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DYNAMICS OF A PREY AND TWO PREDATORS SYSTEM WITH TIME DELAY

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Abstract: The aim of this paper is to study the prey-predator system with delay effects. Initially, the positive equilibrium point of the proposed system is derived and its local stability is discussed using Routh-Hurwitz criterion. A well suited Lyapunov function describes the global asymptotic stability of the system. To preserve the stability of the system without violating its properties the length of time delay is estimated and some important conclusions are made at the end.

AMS Subject Classification: 92D40, 34Dxx
Key Words: prey-predation, mutualism, time delay, global stability

1. Introduction

Prey-predator systems with time delays has gained importance in recent years and these delay models are playing vital role in the study of various ecological phenomenon comprising of interactions between and among the animal or plant species. The scientists Lotka [1] and Volterra [2] are the first to discuss these delay dependent models. In the subsequent period several ecologists and mathematicians like Kapur [3], Freedman [4], Cushing [5] have developed and

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experimented several time delay models. Dominance of biological systems and their dynamics are extensively studied by eminent researchers like Sreehari Rao [6] and Gopalaswamy [7]. Recent works on ecological systems of two and three species with and without time delays are carried by budding researchers like Paparao [8], Ranjith [9] and Vidyanath [10]. However, majority of the authors have studied only two species models with time delay and most of them have not estimated the size of the delay. Also treatises on the dynamical behavior of three species Lotka-Volterra models with time delays are not largely available.

In this work, there are two predators depending on a single prey for their existence, though the three species are survived by an alternative food resource. Moreover, the two predators mutually help each other. A particular case of the proposed model has been taken up and a time delay is introduced in the interaction between the prey and the first predator species. Quasi-linearizaton technique has been implemented to solve the integro-differential equations. Here we have concentrated primarily on the existence of interior equilibrium point i.e., the point where all the three species exist. Local stability as well as global stability satisfying the given parametric conditions are studied. The effect of delay kernels shows the population dynamics of all the three species. Matlab simulations are made at the end to analyze the dynamics of the system and some relevant conclusions are drawn at the end.

2. The model

The following system of integro-differential equations describes the proposed model.

$$\begin{aligned} \frac{dN_1}{dt} =& a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1 \int_{-\infty}^{T} k_2(t-s)N_2(s)ds - \alpha_{13}N_1N_3 \\ \frac{dN_2}{dt} =& a_2N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_2 \int_{-\infty}^{T} k_1(t-s)N_1(s)ds + \alpha_{23}N_2N_3 \\ \frac{dN_3}{dt} =& a_3N_3 - \alpha_{33}N_3^2 + \alpha_{31}N_3N_1 + \alpha_{32}N_3N_2 \end{aligned} \tag{1}$$

Here N_i's are the population densities, a_i's are the growth rates inherent in all the three species and α_{ii}'s are the death rates due to inter competitions for $i = 1, 2, 3$. α_{12} and α_{13} are the decrease rates of N_1 due to attacks by predators N_2 and N_3 respectively. α_{21} and α_{31} are the rates of increase of N_2 and N_3 due

to interaction with N_1 and a_{23}, α_{23} are the growth rates of the predators due to the mutualism nature.
Also $k_1(t-s)$, $k_2(t-s)$ represents the weight factors effecting the population sizes of N_1 & N_2 after a time gap $(t-s)\forall t \geq s$ and for $t-s=z$, $k_1(z)$, $k_2(z) \geq 0$ and normalization gives

$$\int_0^\infty k_1(z)dz = \int_0^\infty k_2(z)dz = 1 \tag{2}$$

Therefore, the system (2.1) is rewritten by implementing the delay kernel conditions:

$$\begin{aligned}
\frac{dN_1}{dt} &= a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1\int_0^\infty k_2(z)N_2(t-z)dz - \alpha_{13}N_1N_3 \\
\frac{dN_2}{dt} &= a_2N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_2\int_0^\infty k_1(z)N_1(t-z)dz + \alpha_{23}N_2N_3 \\
\frac{dN_3}{dt} &= a_3N_3 - \alpha_{33}N_3^2 + \alpha_{31}N_3N_1 + \alpha_{32}N_3N_2
\end{aligned} \tag{3}$$

