the family of semi-open subsets of a space X, that were defined by Levine [7]. Then they work on the family of semi γ-open sets. In this paper we introduce the concept of an operation γ on $SCO(X)$, the family of *sc*-open subsets of a space X that are defined by Khalaf and Ameen [5], and define the concept of *sc*-γ-open sets of (X, τ) by using the operation γ on $SCO(X)$. Also, we define *sc*-closure$_\gamma$, *sc*-γ-closure and *sc*-γ-interior and study their relationships. In Section 4, we introduce the concept of *sc*-γ-generalized closed sets and find its properties. In Section 5, *sc*-γ-T_i spaces where $i \in \{0, \frac{1}{2}, 1, 2\}$ are studied. In the last two sections, some properties of *sc*-(γ, β)-continuous functions with *sc*-β-closed graphs are obtained.

2. Preliminaries

Throughout this paper, the space (X, τ) (or simply X) always mean topological space on which no separation axioms are assumed unless explicitly stated. The symbols $Cl(A)$ and $Int(A)$ are the closure and the interior of $A \subseteq X$, respectively. A subset A of a topological space (X, τ) is called to be semiopen [7] if $A \subseteq Cl(Int(A))$. A semiopen subset A of a topological space (X, τ) is called *sc*-open [5] if for each $x \in A$, there exists a closed set F such that $x \in F \subseteq A$. The complement of *sc*-open is *sc*-closed [5]. The *sc*-closure, denoted by *sc*-$Cl(A)$ [5], of a subset A of X is defined as the intersection of all *sc*-closed sets containing A, and the *sc*-interior, denoted by *sc*-$Int(A)$ [5], of a subset A of X is defined as the union of all *sc*-open sets contained in A [5]. The family of all *sc*-open (resp. semiopen) subsets of a space (X, τ) is denoted by $SCO(X)$ (resp. $SO(X)$).

Definition 2.1. [6] An operation γ on $SO(X)$ is a mapping $\gamma \colon SO(X) \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in SO(X)$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U. A nonempty set A of X is called semi γ-open if for each $x \in A$, there is a semiopen set U such that $x \in U$ and $\gamma(U) \subseteq A$.

Definition 2.2. [6] A point $x \in X$ is in the semi γ-closure of a set $A \subseteq X$ if $\gamma(U) \cap A \neq \phi$ for each semiopen set U containing x. The set of all semi γ-closure points of A, denoted by $sCl_\gamma(A)$, is the semi γ-closure of A.

Definition 2.3. [6] A subset A of (X, τ) with an operation γ on $SO(X)$ is called semi γ-g.closed if $sCl_\gamma(A) \subseteq U$ if $A \subseteq U$ and U is semi γ-open in (X, τ).

Definition 2.4. [6] A topological space (X, τ) with an operation γ on $SO(X)$ is said to be

1. semi γ-T_0 if for any two distinct points x, y in X, there exists a semiopen set U such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.
2. semi γ-T_1 if for any two distinct points x, y in X, there exist two semiopen sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.
3. semi γ-T_2 if for any two distinct points x, y in X, there exist two semiopen sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \phi$.
4. semi γ-$T_{\frac{1}{2}}$ if every semi γ-g.closed set in X is semi γ-closed.

Definition 2.5. A topological space (X, τ) is said to be locally indiscrete [1] if every open subset of X is closed.

Lemma 2.6. *[5] For a topological space (X, τ), the following statements are true:*

1. *If (X, τ) is either T_1 or locally indiscrete, then $SCO(X) = SO(X)$.*
2. *If (X, τ) is regular, then $\tau \subseteq SCO(X)$.*

3. *sc*-γ-Open Sets

Definition 3.1. An operation γ on $SCO(X)$ is a mapping $\gamma\colon SCO(X) \to P(X)$ such that $U \subseteq \gamma(U)$ for every $U \in SCO(X)$, where $P(X)$ is the power set of X and $\gamma(U)$ is the value of γ at U.

From this definition, we can easily find $\gamma(X) = X$ for any operation $\gamma\colon SCO(X) \to P(X)$.

Definition 3.2. Let (X, τ) be a topological space and let $\gamma\colon SCO(X) \to P(X)$ be an operation on $SCO(X)$. A nonempty set A of X is said to be *sc*-γ-open if for each $x \in A$, there exists an *sc*-open set U such that $x \in U$ and $\gamma(U) \subseteq A$. The complement of an *sc*-γ-open set of X is *sc*-γ-closed. Assume that the empty set ϕ is also *sc*-γ-open set for any operation $\gamma\colon SCO(X) \to P(X)$. The family of all *sc*-γ-open subsets of a space (X, τ) is denoted by $SC_\gamma O(X)$.

It is obvious that every *sc*-γ-open set is *sc*-open (that is, $SC_\gamma O(X) \subseteq SCO(X)$), but the converse is not hold as shown in the following example.

Example 3.3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\} = SCO(X)$. Define an operation $\gamma\colon SCO(X) \to P(X)$ as follows:
For every $A \in SCO(X)$

$$A^\gamma = \begin{cases} A & \text{if } a \in A \\ X & \text{if } a \notin A \end{cases}$$

Clearly, $SC_\gamma O(X) = \{\phi, X, \{a\}\}$. Then the set $\{b, c\}$ is *sc*-open, but $\{b, c\}$ is not *sc*-γ-open. Therefore, $SCO(X) \nsubseteq SC_\gamma O(X)$.

Definition 3.4. We can say that a subset A of X is *sc*-*id*-open if and only if A is *sc*-open in X. The identity operation id on $SCO(X)$ is a mapping $id\colon SCO(X) \to P(X)$ such that $A^{id} = A$ for every $A \in SCO(X)$. Then $SC_{id}O(X) = SCO(X)$.

Theorem 3.5. *Let (X, τ) be a topological space and $\gamma\colon SCO(X) \to P(X)$ be an operation on $SCO(X)$. Then the following hold:*

1. *The union of any class of sc-γ-open sets in X is sc-γ-open.*

2. *The intersection of any class of sc-γ-closed sets in X is an sc-γ-closed.*

Proof. (1) Let $x \in \bigcup_{\lambda\in\Lambda}\{A_\lambda\}$, where $\{A_\lambda\}_{\lambda\in\Lambda}$ is a class of *sc*-γ-open sets in X and Λ is an index set. Then $x \in A_\lambda$ for some $\lambda \in \Lambda$. Since A_λ is *sc*-γ-open set in X, then there exists an *sc*-open set V such that $x \in V \subseteq \gamma(V) \subseteq A_\lambda \subseteq \bigcup_{\lambda\in\Lambda}\{A_\lambda\}$. Therefore, $\bigcup_{\lambda\in\Lambda}\{A_\lambda\}$ is *sc*-γ-open in X.
(2) Similar to part (1) using complement. □

Example 3.6. The intersection (resp. the union) of any two *sc*-γ-open (resp. *sc*-γ-closed) sets in (X, τ) is generally not an *sc*-γ-open (resp. *sc*-γ-closed) set. To see this, let $X = \{a, b, c\}$ and $\tau = P(X) = SCO(X)$. Let $\gamma\colon SCO(X) \to P(X)$ be an operation on $SCO(X)$ defined as follows:
For every $A \in SCO(X)$,

$$\gamma(A) = \begin{cases} A & \text{if } A \neq \{c\} \\ \{b, c\} & \text{if } A = \{c\} \end{cases}$$

Thus, $SC_\gamma O(X) = P(X)\backslash\{c\}$. Then $\{a, c\} \in SC_\gamma O(X)$ and $\{b, c\} \in SC_\gamma O(X)$, but $\{a, c\} \cap \{b, c\} = \{c\} \notin SC_\gamma O(X)$. Also, $\{a\}$ and $\{b\}$ are *sc*-γ-closed sets, but $\{a\} \cup \{b\} = \{a, b\}$ is not an *sc*-γ-closed set.

