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# THE RELATION AMONG EULER'S PHI FUNCTION, TAU FUNCTION, AND SIGMA FUNCTION

Apisit Pakapongpun
Department of Mathematics
Faculty of Science
Burapha University
Chonburi, 20131, THAILAND

**Abstract:** The purpose of this study was to investigate the relationship among Euler's phi function, tau function and sigma function. Using knowledge of number theory, the relationship of these functions and provide the proofs was found.

**AMS Subject Classification:** 11A25
**Key Words:** Euler's Phi function, Tau function, Sigma function

## 1. Introduction

**Definition 1.1.** An arithmetic function is a function that is defined for all positive integers.

**Definition 1.2.** An arithmetic function $f$ is called multiplicative if and only if $f(mn) = f(m)f(n)$ where $m$ and $n$ are relatively prime positive integers.

**Definition 1.3.** Euler's phi function denoted by $\phi$ is defined by setting $\phi(n)$ equal to the number of the integer less than or equal to $n$ that are relatively prime to $n$.

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**Theorem 1.4.** *The following statements are true*

1. $\phi$ *is a multiplicative function.*

2. *Let* $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ *is the prime power factorization into distinct primes of the positive integer* $n$. *Then*

$$\phi(n) = n \prod_{i=1}^{k} (1 - \frac{1}{p_i}.)$$

**Definition 1.5.** Tau function or the number of divisors function, denoted by $\tau$ is defined by setting $\tau(n)$ equal to the number of positive divisors of $n$.

**Theorem 1.6.** *The following statements are true*

1. $\tau$ *is a multiplicative function.*

2. *Let* $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ *is the prime power factorization into distinct primes of the positive integer* $n$. *Then*

$$\tau(n) = \prod_{i=1}^{k} (a_i + 1).$$

**Definition 1.7.** Sigma function or the sum of divisors function, denoted by $\sigma$ is defined by setting $\sigma(n)$ equal to the sum of all the positive divisors of $n$.

**Theorem 1.8.** *The following statements are true*

1. $\sigma$ *is a multiplicative function.*

2. *Let* $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ *is the prime power factorization into distinct primes of the positive integer* $n$. *Then*

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

Definition 1.1, 1.2, 1.3, 1.5, 1.7 and theorem 1.4, 1.6, 1.8 are taken from [1], [2].

## 2. Main Results

**Lemma 2.1.** *Let $a$ is a non negative integer and $p$ is a positive prime number. If $n = p^a$, then*

$$\sum_{1\leq k\leq n,(k,n)=1} k = \frac{n\cdot\phi(n)}{2}.$$

*Proof.* Let $A$ be the sum of positive integers less than or equal to $p^a$ and $B$ be the sum of positive integers $r$ less than or equal to $p^a$ and $(r, p^a) \neq 1$.
So

$$\begin{aligned}\sum_{1\leq k\leq n,(k,n)=1} k = A - B &= \frac{p^a(p^a+1)}{2} - \frac{p(p^{a-1})(p^{a-1}+1)}{2} \\ &= \frac{p^a}{2}(p^a + 1 + p^{a-1} - 1) \\ &= \frac{p^a}{2}(p^a - p^{a-1}) \\ &= \frac{n\cdot\phi(n)}{2}.\end{aligned}$$

□

**Theorem 2.2.** *If $k$ and $n$ are positive integers, then*

$$\sum_{1\leq k\leq n,(k,n)=1} k = \frac{n\cdot\phi(n)}{2}.$$

*Proof.* Case I, if $n$ is a prime. Then

$$\sum_{1\leq k\leq n,(k,n)=1} k = 1+2+3+\cdots+(n-1) = \frac{n(n-1)}{2} = \frac{n\cdot\phi(n)}{2}.$$

Case II, if $n$ is not a prime and $n = p_1^{a_1}p_2^{a_2}\cdots p_m^{a_m}$ such that $p_1, p_2, \cdots, p_m$ are distinct primes and $a_1, a_2, \cdots, a_m$ are positive integers, then

$$\begin{aligned}\sum_{1\leq k\leq n,(k,n)=1} k =& 2^{m-1}\Big(\sum_{1\leq k_1\leq p_1^{a_1},(k_1,p_1^{a_1})=1} k_1\Big)\Big(\sum_{1\leq k_2\leq p_2^{a_2},(k_2,p_2^{a_2})=1} k_2\Big) \\ &\cdots\Big(\sum_{1\leq k_m\leq p_m^{a_m},(k_m,p_m^{a_m})=1} k_m\Big) \\ =& 2^{m-1}\Big(\frac{p_1^{a_1}}{2}(p_1^{a_1} - p_1^{a_1-1})\Big)\Big(\frac{p_2^{a_2}}{2}(p_2^{a_2} - p_2^{a_2-1})\Big)\end{aligned}$$

$$\begin{aligned}
&\cdots\big(\frac{p_m^{a_m}}{2}(p_m^{a_m} - p_m^{a_m-1})\big)\\
=&\frac{2^{m-1}}{2^m}(p_1^{a_1}p_2^{a_2}\cdots p_m^{a_m})(p_1^{a_1} - p_1^{a_1-1})(p_2^{a_2} - p_2^{a_2-1})\\
&\cdots(p_m^{a_m} - p_m^{a_m-1})\\
=&\frac{1}{2}(n)\phi(n).
\end{aligned}$$