3. The Positive Equilibrium State

The state in which all the three species exist

$$\overline{N}_1 = \frac{a_1(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) - a_2(\alpha_{12}\alpha_{33} - \alpha_{32}\alpha_{13}) + a_3(\alpha_{13}\alpha_{22} - \alpha_{12}\alpha_{23})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{22})}$$
$$\overline{N}_2 = \frac{a_1(\alpha_{21}\alpha_{33} + \alpha_{31}\alpha_{23}) + a_2(\alpha_{11}\alpha_{33} - \alpha_{31}\alpha_{13}) + a_3(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{22})}$$
$$\overline{N}_3 = \frac{a_1(\alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{22}) + a_2(\alpha_{11}\alpha_{32} - \alpha_{31}\alpha_{12}) + a_3(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{21}\alpha_{33} + \alpha_{31}\alpha_{23}) - \alpha_{13}(\alpha_{21}\alpha_{32} + \alpha_{31}\alpha_{22})}$$

4. The Local Stability of Positive Equilibrium State

Let $N = (N_1, N_2, N_3)^T = \overline{N} + U$
Where $U = (u_1, u_2, u_3)^T$ is the perturbation over the equilibrium state $\overline{N} =$

$(\overline{N}_1, \overline{N}_2, \overline{N}_3)^T$. The basic equations (2.3) are linearized to obtain the equations for the perturbed state which gives

$$\frac{dU}{dt} = AU, \tag{4}$$

where

$$A = \begin{bmatrix} a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}\overline{N}_2 - \alpha_{13}\overline{N}_3 & -\alpha_{12}\overline{N}_1 k_2^*(\lambda) & 13\overline{N}_1 \\ \alpha_{21}\overline{N}_2 k_1^*(\lambda) & a_2 - 2\alpha_{22}\overline{N}_2 + \alpha_{21}\overline{N}_1 + \alpha_{23}\overline{N}_3 & \alpha_{23}\overline{N}_2 \\ \alpha_{31}\overline{N}_3 & \alpha_{32}\overline{N}_3 & a_3 - 2\alpha_{33}\overline{N}_3 + \alpha_{31}\overline{N}_1 + \alpha_{32}\overline{N}_2 \end{bmatrix}$$

The characteristic equation for the system is

$$det[A - \lambda I] = 0 \tag{5}$$

The equilibrium state is stable, if the roots of the equation (4.2) are negative in case they are real or the roots have negative real parts in case they are complex. The linearized equations are

$$\begin{aligned} \frac{du_1}{dt} &= -\alpha_{11}\overline{N}_1 u_1 - \alpha_{12}\overline{N}_1 k_2^*(\lambda) u_2 - \alpha_{13}\overline{N}_1 u_3 \\ \frac{du_2}{dt} &= \alpha_{21}\overline{N}_2 k_1^*(\lambda) u_1 - \alpha_{22}\overline{N}_2 u_2 + \alpha_{23}\overline{N}_2 u_3 \\ \frac{du_3}{dt} &= \alpha_{31}\overline{N}_3 u_1 + \alpha_{32}\overline{N}_3 u_2 - \alpha_{33}\overline{N}_3 u_3 \end{aligned} \tag{6}$$

With the characteristic equation:

$$\begin{aligned} &\lambda^3 + \left(\alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2 + \alpha_{33}\overline{N}_3\right)\lambda^2 \\ &+ \left[(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}k_1^*(\lambda)k_2^*(\lambda))\,\overline{N}_1\overline{N}_2 + (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\,\overline{N}_2\overline{N}_3\right. \\ &\left.+ (\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31})\,\overline{N}_1\overline{N}_3\right]\lambda \\ &+ [\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}\alpha_{21}\alpha_{33}k_1^*(\lambda)k_2^*(\lambda) + \alpha_{12}\alpha_{31}\alpha_{23}k_2^*(\lambda) \\ &+ \alpha_{13}\alpha_{21}\alpha_{32}k_1^*(\lambda) \\ &+ \alpha_{13}\alpha_{31}\alpha_{22}]\overline{N}_1\overline{N}_2\overline{N}_3 = 0 \end{aligned}$$