Remark 3.7. The class of all *sc*-γ-open sets of any topological space (X, τ) need not be a topology on X in general.

Definition 3.8. A topological space (X, τ) with an operation γ on $SCO(X)$ is said to be sc-γ-regular if for every $x \in X$ and for every sc-open set U containing x, there exists an sc-open set W such that $x \in W$ and $\gamma(W) \subseteq U$.

Theorem 3.9. *Let (X, τ) be a topological space and $\gamma\colon SCO(X) \to P(X)$ be an operation on $SCO(X)$. Then the following are equivalent:*

1. $SCO(X) = SC_\gamma O(X)$.
2. *(X, τ) is an sc-γ-regular space.*
3. *For every $x \in X$ and for every sc-open set U of (X, τ) containing x, there exists an sc-γ-open set W of (X, τ) containing x such that $W \subseteq U$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and let U be an sc-open set in X such that $x \in U$. It follows from assumption that U is an sc-γ-open set. This implies that there exists an sc-open set W such that $x \in W$ and $\gamma(W) \subseteq U$. Therefore, the space (X, τ) is sc-γ-regular.

(2) $\Rightarrow$ (3) Let $x \in X$ and let U be sc-open in (X, τ) with $x \in U$. By (2), there exists sc-open W such that $x \in W \subseteq \gamma(W) \subseteq U$. Hence W is an sc-γ-open set containing x such that $W \subseteq U$, which proves (3).

(3) $\Rightarrow$ (1) By (3) and Theorem 3.5 (1), every sc-open set of X is sc-γ-open in X. That is, $SCO(X) \subseteq SC_\gamma O(X)$. But $SC_\gamma O(X) \subseteq SCO(X)$ is always true. Hence $SCO(X) = SC_\gamma O(X)$. □

Remark 3.10. Since every sc-open set is semiopen, then by Definition 3.2 and Definition 2.1, every sc-γ-open set is semi γ-open, but the converse is not true in general. For instance, let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{c\}, \{d\}, \{c, d\}\}$. Then $SO(X) = \{\phi, X, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $SCO(X) = \{\phi, X, \{a, b, c\}, \{a, b, d\}\}$. Define an operation $\gamma\colon SCO(X) \to P(X)$ by $\gamma(A) = A$ for all $A \in SCO(X)$. Here, $SC_\gamma O(X) = SCO(X)$ and $SO(X)_\gamma = SO(X)$. Therefore, the set $\{c\}$ is semi γ-open but not sc-γ-open.

Lemma 3.11. *The following properties are true for any topological space (X, τ):*

1. *If (X, τ) is either T_1 or locally indiscrete, then the operations γ on $SCO(X)$ and γ on $SO(X)$ are equivalent and hence the concept of sc-γ-open set and semi γ-open set coincide (That is $SC_\gamma O(X) = SO(X)_\gamma$).*

2. *If (X, τ) is regular, then every γ-open set is sc-γ-open.*

Proof. (1) Follows from their definitions and Lemma 2.6 (1).

(2) Follows from their definitions and Lemma 2.6 (2). □

Definition 3.12. Let (X, τ) be any topological space. An operation γ on $SCO(X)$ is said to be

1. *sc*-open if for each $x \in X$ and for every *sc*-open set U containing x, there exists an *sc*-γ-open set W containing x such that $W \subseteq \gamma(U)$.
2. *sc*-regular if for each $x \in X$ and for every pair of *sc*-open sets U_1 and U_2 such that both containing x, there exists an *sc*-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$.

For example, the mapping $\gamma \colon SCO(X) \to P(X)$ defined in Example 3.3 is *sc*-regular operation.

Theorem 3.13. *Let a mapping γ be sc-regular operation on $SCO(X)$. If the subsets A and B are sc-γ-open in (X, τ), then $A \cap B$ is sc-γ-open in (X, τ).*

Proof. Suppose $x \in A \cap B$ for any *sc*-γ-open subsets A and B in (X, τ) both containing x. Then there exist *sc*-open sets U_1 and U_2 such that $x \in U_1 \subseteq A$ and $x \in U_2 \subseteq B$. Since γ is an *sc*-regular operation on $SCO(X)$, there exists an *sc*-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2) \subseteq A \cap B$. Therefore, $A \cap B$ is *sc*-γ-open in (X, τ). □

Remark 3.14. Theorem 3.13 shows that $SC_\gamma O(X)$ forms a topology on X for any *sc*-regular operation γ on $SCO(X)$.

Definition 3.15. The point $x \in X$ is in the *sc*-closure$_\gamma$ of a set A if $\gamma(U) \cap A \neq \phi$ for every *sc*-open set U containing x. The set of all *sc*-closure$_\gamma$ points of A is called *sc*-closure$_\gamma$ of A, and is denoted by $sc\text{-}Cl_\gamma(A)$.

Definition 3.16. Let A be a subset of a topological space (X, τ) and let γ be an operation on $SCO(X)$. The *sc*-γ-closure of A is defined as the intersection of all *sc*-γ-closed sets of X containing A and is denoted by $sc_\gamma Cl(A)$. That is,

$$sc_\gamma Cl(A) = \bigcap \{F : A \subseteq F,\ X \backslash F \in SC_\gamma O(X)\}.$$

Theorem 3.17. *Assume that A is a subset of a space X and γ is an operation on $SCO(X)$. Then $x \in sc_\gamma Cl(A)$ if and only if $A \cap U \neq \phi$ for every sc-γ-open set U of X containing x.*

Proof. Let $x \in sc_\gamma Cl(A)$ and let $A \cap U = \phi$ for some sc-γ-open set U of X containing x. Then $A \subseteq X \backslash U$ and $X \backslash U$ is sc-γ-closed set in X. So $sc_\gamma Cl(A) \subseteq X \backslash U$. Thus, $x \in X \backslash U$. This is a contradiction. Hence $A \cap U \neq \phi$ for every sc-γ-open set U of X containing x.

Conversely, suppose that $x \notin sc_\gamma Cl(A)$. So there exists an sc-γ-closed set F such that $A \subseteq F$ and $x \notin F$. Then $X \backslash F$ is an sc-γ-open set such that $x \in X \backslash F$ and $A \cap (X \backslash F) = \phi$. This contradicts the hypothesis. Therefore, $x \in sc_\gamma Cl(A)$. □

Lemma 3.18. *The following statements hold for any subsets A and B of a topological space (X, τ) with an operation γ on $SCO(X)$.*

1. $sc_\gamma Cl(A)$ *is* sc-γ-*closed in* X *and* sc-$Cl_\gamma(A)$ *is* sc-*closed in* X.
2. $A \subseteq sc\text{-}Cl(A) \subseteq sc\text{-}Cl_\gamma(A) \subseteq sc_\gamma Cl(A)$.
3. $sc_\gamma Cl(\phi) = sc\text{-}Cl_\gamma(\phi) = \phi$ *and* $sc_\gamma Cl(X) = sc\text{-}Cl_\gamma(X) = X$.
4. (a) A *is* sc-γ-*closed if and only if* $sc_\gamma Cl(A) = A$ *and,*
   (b) A *is* sc-γ-*closed if and only if* $sc\text{-}Cl_\gamma(A) = A$.
5. *If* $A \subseteq B$, *then* $sc_\gamma Cl(A) \subseteq sc_\gamma Cl(B)$ *and* $sc\text{-}Cl_\gamma(A) \subseteq sc\text{-}Cl_\gamma(B)$.
6. (a) $sc_\gamma Cl(A \cap B) \subseteq sc_\gamma Cl(A) \cap sc_\gamma Cl(B)$ *and,*
   (b) $sc\text{-}Cl_\gamma(A \cap B) \subseteq sc\text{-}Cl_\gamma(A) \cap sc\text{-}Cl_\gamma(B)$.
7. (a) $sc_\gamma Cl(A) \cup sc_\gamma Cl(B) \subseteq sc_\gamma Cl(A \cup B)$ *and,*
   (b) $sc\text{-}Cl_\gamma(A) \cup sc\text{-}Cl_\gamma(B) \subseteq sc\text{-}Cl_\gamma(A \cup B)$.
8. $sc_\gamma Cl(sc_\gamma Cl(A)) = sc_\gamma Cl(A)$.