□

**Theorem 2.3.** *Let $a$ and $n$ are positive integers. If $n = 2^a$ and $2^{a+1} - 1$ is a prime number, then*

$$\sigma(\sigma(n)) = 2n = 2^{\tau(n)}.$$

*Proof.* Since

$$\sigma(n) = \sigma(2^a) = 2^{a+1} - 1,$$

hence

$$\sigma(\sigma(n)) = \sigma(2^{a+1} - 1) = 2^{a+1} = 2n$$

and

$$\tau(n) = \tau(2^a) = a + 1.$$

Therefore

$$\sigma(\sigma(n)) = 2n = 2 \cdot 2^a = 2^{a+1} = 2^{\tau(n)}.$$

□

**Theorem 2.4.** *If $p$ is a prime number, then*

$$\sigma(p) = \phi(p) + \tau(p).$$

| p | $\phi(p)$ | $\tau(p)$ | $\sigma(p)$ | $\phi(p) + \tau(p)$ |
|---|---|---|---|---|
| 2 | 1 | 2 | 3 | 3 |
| 3 | 2 | 2 | 4 | 4 |
| 5 | 4 | 2 | 6 | 6 |
| 7 | 6 | 2 | 8 | 8 |
| 11 | 10 | 2 | 12 | 12 |

Table 1: some $p$ of $\sigma(p) = \phi(p) + \tau(p)$.

*Proof.* Let $p$ is a prime number. Then

$$\sigma(p) = p + 1, \ \phi(p) = p - 1 \text{ and } \tau(p) = 2,$$

hence

$$\phi(p) + \tau(p) = p - 1 + 2 = p + 1 = \sigma(p).$$

Therefore

$$\sigma(p) = \phi(p) + \tau(p) \quad \text{where} \quad p \text{ is a prime number.}$$

□

**Theorem 2.5.** *If $n = 2p$ and $p$ is an odd prime number, then*

$$\sigma(n) = n + \phi(n) + \tau(n).$$

| $p$ | $n = 2p$ | $\phi(n)$ | $\tau(n)$ | $\sigma(n)$ | $n + \phi(n) + \tau(n)$ |
|---|---|---|---|---|---|
| 3 | 6 | 2 | 4 | 12 | 12 |
| 5 | 10 | 4 | 4 | 18 | 18 |
| 7 | 14 | 6 | 4 | 24 | 24 |
| 11 | 22 | 10 | 4 | 36 | 36 |
| 13 | 26 | 12 | 4 | 42 | 42 |

Table 2: some $n$ of $\sigma(n) = n + \phi(n) + \tau(n)$.

*Proof.* Let $n = 2p$ and $p$ is an odd prime number. Then

$$\sigma(n) = \sigma(2p) = \sigma(2)\sigma(p) = 3(p + 1) = 3p + 3,$$

$$\phi(n) = \phi(2p) = \phi(2)\phi(p) = p - 1$$

and

$$\tau(n) = \tau(2p) = \tau(2)\tau(p) = 2(2) = 4,$$

hence

$$n + \phi(n) + \tau(n) = 3p + 3.$$

Therefore

$$\sigma(n) = n + \phi(n) + \tau(n)$$

where $n = 2p$ and $p$ is an odd prime number. □

**Theorem 2.6.** *If $n = 3p$ and $p$ is a prime number not equal to 3, then*

$$\sigma(n) = 2(\phi(n) + \tau(n)).$$

| $p$ | $n = 3p$ | $\phi(n)$ | $\tau(n)$ | $\sigma(n)$ | $2(\phi(n) + \tau(n))$ |
|---|---|---|---|---|---|
| 2 | 6 | 2 | 4 | 12 | 12 |
| 5 | 15 | 8 | 4 | 24 | 24 |
| 7 | 21 | 12 | 4 | 32 | 32 |
| 11 | 33 | 20 | 4 | 48 | 48 |
| 13 | 39 | 24 | 4 | 56 | 56 |

Table 3: some $n$ of $\sigma(n) = 2(\phi(n) + \tau(n))$.