Let

$$h_1 = \alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2 + \alpha_{33}\overline{N}_3$$
$$h_2 = (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}k_1^*(\lambda)k_2^*(\lambda))\,\overline{N}_1\overline{N}_2 + (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\,\overline{N}_2\overline{N}_3 + (\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31})\,\overline{N}_1\overline{N}_3$$
$$h_3 = [\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}\alpha_{21}\alpha_{33}k_1^*(\lambda)k_2^*(\lambda) + \alpha_{12}\alpha_{31}\alpha_{23}k_2^*(\lambda) + \alpha_{13}\alpha_{21}\alpha_{32}k_1^*(\lambda) + \alpha_{13}\alpha_{31}\alpha_{22}]\overline{N}_1\overline{N}_2\overline{N}_3$$

By Routh-Hurwitz criteria, the system is stable if, $D_1 = h_1 > 0$,
$D_2 = (h_1h_2 - h_3) > 0$ and $D_3 = h_3(h_1h_2 - h_3) > 0$.
Clearly $h_1 > 0$ and also by certain algebraic deductions we have,
$D_2 = h_1h_2 - h_3 > 0$.
Therefore, the positive equilibrium point is locally asymptotically stable.

5. Global Stability

The following Lyapunov function is chosen for the interior equilibrium point:

$$V(\overline{N}_1, \overline{N}_2, \overline{N}_3) = \sum_{i=1}^{3} N_i - \overline{N}_i \ln(\frac{N_i}{\overline{N}_i}) + \frac{1}{2}\alpha_{12}\int_0^\infty k_2^*(z)\int_{t-z}^{t}[N_2 - \overline{N}_2]^2 du dz + \frac{1}{2}\alpha_{21}\int_0^\infty k_1^*(z)\int_{t-z}^{t}[N_1 - \overline{N}_1]^2 du dz \quad (7)$$

Then calculate $\frac{dV}{dt}$ which is as follows

$$\frac{dV}{dt} = \sum_{i=1}^{3}(N_i - \overline{N}_i)\frac{1}{N_i}\frac{dN_i}{dt} + \frac{1}{2}\alpha_{12}\int_0^\infty k_2^*(z)[N_2 - \overline{N}_2]^2 dz - \frac{1}{2}\alpha_{12}\int_0^\infty k_2^*(z)[N_2(t-z) - \overline{N}_2]^2 dz + \frac{1}{2}\alpha_{21}\int_0^\infty k_1^*(z)[N_1 - \overline{N}_1]^2 dz - \frac{1}{2}\alpha_{21}\int_0^\infty k_1^*(z)[N_1(t-z) - \overline{N}_1]^2 dz$$

Using the model equations (2.1) we get,

$$\begin{aligned}\frac{dV}{dt} =& \left[N_1 - \overline{N}_1\right] \left[a_1 - \alpha_{11}N_1 - \alpha_{12} \int\limits_0^\infty k_2^*(z)N_2(t-z)dz - \alpha_{13}N_3\right] \\ &+ \left[N_2 - \overline{N}_2\right] \left[a_2 - \alpha_{22}N_2 + \alpha_{21} \int\limits_0^\infty k_1^*(z)N_1(t-z)dz + \alpha_{23}N_3\right] \\ &+ \left[N_3 - \overline{N}_3\right] \left[a_3 - \alpha_{33}N_3 + \alpha_{31}N_1 + \alpha_{32}N_2\right] \\ &+ \frac{1}{2}\alpha_{12}\left[N_2 - \overline{N}_2\right]^2 - \frac{1}{2}\alpha_{12} \int\limits_0^\infty k_2^*(z)\left[N_2(t-z) - \overline{N}_2\right]^2 dz \\ &+ \frac{1}{2}\alpha_{21}\left[N_1 - \overline{N}_1\right]^2 - \frac{1}{2}\alpha_{21} \int\limits_0^\infty k_1^*(z)\left[N_1(t-z) - \overline{N}_1\right]^2 dz\end{aligned}$$