Proof. Straightforward. □

Theorem 3.19. *For any subsets A, B of a topological space (X, τ), if γ is an sc-regular operation on $SCO(X)$, then*

1. $sc_\gamma Cl(A) \cup sc_\gamma Cl(B) = sc_\gamma Cl(A \cup B)$.
2. $sc\text{-}Cl_\gamma(A) \cup sc\text{-}Cl_\gamma(B) = sc\text{-}Cl_\gamma(A \cup B)$.

Proof. (1) It is enough to proof that $sc_\gamma Cl(A \cup B) \subseteq sc_\gamma Cl(A) \cup sc_\gamma Cl(B)$ since the other part follows directly from Lemma 3.18 (7). Let $x \notin sc_\gamma Cl(A) \cup sc_\gamma Cl(B)$. Then there exist two sc-γ-open sets U and V containing x such that $A \cap U = \phi$ and $B \cap V = \phi$. Since γ is an sc-regular operation on $SCO(X)$, by Theorem 3.13, $U \cap V$ is sc-γ-open in X such that

$$(U \cap V) \cap (A \cup B) = \phi.$$

Therefore, we have $x \notin sc_\gamma Cl(A \cup B)$ and hence

$$sc_\gamma Cl(A \cup B) \subseteq sc_\gamma Cl(A) \cup sc_\gamma Cl(B).$$

(2) Let $x \notin sc\text{-}Cl_\gamma(A) \cup sc\text{-}Cl_\gamma(B)$. Then there exist sc-open sets U_1 and U_2 such that $x \in U_1$, $x \in U_2$, $A \cap \gamma(U_1) = \phi$ and $A \cap \gamma(U_2) = \phi$. Since γ is an sc-regular operation on $SCO(X)$, then there exists an sc-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$. Thus, we have

$$(A \cup B) \cap \gamma(W) \subseteq (A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)).$$

This implies that $(A \cup B) \cap \gamma(W) = \phi$ since $(A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)) = \phi$. This means that $x \notin sc\text{-}Cl_\gamma(A \cup B)$ and hence $sc\text{-}Cl_\gamma(A \cup B) \subseteq sc\text{-}Cl_\gamma(A) \cup sc\text{-}Cl_\gamma(B)$. Using Lemma 3.18 (7), we have the equality. □

Theorem 3.20. *Suppose that A is a subset of a space X. If γ is an sc-open operation on $SCO(X)$, then $sc\text{-}Cl_\gamma(A) = sc_\gamma Cl(A)$, $sc\text{-}Cl_\gamma(sc\text{-}Cl_\gamma(A)) = sc\text{-}Cl_\gamma(A)$ and $sc\text{-}Cl_\gamma(A)$ is an sc-γ-closed set in X.*

Proof. We only need to show that $sc_\gamma Cl(A) \subseteq sc\text{-}Cl_\gamma(A)$ because by Lemma 3.18 (2), we have that $sc\text{-}Cl_\gamma(A) \subseteq sc_\gamma Cl(A)$. Now, let $x \notin sc\text{-}Cl_\gamma(A)$. So there is sc-open U containing x such that $A \cap \gamma(U) = \phi$. Since γ is an sc-open on $SCO(X)$, then there exists an sc-γ-open set W containing x such that $W \subseteq \gamma(U)$. Therefore $A \cap W = \phi$ and so $x \notin sc_\gamma Cl(A)$, by Theorem 3.17. This implies that $sc_\gamma Cl(A) \subseteq sc\text{-}Cl_\gamma(A)$. Hence $sc\text{-}Cl_\gamma(A) = sc_\gamma Cl(A)$. Moreover, using the above result and Lemma 3.18 (8), we get $sc\text{-}Cl_\gamma(sc\text{-}Cl_\gamma(A)) = sc\text{-}Cl_\gamma(A)$ and then by Lemma 3.18 (4b), we obtain $sc\text{-}Cl_\gamma(A)$ is sc-γ-closed in X. □

Theorem 3.21. *Let A be a subset of a topological space (X, τ) and let γ be an operation on $SCO(X)$. Then the following statements are equivalent:*

1. *A is sc-γ-open.*
2. *$sc\text{-}Cl_\gamma(X \backslash A) = X \backslash A$.*
3. *$sc_\gamma Cl(X \backslash A) = X \backslash A$.*
4. *$X \backslash A$ is sc-γ-closed.*

Proof. Clear. □

Definition 3.22. A subset N of a topological space (X, τ) is called an sc-γ-neighbourhood of a point $x \in X$, if there exists an sc-γ-open set U in X such that $x \in U \subseteq N$.

Lemma 3.23. *A subset $U \subseteq (X, \tau)$ is sc-γ-open if and only if it is an sc-γ-neighbourhood of each of its points.*

Proof. Let U be an sc-γ-open set in (X, τ). By Definition 3.22, U is an sc-γ-neighbourhood of each of its points, since for every $x \in U$, $x \in U \subseteq U$ and $U \in SC_\gamma O(X)$.

Conversely, suppose that U is an sc-γ-neighbourhood of each of its points. Then for each $x \in U$, there exists an sc-γ-open set V_x containing x such that $V_x \subseteq U$. Set $U = \bigcup_{x \in U} V_x$. Since each V_x is sc-γ-open, by Theorem 3.5 (1), U is sc-γ-open in X. □

Definition 3.24. Let A be a subset of a topological space (X, τ) and let γ be an operation on $SCO(X)$. The sc-γ-interior of A is defined as the union of all sc-γ-open sets of X contained in A, and is denoted by $sc_\gamma Int(A)$. That is,

$$sc_\gamma Int(A) = \bigcup \{U : U \in SC_\gamma O(X) \text{ and } U \subseteq A\}.$$

Some important properties of sc-γ-interior operator will be given in Lemma 3.25.

Lemma 3.25. *Let A and B be subsets of a topological space (X, τ) and let γ be an operation on $SCO(X)$. Then the following statements hold:*

1. *$sc_\gamma Int(A)$ is sc-γ-open in X and $sc_\gamma Int(A) \subseteq sc\text{-}Int(A) \subseteq A$.*
2. *$sc_\gamma Int(\phi) = \phi$ and $sc_\gamma Int(X) = X$.*
3. *A is sc-γ-open if and only if $sc_\gamma Int(A) = A$.*
4. *If $A \subseteq B$, then $sc_\gamma Int(A) \subseteq sc_\gamma Int(B)$.*
5. *$sc_\gamma Int(A \cap B) \subseteq sc_\gamma Int(A) \cap sc_\gamma Int(B)$.*
6. *$sc_\gamma Int(A) \cup sc_\gamma Int(B) \subseteq sc_\gamma Int(A \cup B)$.*
7. *$sc_\gamma Int(sc_\gamma Int(A)) = sc_\gamma Int(A)$.*
8. *$sc_\gamma Int(X \backslash A) = X \backslash sc_\gamma Cl(A)$.*

Proof. Straightforward. □

Theorem 3.26. *If γ is an sc-regular operation on $SCO(X)$, then for any subsets A, B of X,*

$$sc_\gamma Int(A) \cap sc_\gamma Int(B) = sc_\gamma Int(A \cap B).$$

Proof. Follows from Theorem 3.19 (1) and Lemma 3.25 (8). □

Lemma 3.27. *Let (X, τ) be a topological space and let γ be an sc-regular operation on $SCO(X)$. Then $sc_\gamma Cl(A) \cap U \subseteq sc_\gamma Cl(A \cap U)$ holds for every sc-γ-open set U and every subset A of X.*

Proof. Suppose that $x \in sc_\gamma Cl(A) \cap U$ for every sc-γ-open set U. Then $x \in sc_\gamma Cl(A)$ and $x \in U$. Let V be an sc-γ-open set of X containing x. Since γ is sc-regular on $SCO(X)$, by Theorem 3.13, $U \cap V$ is sc-γ-open set containing x. But $x \in sc_\gamma Cl(A)$. Then, by Theorem 3.17, we have $A \cap (U \cap V) \neq \phi$. This implies that $(A \cap U) \cap V \neq \phi$. Therefore, by Theorem 3.17, $x \in sc_\gamma Cl(A \cap U)$. Thus, $sc_\gamma Cl(A) \cap U \subseteq sc_\gamma Cl(A \cap U)$. □

The proof of the following lemma is similar to Lemma 3.27 using Lemma 3.25 (8).