*Proof.* Let $n = 3p$ and $p$ is a prime number not equal to 3. Then

$$\sigma(n) = \sigma(3p) = \sigma(3)\sigma(p) = 4(p+1) = 4p + 4,$$

$$\phi(n) = \phi(3p) = \phi(3)\phi(p) = 2p - 2$$

and

$$\tau(n) = \tau(3p) = \tau(3)\tau(p) = 2(2) = 4,$$

hence

$$2(\phi(n) + \tau(n)) = 2(2p - 2 + 4) = 4p + 4.$$

Therefore

$$\sigma(n) = 2(\phi(n) + \tau(n))$$

where $n = 3p$ and $p$ is a prime number not equal to 3. □

**Theorem 2.7.** *If $n = 2^k$ and $k$ is a non negative integer, then*

$$\sigma(n) = 2n - 1.$$

*Proof.* Let $n = 2^k$ and $k$ is a non negative integer. Then

$$\sigma(n) = \sigma(2^k) = \frac{2^{k+1} - 1}{2 - 1} = 2n - 1.$$

Therefore

$$\sigma(n) = 2n - 1.$$

□

| $k$ | $n = 2^k$ | $\sigma(n)$ | $2n - 1$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 2 | 3 | 3 |
| 2 | 4 | 7 | 7 |
| 3 | 8 | 15 | 15 |
| 4 | 16 | 31 | 31 |

Table 4: some $n$ of $\sigma(n) = 2n - 1$.

**Theorem 2.8.** *If $n = 1$ or $n = p$ is a prime number, then*

$$\sigma(n) + \phi(n) = 2n.$$

| $n$ | 2n | $\phi(n)$ | $\sigma(n)$ | $\phi(n) + \sigma$ |
|---|---|---|---|---|
| 1 | 2 | 1 | 1 | 2 |
| 2 | 4 | 1 | 3 | 4 |
| 3 | 6 | 2 | 4 | 6 |
| 5 | 10 | 4 | 6 | 10 |
| 7 | 14 | 6 | 8 | 14 |

Table 5: some $n$ of $\sigma(n) + \phi(n) = 2n$.

*Proof.* Where $n = 1$ is obvious. If $n = p$ is a prime number, then

$$\sigma(n) = \sigma(p) = p + 1$$

and

$$\phi(n) = \phi(p) = p - 1.$$

Therefore

$$\sigma(n) + \phi(n) = 2p = 2n$$

where $n = 1$ or $n = p$ is a prime number. □

**Theorem 2.9.** *If $p$ is a prime number, then*

$$\phi(p) = p - (\tau(p))^2 + 3.$$

| $p$ | $\tau(p)$ | $(\tau(p))^2$ | $\phi(p)$ | $p-(\tau(p))^2+3$ |
|---|---|---|---|---|
| 2 | 2 | 4 | 1 | 1 |
| 3 | 2 | 4 | 2 | 2 |
| 5 | 2 | 4 | 4 | 4 |
| 7 | 2 | 4 | 6 | 6 |
| 11 | 2 | 4 | 10 | 10 |

Table 6: some $p$ of $\phi(p) = p - (\tau(p))^2 + 3$.

*Proof.* Let $p$ is a prime number. Then

$$\phi(p) = p - 1 \text{ and } \tau(p) = 2,$$

hence

$$p - (\tau(p))^2 + 3 = p - 2^2 + 3 = p - 1.$$

Therefore

$$\phi(p) = p - (\tau(p))^2 + 3 \text{ where } p \text{ is a prime number.}$$

□

**Theorem 2.10.** *If $n = 2p$ and $p$ is an odd prime number, then*

$$\phi(n) = \frac{n}{2} - 1.$$

| $p$ | $n=2p$ | $\phi(n)$ | $\frac{n}{2}-1$ |
|---|---|---|---|
| 3 | 6 | 2 | 2 |
| 5 | 10 | 4 | 4 |
| 7 | 14 | 6 | 6 |
| 11 | 22 | 10 | 10 |
| 13 | 26 | 12 | 12 |

Table 7: some $n$ of $\phi(n) = \frac{n}{2} - 1$.

*Proof.* Let $n = 2p$ and $p$ is an odd prime number. Then

$$\phi(n) = \phi(2p) = p - 1 = \frac{2p}{2} - 1 = \frac{n}{2} - 1.$$

Therefore

$$\phi(n) = \frac{n}{2} - 1$$

where $n = 2p$ and $p$ is an odd prime number. □

**Theorem 2.11.** *If $n = 4p$ and $p$ is an odd prime number, then*

$$\phi(n) = \frac{n}{2} - 2.$$

| $p$ | $n = 4p$ | $\phi(n)$ | $\frac{n}{2} - 2$ |
|---|---|---|---|
| 3 | 12 | 4 | 4 |
| 5 | 20 | 8 | 8 |
| 7 | 28 | 12 | 12 |
| 11 | 44 | 20 | 20 |
| 13 | 52 | 24 | 24 |

Table 8: some $n$ of $\phi(n) = \frac{n}{2} - 2$.

*Proof.* Let $n = 4p$ and $p$ is an odd prime number. Then

$$\phi(n) = \phi(4p) = 2(p-1) = 2p - 2 = \frac{4p}{2} - 2 = \frac{n}{2} - 2.$$

Therefore

$$\phi(n) = \frac{n}{2} - 2$$

where $n = 4p$ and $p$ is an odd prime number. □

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