Choosing,

$$\begin{aligned}a_1 =& \alpha_{11}\overline{N}_1 + \alpha_{12} \int\limits_0^\infty k_2^*(z)N_2(t-z) + \alpha_{13}\overline{N}_3, \\ a_2 =& \alpha_{22}\overline{N}_2 - \alpha_{21} \int\limits_0^\infty k_1^*(z)N_1(t-z)dz - \alpha_{23}\overline{N}_3, \\ a_3 =& \alpha_{33}\overline{N}_3 - \alpha_{31}\overline{N}_1 - \alpha_{32}\overline{N}_2\end{aligned}$$

and using the inequality, $ab \leq \frac{a^2+b^2}{2}$, we get

$$\int_0^\infty k_2(z)\left[N_2(t-z) - \overline{N}_2\right]^2 dz \leq \int_0^\infty k_2(z)dz = 1$$
$$\int_0^\infty k_1(z)\left[N_1(t-z) - \overline{N}_1\right]^2 dz \leq \int_0^\infty k_1(z)dz = 1$$

We have

$$\begin{aligned}\frac{dv}{dt} \leq& -\left[\alpha_{11} - \frac{1}{2}(\alpha_{31} + \alpha_{21} - \alpha_{13})\right]\left[N_1 - \overline{N}_1\right]^2 \\ &- \left[\alpha_{22} - \frac{1}{2}(\alpha_{23} + \alpha_{32} - \alpha_{12})\right]\left[N_2 - \overline{N}_2\right]^2 \\ &- \left[\alpha_{33} - \frac{1}{2}(\alpha_{31} + \alpha_{23} + \alpha_{32} - \alpha_{13})\right]\left[N_3 - \overline{N}_3\right]^2 - \frac{1}{2}(\alpha_{12} + \alpha_{21}) \\ &\leq 0\end{aligned} \tag{8}$$

This proves that the system is asymptotically globally stable at the interior equilibrium point.

6. Numerical Simulation

Case (A): The following set of parametric values are considered for analyzing the dynamics of the system with the initial strengths of the species as 100, 75, 75 respectively.

a1=100; α_{11}=0.01; α_{12}=0.5; α_{13}=0.25; a2=5; α_{21}=0.15; α_{22}=0.75; α_{23}=0.5; a3=2; α_{31}=0.25; α_{32}=0.5; α_{33}=0.75;

The Fig:1 below shows the time series analysis and phase space trajectories of the three species without delay kernels. The strengths of all the three species grow drastically and the system shows asymptotic stability and converges to the equilibrium point (148.8, 128.2, 137.8).

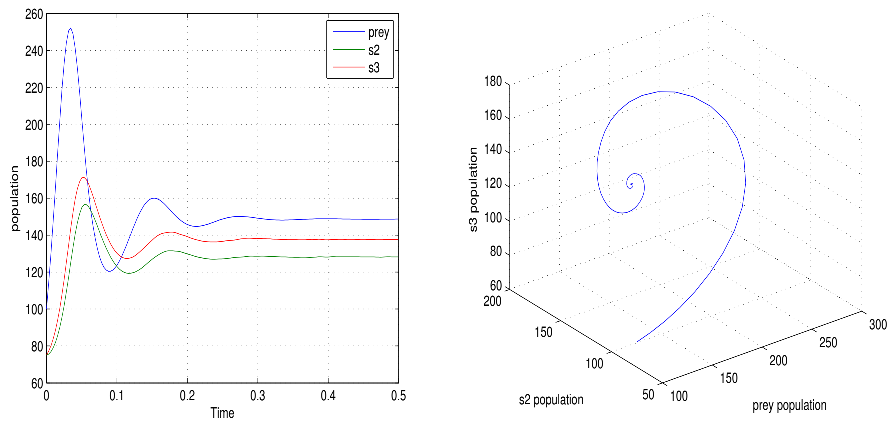

Figure 1

Case (B): Keeping all the parameters as in case (A) fixed and varying the delay kernels the graphs are traced for some specific values of α and β in the interval 0 and 1. With the help of time series graphs and phase space portraits the stability nature of the proposed system can be analyzed. The following table shows the description in detail.