Lemma 3.28. *Let (X, τ) be a topological space and let γ be an sc-regular operation on $SCO(X)$. Then $sc_\gamma Int(A \cup F) \subseteq sc_\gamma Int(A) \cup F$ holds for every sc-γ-closed set F and every subset A of X.*

4. *sc*-γ-*g*.Closed Sets

Definition 4.1. A subset A of a topological space (X, τ) with an operation γ on $SCO(X)$ is said to be sc-γ-generalized closed (briefly sc-γ-g.closed) if sc-$Cl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is an sc-γ-open set in X.

Lemma 4.2. *Let (X, τ) be a topological space and let γ be an operation on $SCO(X)$. A set A in (X, τ) is sc-γ-g.closed if and only if $A \cap sc_\gamma Cl(\{x\}) \neq \phi$ for every $x \in sc$-$Cl_\gamma(A)$.*

Proof. Suppose that A is sc-γ-g.closed set in X and suppose (if possible) that there exists an element $x \in sc$-$Cl_\gamma(A)$ such that $A \cap sc_\gamma Cl(\{x\}) = \phi$. This follows that $A \subseteq X \backslash sc_\gamma Cl(\{x\})$. Since $sc_\gamma Cl(\{x\})$ is sc-γ-closed and A is an sc-γ-g.closed set in X, then $X \backslash sc_\gamma Cl(\{x\})$ is sc-γ-open and so sc-$Cl_\gamma(A) \subseteq X \backslash sc_\gamma Cl(\{x\})$. This means that $x \notin sc$-$Cl_\gamma(A)$, which is a contradiction. Hence $A \cap sc_\gamma Cl(\{x\}) \neq \phi$.

Conversely, let $U \in SC_\gamma O(X)$ such that $A \subseteq U$. To show that sc-$Cl_\gamma(A) \subseteq U$, let $x \in sc$-$Cl_\gamma(A)$. By hypothesis, $A \cap sc_\gamma Cl(\{x\}) \neq \phi$. So there exists an element $y \in A \cap sc_\gamma Cl(\{x\})$. Therefore $y \in A \subseteq U$ and $y \in sc_\gamma Cl(\{x\})$. By Theorem 3.17, $\{x\} \cap U \neq \phi$. Hence $x \in U$ and so sc-$Cl_\gamma(A) \subseteq U$. Thus, A is sc-γ-g.closed set in (X, τ). □

Theorem 4.3. *Let A be a subset of topological space (X, τ) and let γ be an operation on $SCO(X)$. If A is sc-γ-g.closed, then sc-$Cl_\gamma(A)\backslash A$ does not contain any nonempty sc-γ-closed set.*

Proof. Let F be a nonempty *sc*-γ-closed set in X such that $F \subseteq$ *sc*-$Cl_\gamma(A)\backslash A$. Then $F \subseteq X\backslash A$ implies that $A \subseteq X\backslash F$. Since $X\backslash F$ is *sc*-γ-open and A is *sc*-γ-g.closed, then *sc*-$Cl_\gamma(A) \subseteq X\backslash F$. That is $F \subseteq X\backslash$*sc*-$Cl_\gamma(A)$. Hence $F \subseteq X\backslash$*sc*-$Cl_\gamma(A) \cap$ *sc*-$Cl_\gamma(A)\backslash A \subseteq X\backslash$*sc*-$Cl_\gamma(A) \cap$ *sc*-$Cl_\gamma(A) = \phi$. This shows that $F = \phi$, which is a contradiction. Therefore, $F \nsubseteq$ *sc*-$Cl_\gamma(A)\backslash A$. □

Theorem 4.4. *If $\gamma\colon SCO(X) \to P(X)$ is an sc-open operation, then the converse of the Theorem 4.3 is true.*

Proof. Let U be an *sc*-γ-open set in (X, τ) such that $A \subseteq U$. Since $\gamma\colon SCO(X) \to P(X)$ is an *sc*-open operation, by Theorem 3.20, *sc*-$Cl_\gamma(A)$ is *sc*-γ-closed in X. Thus, using Theorem 3.5 (1), we have *sc*-$Cl_\gamma(A) \cap X\backslash U$ is an *sc*-γ-closed set in (X, τ). Since $X\backslash U \subseteq X\backslash A$, then *sc*-$Cl_\gamma(A) \cap X\backslash U \subseteq$ *sc*-$Cl_\gamma(A)\backslash A$. Using the assumption of the converse of Theorem 4.3, *sc*-$Cl_\gamma(A) \subseteq U$. Therefore, A is *sc*-γ-g.closed in (X, τ). □

Corollary 4.5. *Let A be an sc-γ-g.closed subset of topological space (X, τ) and let γ be an operation on $SCO(X)$. Then A is sc-γ-closed if and only if sc-$Cl_\gamma(A)\backslash A$ is sc-γ-closed.*

Proof. Let A be an *sc*-γ-closed set in (X, τ). By Lemma 3.18 (4b), *sc*-$Cl_\gamma(A) = A$ and so *sc*-$Cl_\gamma(A)\backslash A = \phi$ which is *sc*-γ-closed.

Conversely, suppose that *sc*-$Cl_\gamma(A)\backslash A$ is *sc*-γ-closed and A is *sc*-γ-g.closed. By Theorem 4.3, *sc*-$Cl_\gamma(A)\backslash A$ does not contain any nonempty *sc*-γ-closed set and since *sc*-$Cl_\gamma(A)\backslash A$ is *sc*-γ-closed subset of itself, then *sc*-$Cl_\gamma(A)\backslash A = \phi$ implies that *sc*-$Cl_\gamma(A) \cap X\backslash A = \phi$. So *sc*-$Cl_\gamma(A) = A$. Hence A is *sc*-γ-closed in (X, τ). □

Theorem 4.6. *Let (X, τ) be a topological space and let γ be an operation on $SCO(X)$. If a subset A of X is sc-γ-g.closed and sc-γ-open, then A is sc-γ-closed.*

Proof. Let A be *sc*-γ-g.closed and *sc*-γ-open in X, then *sc*-$Cl_\gamma(A) \subseteq A$ and so, by Lemma 3.18 (4b), A is *sc*-γ-closed. □

Theorem 4.7. *Let (X, τ) be a topological space with an operation γ on $SCO(X)$. For an element $x \in X$, the set $X\backslash\{x\}$ is either sc-γ-g.closed or sc-γ-open.*

Proof. Suppose that $X\backslash\{x\}$ is not *sc*-γ-open. Then X is the only *sc*-γ-open set containing $X\backslash\{x\}$. This implies that *sc*-$Cl_\gamma(X\backslash\{x\}) \subseteq X$. Thus $X\backslash\{x\}$ is an *sc*-γ-g.closed set in X. □

Corollary 4.8. *Let (X,τ) be a topological space with an operation γ on $SCO(X)$. For an element $x \in X$, either the set $\{x\}$ is sc-γ-closed or the set $X\backslash\{x\}$ is sc-γ-g.closed.*

Proof. Suppose that $\{x\}$ is not *sc*-γ-closed, then $X\backslash\{x\}$ is not *sc*-γ-open. By Theorem 4.7, $X\backslash\{x\}$ is *sc*-γ-g.closed in X. □

Definition 4.9. For a subset A of a topological space (X,τ) with an operation γ on $SCO(X)$, the $SC_\gamma O(X)$-kernel of A, denoted by $SC_\gamma O(X)$-ker(A), is defined by

$$SC_\gamma O(X)\text{-ker}(A)= \cap\{U : A \subseteq U \text{ and } U \in SC_\gamma O(X)\}$$

In other words, $SC_\gamma O(X)$-ker(A) is the intersection of all *sc*-γ-open sets of (X,τ) containing A.

Theorem 4.10. *Let $A \subseteq (X,\tau)$ and let γ be an operation on $SCO(X)$. Then A is sc-γ-g.closed if and only if sc-$Cl_\gamma(A) \subseteq SC_\gamma O(X)$-ker$(A)$.*

Proof. Suppose that A is *sc*-γ-g.closed. Then *sc*-$Cl_\gamma(A) \subseteq U$, whenever $A \subseteq U$ and U is *sc*-γ-open. Let $x \in$ *sc*-$Cl_\gamma(A)$. By Lemma 4.2, $A \cap sc_\gamma Cl(\{x\}) \neq \phi$. So there exists a point z in X such that $z \in A \cap sc_\gamma Cl(\{x\})$ which implies that $z \in A \subseteq U$ and $z \in sc_\gamma Cl(\{x\})$. By Theorem 3.17, $\{x\} \cap U \neq \phi$. This concludes that $x \in SC_\gamma O(X)$-ker(A). Therefore, *sc*-$Cl_\gamma(A) \subseteq SC_\gamma O(X)$-ker$(A)$.

Conversely, let *sc*-$Cl_\gamma(A) \subseteq SC_\gamma O(X)$-ker$(A)$. Let U be an *sc*-γ-open set containing A. Let x be a point in X such that $x \in$ *sc*-$Cl_\gamma(A)$. Then $x \in SC_\gamma O(X)$-ker(A). Now, we have $x \in U$, because $A \subseteq U$ and $U \in SC_\gamma O(X)\}$. Therefore *sc*-$Cl_\gamma(A) \subseteq SC_\gamma O(X)$-ker$(A) \subseteq U$. Thus A is *sc*-γ-g.closed set in X. □

5. sc-γ-T_i Spaces for $i \in \{0, \frac{1}{2}, 1, 2\}$

In this section, we introduce types of *sc*-γ- separation axioms called *sc*-γ-T_i for $i \in \{0, \frac{1}{2}, 1, 2\}$, and some basic properties of such spaces are obtained.

Definition 5.1. A topological space (X,τ) with an operation γ on $SCO(X)$ is said to be *sc*-γ-T_0 if for every two distinct points x, y in X, there exists an *sc*-open set U such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.

Definition 5.2. A topological space (X, τ) with an operation γ on $SCO(X)$ is said to be *sc*-γ-T_1 if for every two distinct points x, y in X, there exist two *sc*-open sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.

Definition 5.3. A topological space (X, τ) with an operation γ on $SCO(X)$ is said to be *sc*-γ-T_2 if for every two distinct points x, y in X, there exist two *sc*-open sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \phi$.

Definition 5.4. A topological space (X, τ) with an operation γ on $SCO(X)$ is said to be *sc*-γ-$T_{\frac{1}{2}}$ if every *sc*-γ-g.closed set in X is *sc*-γ-closed.

Theorem 5.5. *For any topological space (X, τ) with an operation γ on $SCO(X)$, (X, τ) is sc-γ-$T_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is sc-γ-closed or sc-γ-open.*

Proof. Let X be an *sc*-γ-$T_{\frac{1}{2}}$ space and let $\{x\}$ be not an *sc*-γ-closed set in (X, τ). By Corollary 4.8, $X \backslash \{x\}$ is *sc*-γ-g.closed. Since (X, τ) is *sc*-γ-$T_{\frac{1}{2}}$, then $X \backslash \{x\}$ is *sc*-γ-closed which means that $\{x\}$ is *sc*-γ-open in X.

Conversely, let F be an *sc*-γ-g.closed set in (X, τ). We have to show that F is *sc*-γ-closed (that is *sc*-$Cl_\gamma(F) = F$ (by Lemma 3.18 (4b))). It is sufficient to show that *sc*-$Cl_\gamma(F) \subseteq F$. Let $x \in$ *sc*-$Cl_\gamma(F)$. By hypothesis $\{x\}$ is *sc*-γ-closed or *sc*-γ-open for each $x \in X$. We consider two cases:

Case (1): Let $\{x\}$ be an *sc*-γ-closed set. Suppose that $x \notin F$, then $x \in$ *sc*-$Cl_\gamma(F) \backslash F$ contains a nonempty *sc*-γ-closed set $\{x\}$. Since F is *sc*-γ-g.closed set, so this leads us to contradiction according to Theorem 4.3. Thus $x \in F$. Therefore *sc*-$Cl_\gamma(F) \subseteq F$ and so *sc*-$Cl_\gamma(F) = F$. This means that F is *sc*-γ-closed in (X, τ). Hence (X, τ) is *sc*-γ-$T_{\frac{1}{2}}$ space.

Case (2): let $\{x\}$ be an *sc*-γ-open set. By Theorem 3.17, $F \cap \{x\} \neq \phi$ which implies that $x \in F$. So *sc*-$Cl_\gamma(F) \subseteq F$. By Lemma 3.18 (4b), F is *sc*-γ-closed. Therefore, (X, τ) is *sc*-γ-$T_{\frac{1}{2}}$ space. □

Theorem 5.6. *Let γ be an sc-open operation on $SCO(X)$. Then (X, τ) is sc-γ-T_0 if and only if sc-$Cl_\gamma(\{x\}) \neq$ sc-$Cl_\gamma(\{y\})$, for every pair x, y of X with $x \neq y$.*

Proof. Let x, y be any two distinct points of an *sc*-γ-T_0 space (X, τ). By definition, there exists an *sc*-γ-open set U such that $x \in U$ and $y \notin \gamma(U)$. Since γ is an *sc*-open operation on $SCO(X)$, then there exists an *sc*-γ-open set W such that $x \in W$ and $W \subseteq \gamma(U)$. Hence $y \in X \backslash \gamma(U) \subseteq X \backslash W$. Since $X \backslash W$

is sc-γ-closed in (X,τ), then sc-$Cl_\gamma(\{y\}) \subseteq X\backslash W$ and so sc-$Cl_\gamma(\{x\}) \neq sc$-$Cl_\gamma(\{y\})$.