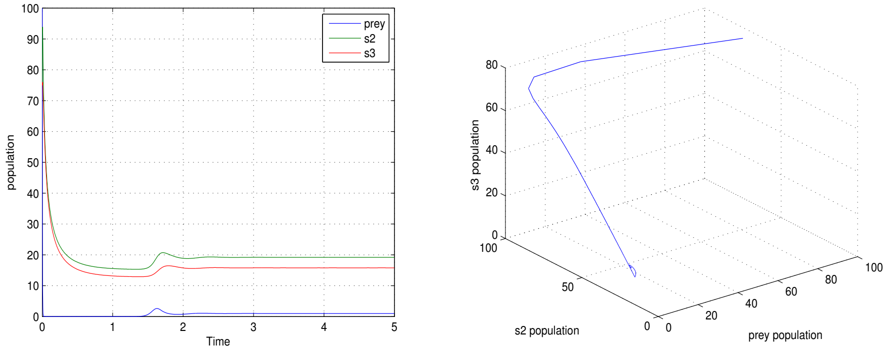

Figure 2

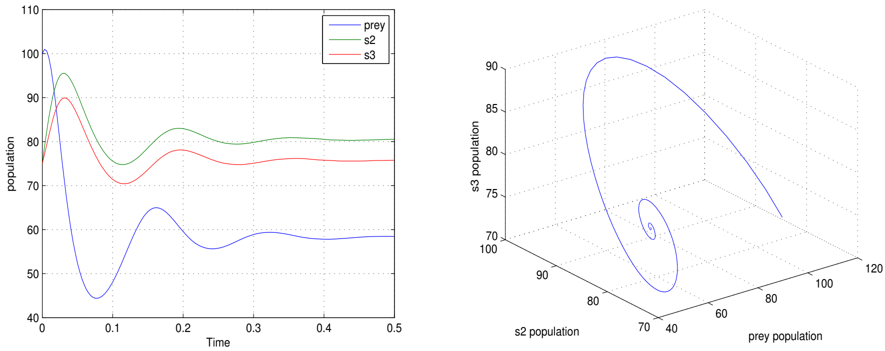

Figure 3

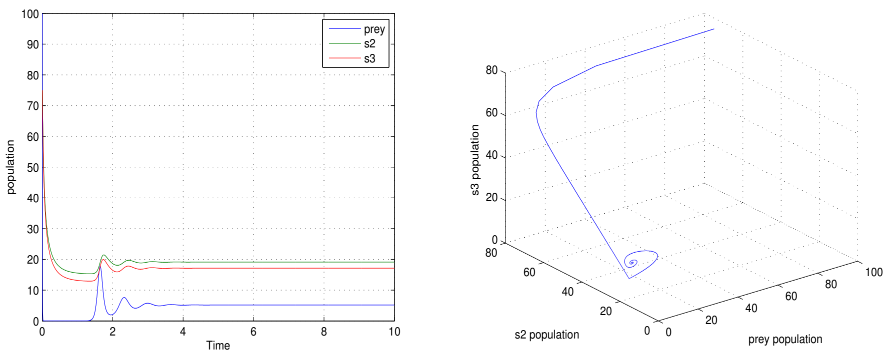

Figure 4

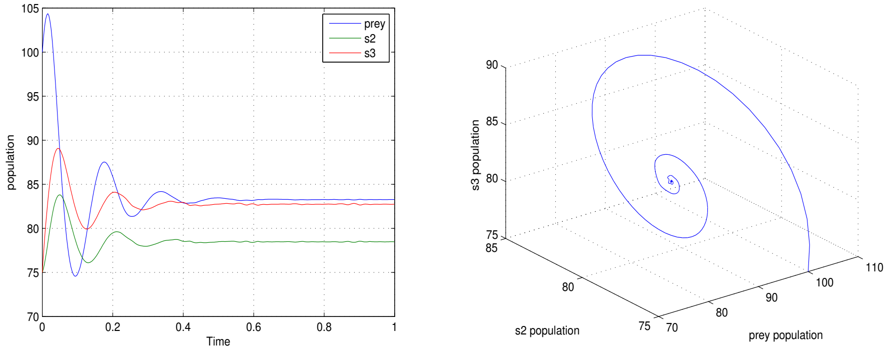

Figure 5

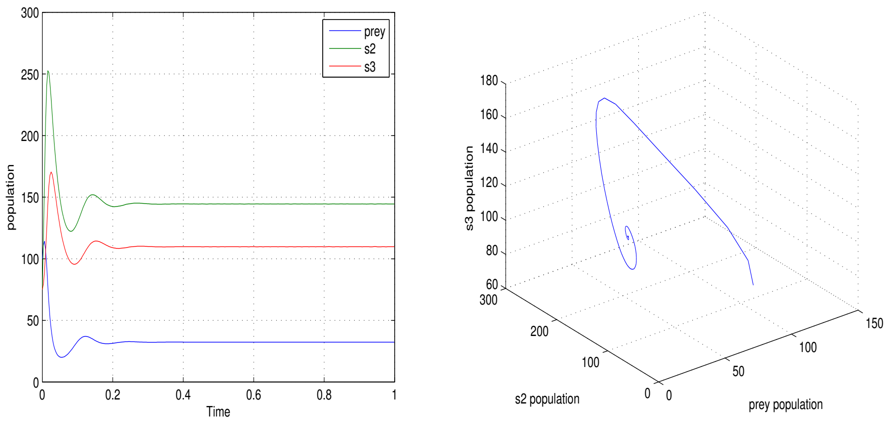

Figure 6

Delay Kernels	Figure Number	Nature of the System
$\alpha = 0.1$, $\beta = 0.1$	2	Delay kernels effect the population growth. The prey species population is almost extinct and also it effects the growth of predator species also. The system is able to survive in this case and reaches the equilibrium state (1.103, 19.19, 15.7).
$\alpha = 0.5$, $\beta = 0.5$	3	With equal delay kernels the system is globally asymptotically stable and stretches to the equilibrium point (58.42, 80.41, 75.83).
$\alpha = 1$, $\beta = 0.1$	4	With the effect of only , the prey population reduces initially and within a short period of time it gains its strength and leads to a stable system and (5.201, 19.28, 17.39) is the equilibrium state.
$\alpha = 1$, $\beta = 0.5$	5	Increase in the delay effect further shows the variations in all the three species and the system is asymptotically globally stable and stabilizes at the equilibrium (83.35,78.52,82.82).
$\alpha = 0.1$, $\beta = 1$	6	The effect of delay kernel is clearly shown here. The predator populations are increasing on a fast note and hence lead to decrease in the prey population. The models sustains and lands at the equilibrium point (32.55, 145.1, 110.1).

7. Concluding Remarks

The present study deals with a three species eco-system comprising of a prey and two predators. The proposed system shows that the predators are mutually helping each other in addition to the food resource available in the form of a single prey. And hence, as a particular case study we have imposed continuous delay in the interaction between the prey and the first predator. The system shows a global asymptotic stability at the interior equilibrium point which was derived using suitable Lyapunov function. The effect of delay kernels on the populations of all the three species are distinguished and clearly understood. Analyzing the graphs obtained by varying the delay kernels we observe that the model sustains and remains stable under the conditions that $\alpha > 0.01$ and $\beta > 0.1$. Also we noted that mutualism between the predators is playing a crucial role here and is effecting the population of the prey species which forces us to assume a large value as the natural growth rate for the prey species.

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