Conversely, suppose that for any $x, y \in X$ with $x \neq y$, we have sc-$Cl_\gamma(\{x\}) \neq sc$-$Cl_\gamma(\{y\})$. Now, we assume that there exists $z \in X$ such that $z \in sc$-$Cl_\gamma(\{x\})$, but $z \notin sc$-$Cl_\gamma(\{y\})$. If $x \in sc$-$Cl_\gamma(\{y\})$, then $\{x\} \subseteq sc$-$Cl_\gamma(\{y\})$, which implies that sc-$Cl_\gamma(\{x\}) \subseteq sc$-$Cl_\gamma(\{y\})$ (by Lemma 3.18 (5)). This implies that $z \in sc$-$Cl_\gamma(\{y\})$. This contradiction shows that $x \notin sc$-$Cl_\gamma(\{y\})$. By Definition 3.15, there exists an sc-open set U such that $x \in U$ and $\gamma(U) \cap \{y\} = \phi$. Therefore, we have that $x \in U$ and $y \notin \gamma(U)$. This proves that the space (X,τ) is sc-γ-T_0. □

Theorem 5.7. *The space (X,τ) is sc-γ-T_1 if and only if for every point $x \in X$, $\{x\}$ is an sc-γ-closed set.*

Proof. Let X be sc-γ-T_1 and let x be a point in (X,τ). Then for any point $y \in X$ such that $x \neq y$, there exists an sc-open set V_y such that $y \in V_y$ but $x \notin \gamma(V_y)$. Thus, $y \in \gamma(V_y) \subseteq X\backslash\{x\}$. This implies that $X\backslash\{x\} = \cup\{\gamma(V_y) : y \in X\backslash\{x\}\}$. So $X\backslash\{x\}$ is sc-γ-open in (X,τ). Hence $\{x\}$ is sc-γ-closed in (X,τ).

Conversely, let $x, y \in X$ such that $x \neq y$. By hypothesis, $X\backslash\{y\}$ and $X\backslash\{x\}$ are sc-γ-open sets such that $x \in X\backslash\{y\}$ and $y \in X\backslash\{x\}$. Therefore, there exist sc-open sets U and V such that $x \in U$, $y \in V$, $\gamma(U) \subseteq X\backslash\{y\}$ and $\gamma(V) \subseteq X\backslash\{x\}$. So, $y \notin \gamma(U)$ and $x \notin \gamma(V)$. This shows that (X,τ) is sc-γ-T_1. □

Theorem 5.8. *For any topological space (X,τ) and any operation γ on $SCO(X)$, the following properties hold.*

1. *Every sc-γ-T_2 space is sc-γ-T_1.*
2. *Every sc-γ-T_1 space is sc-γ-$T_{\frac{1}{2}}$.*
3. *Every sc-γ-$T_{\frac{1}{2}}$ space is sc-γ-T_0.*

Proof. The proofs can be followed from their definitions. □

Remark 5.9. By Theorem 5.8, Remark 3.10 and Remark 4.12 [6], we obtain the following diagram. Moreover, the following Examples 5.10, 5.11, 5.12, and 5.13 below show that the reverse implications are not true in general.

$$\begin{array}{ccccccc}
sc\text{-}\gamma\text{-}T_2 & \longrightarrow & sc\text{-}\gamma\text{-}T_1 & \longrightarrow & sc\text{-}\gamma\text{-}T_{\frac{1}{2}} & \longrightarrow & sc\text{-}\gamma\text{-}T_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{semi } \gamma\text{-}T_2 & \longrightarrow & \text{semi } \gamma\text{-}T_1 & \longrightarrow & \text{semi } \gamma\text{-}T_{\frac{1}{2}} & \longrightarrow & \text{semi } \gamma\text{-}T_0
\end{array}$$

Example 5.10. The space (X, τ) in the example appeared in Remark 3.10 is semi γ-T_i but not sc-γ-T_i for $i \in \{0, \frac{1}{2}, 1, 2\}$.

Example 5.11. Let $X = \{a, b, c\}$ and let $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Define an operation $\gamma\colon SCO(X) \to P(X)$ by $\gamma(A) = A$ for all $A \in SCO(X)$. Thus, $SC_\gamma O(X) = SCO(X) = \{\phi, X, \{b\}, \{a, c\}\}$. Then (X, τ) is sc-γ-T_0 but not sc-γ-$T_{\frac{1}{2}}$.

Example 5.12. Let $X = \{a, b, c\}$ and let $\tau = P(X) = SCO(X)$. Let $\gamma\colon SCO(X) \to P(X)$ be an operation on $SCO(X)$ defined by
For every set $A \in SCO(X)$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a\} \text{ or } \{b\} \text{ or } \{a, b\} \text{ or } \{b, c\} \\ X & \text{otherwise} \end{cases}$$

Clearly, $SC_\gamma O(X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Thus (X, τ) is sc-γ-$T_{\frac{1}{2}}$ but not sc-γ-T_1.

Example 5.13. Suppose $X = \{a, b, c\}$ and let τ be the discrete topology on X. Define an operation γ on $SCO(X)$ as follows: For every $A \in SCO(X)$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \text{ or } \{b, c\} \\ X & \text{otherwise} \end{cases}$$

Then (X, τ) is sc-γ-T_1 space, but (X, τ) is not sc-γ-T_2.

Lemma 5.14. *Let (X, τ) be either T_1 or locally indiscrete space. Then (X, τ) is sc-γ-T_i if and only if it is semi γ-T_i, where $i \in \{0, \frac{1}{2}, 1, 2\}$.*

Proof. Follows from Lemma 2.6 (1). □

6. *sc*-(γ, β)-Continuous Functions

Throughout Section 6 and Section 7, let (X, τ) and (Y, σ) be two topological spaces and let $\gamma\colon SCO(X) \to P(X)$ and $\beta\colon SCO(Y) \to P(Y)$ be operations on $SCO(X)$ and $SCO(Y)$ respectively. In this section, we introduce a new class of functions called *sc*-(γ, β)-continuous. Several characterizations and properties of this class are mentioned.

Definition 6.1. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be *sc*-(γ, β)-continuous if for every $x \in X$ and every *sc*-open set V containing $f(x)$, there exists an *sc*-open set U containing x such that $f(\gamma(U)) \subseteq \beta(V)$.

Theorem 6.2. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be an sc-(γ, β)-continuous function. Then,*

1. $f(sc\text{-}Cl_\gamma(A)) \subseteq sc\text{-}Cl_\beta(f(A))$, *for every* $A \subseteq (X, \tau)$.
2. $f^{-1}(F)$ *is sc-γ-closed in* (X, τ), *for every sc-β-closed set F of* (Y, σ).

Proof. (1) Let $y \in f(sc\text{-}Cl_\gamma(A))$ and V be any *sc*-open set containing y. By hypothesis, there exists $x \in X$ and *sc*-open set U containing x such that $f(x) = y$ and $f(\gamma(U)) \subseteq \beta(V)$. Since $x \in sc\text{-}Cl_\gamma(A)$, we have $\gamma(U) \cap A \neq \phi$. So $\phi \neq f(\gamma(U) \cap A) \subseteq f(\gamma(U)) \cap f(A) \subseteq \beta(V) \cap f(A)$. This implies that $y \in sc\text{-}Cl_\beta(f(A))$. Therefore, $f(sc\text{-}Cl_\gamma(A)) \subseteq sc\text{-}Cl_\beta(f(A))$.

(2) Let F be any *sc*-β-closed set of (Y, σ). By (1), we have $f(sc\text{-}Cl_\gamma(f^{-1}(F))) \subseteq sc\text{-}Cl_\beta(F) = F$. Therefore, $sc\text{-}Cl_\gamma(f^{-1}(F)) = f^{-1}(F)$. Hence $f^{-1}(F)$ is *sc*-γ-closed in (X, τ). □

Theorem 6.3. *In Theorem* 6.2, *the properties of sc-(γ, β)-continuity of f, (1) and (2) are equivalent if either the space (Y, σ) is sc-β-regular or the operation β is sc-open.*

Proof. It follows from the proof of Theorem 6.2 that the following implications: "*sc*-(γ, β)-continuity of f" $\Rightarrow$ (1) $\Rightarrow$ (2) are already known. So we only need to prove that (2) $\Rightarrow$ *sc*-(γ, β)-continuity of f, when the space (Y, σ) is *sc*-β-regular. Let $x \in X$ and let $V \in SCO(Y)$ such that $f(x) \in V$. Since (Y, σ) is an *sc*-β-regular space, by Theorem 3.9, $V \in SC_\beta O(Y)$. By Theorem 6.2 (1), $f^{-1}(V) \in SC_\gamma O(X)$ and $x \in f^{-1}(V)$. So there exists an *sc*-open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(V)$. This implies that $f(\gamma(U)) \subseteq V \subseteq \beta(V)$. Therefore, f is *sc*-(γ, β)-continuous.

Now, we prove the implication: (2) $\Rightarrow$ *sc*-(γ, β)-continuity of f when β is an *sc*-open operation. Let $x \in X$ and let $V \in SCO(Y)$ such that $f(x) \in V$. Since

β is an *sc*-open operation, then there exists $W \in SC_\beta O(Y)$ such that $f(x) \in W$ and $W \subseteq \beta(V)$. By Theorem 6.2 (2), $f^{-1}(W) \in SC_\gamma O(X)$ and $x \in f^{-1}(W)$. So there exists an *sc*-open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(W) \subseteq f^{-1}(\beta(V))$. Therefore $f(\gamma(U)) \subseteq \beta(V)$. Hence f is *sc*-(γ, β)-continuous. □

Definition 6.4. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

1. *sc*-(γ, β)-closed if the image of every *sc*-γ-closed set of X is *sc*-β-closed in Y.

2. *sc*-(id, β)-closed if the image of every *sc*-closed set of X is *sc*-β-closed in Y.

Theorem 6.5. *Suppose that a function $f : (X, \tau) \to (Y, \sigma)$ is both sc-(γ, β)-continuous and sc-(id, β)-closed, then:*

1. *For every sc-γ-g.closed set A of (X, τ), the image $f(A)$ is sc-β-g.closed in (Y, σ).*

2. *For every sc-β-g.closed set B of (Y, σ), the inverse set $f^{-1}(B)$ is sc-γ-g.closed in (X, τ).*

Proof. (1) Let A be an *sc*-γ-g.closed set and let V be any *sc*-β-open set in (Y, σ) such that $f(A) \subseteq V$. Since f is *sc*-(γ, β)-continuous function, by Theorem 6.2 (2), $f^{-1}(V)$ is *sc*-γ-open in (X, τ). By assumption A is *sc*-γ-g.closed and we have $A \subseteq f^{-1}(V)$, then $sc\text{-}Cl_\gamma(A) \subseteq f^{-1}(V)$, and so $f(sc\text{-}Cl_\gamma(A)) \subseteq V$. By Lemma 3.18 (1), $sc\text{-}Cl_\gamma(A)$ is *sc*-closed. Since f is *sc*-(id, β)-closed, then $f(sc\text{-}Cl_\gamma(A))$ is *sc*-β-closed in Y. Therefore, $sc\text{-}Cl_\beta(f(A)) \subseteq sc\text{-}Cl_\beta(f(sc\text{-}Cl_\gamma(A))) = f(sc\text{-}Cl_\gamma(A)) \subseteq V$. This implies that $f(A)$ is *sc*-β-g.closed in (Y, σ).

(2) Let U be an *sc*-γ-open set of (X, τ) such that $f^{-1}(B) \subseteq U$. Let $C = sc\text{-}Cl_\gamma(f^{-1}(B)) \cap (X \backslash U)$, by Lemma 3.18 (1), C is *sc*-closed set in (X, τ). Then $f(C)$ is *sc*-β-closed in (Y, σ) as f is *sc*-(id, β)-closed. Since f is an *sc*-(γ, β)-continuous function, by Theorem 6.2 (1), we have $f(C) = f(sc\text{-}Cl_\gamma(f^{-1}(B))) \cap f(X \backslash U) \subseteq sc\text{-}Cl_\beta(B) \cap f(X \backslash U) \subseteq sc\text{-}Cl_\beta(B) \cap (Y \backslash B) = sc\text{-}Cl_\beta(B) \backslash B$. It follows from Theorem 4.3 that $f(C) = \phi$, and so $C = \phi$. Therefore $sc\text{-}Cl_\gamma(f^{-1}(B)) \subseteq U$. Hence $f^{-1}(B)$ is *sc*-γ-g.closed in (X, τ). □

Theorem 6.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be an injective, sc-(γ, β)-continuous and sc-(id, β)-closed function. If (Y, σ) is sc-β-$T_{\frac{1}{2}}$, then (X, τ) is sc-γ-$T_{\frac{1}{2}}$.*

Proof. Let U be any sc-γ-g.closed set of (X,τ). By Theorem 6.5 (1), $f(U)$ is sc-β-g.closed in (Y,σ) as f is sc-(γ,β)-continuous and sc-(id,β)-closed. Since (Y,σ) is sc-β-$T_{\frac{1}{2}}$, then $f(U)$ is sc-β-closed in Y. Again, since f is sc-(γ,β)-continuous, by Theorem 6.2 (2), $f^{-1}(f(U))$ is sc-γ-closed in X. Then U is sc-γ-closed in X because f is injective. Therefore, (X,τ) is sc-γ-$T_{\frac{1}{2}}$. □

Theorem 6.7. *Let $f\colon (X,\tau)\to(Y,\sigma)$ be a surjective, sc-(γ,β)-continuous and sc-(id,β)-closed function. If (X,τ) is sc-γ-$T_{\frac{1}{2}}$, then (Y,σ) is sc-β-$T_{\frac{1}{2}}$.*

Proof. Let V be an sc-β-g.closed set of (Y,σ). By Theorem 6.5 (2), $f^{-1}(V)$ is sc-γ-g.closed in (X,τ) because f is sc-(γ,β)-continuous and sc-(id,β)-closed. Since (X,τ) is sc-γ-$T_{\frac{1}{2}}$, then, $f^{-1}(V)$ is sc-γ-closed set in X. Again, since f is sc-(id,β)-closed, then $f(f^{-1}(V))$ is sc-β-closed in Y. Therefore, V is sc-β-closed in Y as f is surjective. Thus (Y,σ) is sc-β-$T_{\frac{1}{2}}$. □

Theorem 6.8. *If a function $f\colon (X,\tau)\to(Y,\sigma)$ is injective sc-(γ,β)-continuous and the space (Y,σ) is sc-β-T_2, then the space (X,τ) is sc-γ-T_2.*

Proof. Let x_1 and x_2 be any two distinct points of (X,τ). Since f is an injective function and (Y,σ) is sc-β-T_2, then there exist two sc-open sets U_1 and U_2 in Y such that $f(x_1)\in U_1$, $f(x_2)\in U_2$ and $\beta(U_1)\cap\beta(U_2)=\phi$. Since f is sc-(γ,β)-continuous, there exist sc-open sets V_1 and V_2 in X such that $x_1\in V_1$, $x_2\in V_2$, $f(\gamma(V_1))\subseteq\beta(U_1)$ and $f(\gamma(V_2))\subseteq\beta(U_2)$. Therefore $\beta(U_1)\cap\beta(U_2)=\phi$. Hence (X,τ) is sc-γ-T_2. □

Theorem 6.9. *If a function $f\colon (X,\tau)\to(Y,\sigma)$ is injective sc-(γ,β)-continuous and the space (Y,σ) is sc-β-T_i, then (X,τ) is sc-γ-T_i for $i\in\{0,1\}$.*

Proof. Similar to Theorem 6.8. □

Definition 6.10. A function $f\colon (X,\tau)\to(Y,\sigma)$ is said to be sc-(γ,β)-homeomorphism if f is bijective, sc-(γ,β)-continuous and f^{-1} is sc-(β,γ)-continuous.

Theorem 6.11. *Assume that $f\colon (X,\tau)\to(Y,\sigma)$ is an sc-(γ,β)- homeomorphism function. If (X,τ) is sc-γ-$T_{\frac{1}{2}}$, then (Y,σ) is sc-β-$T_{\frac{1}{2}}$.*

Proof. Let $\{y\}$ be a singleton set of (Y,σ). Then there exists an element x of X such that $y=f(x)$. By hypothesis and Theorem 5.5, $\{x\}$ is sc-γ-closed or sc-γ-open set in X. By Theorem 6.2, $\{y\}$ is sc-β-closed or sc-β-open set. Therefore, by Theorem 5.5, (Y,σ) is sc-β-$T_{\frac{1}{2}}$. □

7. Functions with *sc*-β-Closed Graphs

For a function $f\colon (X,\tau) \to (Y,\sigma)$, the subset $\{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f, and is denoted by $G(f)$ [2]. In this section, we further investigate general operator approaches of closed graphs of functions. Let $\rho\colon (\tau \times \sigma)_g \to P(X \times Y)$ be an operation on $(\tau \times \sigma)_g$.

Definition 7.1. The graph $G(f)$ of $f\colon (X,\tau) \to (Y,\sigma)$ is called *sc*-β-closed if for each $(x,y) \in (X \times Y)\backslash G(f)$, there exist *sc*-open sets $U \subseteq X$ and $V \subseteq Y$ containing x and y, respectively, such that $(U \times \beta(V)) \cap G(f) = \phi$.

The proof of the following lemma follows directly from the above definition.

Lemma 7.2. *A function $f\colon (X,\tau) \to (Y,\sigma)$ has an sc-β-closed graph if and only if for every $(x,y) \in (X\times Y)\backslash G(f)$, there exist $U \in SCO(X)$ containing x and $V \in SCO(Y)$ containing y such that $f(U) \cap \beta(V) = \phi$.*

Definition 7.3. An operation $\rho\colon (\tau \times \sigma)_g \to P(X \times Y)$ is said to be *sc*-associated with γ and β if $\rho(U\times V) = \gamma(U)\times\beta(V)$ holds for every $U \in SCO(X)$ and $V \in SCO(Y)$.

Definition 7.4. The operation $\rho\colon (\tau \times \sigma)_g \to P(X \times Y)$ is said to be *sc*-regular with respect to γ and β if for each $(x,y) \in X \times Y$ and each *sc*-open set W containing (x,y), there exist *sc*-open sets U in X and V in Y such that $x \in U$, $y \in V$ and $\gamma(U) \times \beta(V) \subseteq \rho(W)$.

Theorem 7.5. *Let $\rho\colon (\tau\times\tau)_g \to P(X \times X)$ be an sc-associated operation with γ and γ. If $f\colon (X,\tau) \to (Y,\sigma)$ is an sc-(γ,β)-continuous function and (Y,σ) is an sc-β-T_2 space, then the set $A = \{(x,y) \in X \times X : f(x) = f(y)\}$ is sc-ρ-closed of $(X \times X, \tau \times \tau)$.*

Proof. We want to prove that $sc\text{-}Cl_\rho(A) \subseteq A$. Let $(x,y) \in (X \times X)\backslash A$. Since (Y,σ) is *sc*-β-T_2, then there exist two *sc*-open sets U and V in (Y,σ) such that $f(x) \in U$, $f(y) \in V$ and $\beta(U) \cap \beta(V) = \phi$. Moreover, for U and V there exist *sc*-open sets R and S in (X,τ) such that $x \in R$, $y \in S$ and $f(\gamma(R)) \subseteq \beta(U)$ and $f(\gamma(S)) \subseteq \beta(V)$ as f is *sc*-(γ,β)-continuous. Therefore we have $(x,y) \in \gamma(R) \times \gamma(S) = \rho(R \times S) \cap A = \phi$ because $R \times S \in (\tau \times \tau)_g$. This shows that $(x,y) \notin sc\text{-}Cl_\rho(A)$. □

Corollary 7.6. *Suppose $\rho\colon (\tau\times\tau)_g \to P(X\times X)$ is sc-associated operation with γ and γ, and it is sc-regular with γ and γ. A space (X,τ) is sc-γ-T_2 if and only if the diagonal set $\triangle = \{(x,x) : x \in X\}$ is sc-ρ-closed of $(X \times X, \tau \times \tau)$.*

Theorem 7.7. *Let $\rho\colon (\tau\times\sigma)_g \to P(X\times Y)$ be an sc-associated operation with γ and β. If $f\colon (X,\tau)\to(Y,\sigma)$ is sc-(γ,β)-continuous and (Y,σ) is sc-β-T_2, then the graph of f, $G(f)=\{(x,f(x))\in X\times Y\}$ is an sc-ρ-closed set of $(X\times Y,\tau\times\sigma)$.*

Proof. Similar to Theorem 7.5. □

Definition 7.8. Let (X,τ) be a topological space and let γ be an operation on $SCO(X)$. A subset S of X is said to be *sc*-γ-compact if for every *sc*-open cover $\{U_i, i\in\mathbb{N}\}$ of S, there exists a finite subfamily $\{U_1,U_2,...,U_n\}$ such that $S\subseteq\gamma(U_1)\cup\gamma(U_2)\cup...\cup\gamma(U_n)$.

Theorem 7.9. *Suppose that γ is sc-regular and $\rho\colon(\tau\times\sigma)_g\to P(X\times Y)$ is sc-regular with respect to γ and β. Let $f\colon(X,\tau)\to(Y,\sigma)$ be a function whose graph $G(f)$ is sc-ρ-closed in $(X\times Y,\tau\times\sigma)$. If a subset S is sc-β-compact in (Y,σ), then $f^{-1}(S)$ is sc-γ-closed in (X,τ).*

Proof. Suppose that $f^{-1}(S)$ is not *sc*-γ-closed, then there exist a point x such that $x\in sc\text{-}Cl_\gamma(f^{-1}(S))$ and $x\notin f^{-1}(S)$. Since $(x,s)\notin G(f)$ for each $s\in S$ and $sc\text{-}Cl_\rho(G(f))\subseteq G(f)$, there exists an *sc*-open set W of $(X\times Y,\tau\times\sigma)$ such that $(x,s)\in W$ and $\beta(W)\cap G(f)=\phi$. By *sc*-regularity of ρ, for each $s\in S$ we can take two *sc*-open sets $U(s)$ and $V(s)$ in (Y,σ) such that $x\in U(s)$, $s\in V(s)$ and $\gamma(U(s))\times\beta(V(s))\subseteq\rho(W)$. Then we have $f(\gamma(U(s)))\cap\beta(V(s))=\phi$. Since $\{V(s): s\in S\}$ is *sc*-open cover of S, then by *sc*-γ-compactness there exists a finite number $s_1,s_2,...,s_n\in S$ such that $S\subseteq\beta(V(s_1))\cup\beta(V(s_2))\cup...\cup\beta(V(s_n))$. By the *sc*-regularity of γ, there exist an *sc*-open set U such that $x\in U$, $\gamma(U)\subseteq\gamma(U(s_1))\cap\gamma(U(s_2))\cap...\cap\gamma(U(s_n))$. So, we have $\gamma(U)\cap f^{-1}(S)\subseteq U(s_i)\cap f^{-1}(\beta(V(s_i)))=\phi$. This shows that $x\notin sc\text{-}Cl_\gamma(f^{-1}(S))$. This is a contradiction. Thus, $f^{-1}(S)$ is *sc*-γ-closed. □

Theorem 7.10. *Suppose that the following condition hold:*

1. *$\gamma\colon SCO(X)\to P(X)$ is sc-open*
2. *$\beta\colon SCO(Y)\to P(Y)$ is sc-regular, and*
3. *$\rho\colon(\tau\times\sigma)_g\to P(X\times Y)$ is associated with γ and β, and ρ is sc-regular with respect to γ and β.*

Let $f\colon(X,\tau)\to(Y,\sigma)$ be a function whose graph $G(f)$ is sc-ρ-closed in $(X\times Y,\tau\times\sigma)$. If every cover of A by sc-γ-open sets of (X,τ) has finite sub cover, then $f(A)$ is sc-β-closed in (Y,σ).

Proof. Similar to Theorem 7.9. □

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