International Journal of Pure and Applied Mathematics
Volume 118 No. 3 2018, 751-765
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v118i3.21

DYNAMICAL ANALYSIS OF FRACTIONAL-ORDER PREDATOR-PREY MODEL WITH PREY REFUGE AND ADDITIONAL FOOD FOR PREDATOR

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Abstract: In this work, a fractional-order predator-prey model with prey refuge and additional food for predator is presented. First, the non-negativity, boundedness, existence and uniqueness of the solution for the model are discussed. We also determine the equilibrium points and their existence conditions as well as their stability behaviour, both locally and globally. Some numerical simulations are performed to illustrate analytical results.

AMS Subject Classification: 34A08, 34D23, 92D25
Key Words: fractional-order model, prey refuge, additional food, global stability, Lyapunov function, Grünwald-Letnikov method

1. Introduction

In recent decades, the dynamics of predator-prey interaction has gained an important role in ecology. The first predator-prey interaction model was introduced by Lotka-Volterra [1]. One of ecological factors that affects the predator-prey interaction is refuge, reffering to a places or situations where predation risk is somehow reduced [2]. The refuge has been shown to have a stabilizing effect on the dynamical system as well as to reduce the risk of prey extinction due to predation [2-4].

Received: 2017-12-07
Revised: 2018-02-05
Published: April 18, 2018

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Under limited resources, predator may deviate from their usual diet and exhibit some inclination toward an additional food to prolong their survival [5]. Many research have studied the effect of additional food for predator in predator-prey interaction. It is shown that handling time (time required to eat or consume additional food) plays a key role in determining the eventual state of the ecosystem [6]. Sahoo [7] has shown that the predator population never goes extinct in the presence of additional food. According to Prasad et al. [8], the availability of additional food can be used as a control to either eliminate any of the species or to have coexistence. The coexistence of predator and prey depends on the magnitude of the additional food characteristic and harvesting intensity [9]. Furthermore, Ulfa et al. have shown that additional food for predators may destabilize the extinction of prey point and at the same time stabilize the coexistence point, see [10].

Recently, Ghosh et al. [11] have investigated predator-prey model with prey refuge and additional food for predators as follows

$$\begin{aligned} \frac{dx}{dt} &= x\left[\left(1-\frac{x}{\gamma}\right)-\frac{(1-c')y}{1+\theta\xi+x}\right] \\ \frac{dy}{dt} &= y\left[\frac{\beta[(1-c')x+\xi]}{1+\theta\xi+x}-\delta\right], \end{aligned} \tag{1}$$

where $x, y, \gamma, \theta, \xi, \beta, \delta$ are all positive parameters and $0 < c' < 1$. Parameter θ and ξ represent the quality and quantity of additional food, respectively. It is clearly seen that the growth rate of population depends only on the current state. In fact, the growth rate of population is also dependent on the history of variable or its memory. To include the dependency of its memory, many researchers have implemented the fractional-order derivative instead of the first order derivative [12-15]. One of commonly used fractional-order derivative is the Caputo fractional derivative which is defined as follows

Definition 1 (see [17])**.** The Caputo fractional differential operator of order α, with $m-1<\alpha<m, m\in N$, is defined by

$$\begin{aligned} {}_a^C D_t^\alpha u(x) &= J^{m-\alpha}u^{(m)}(x) \\ &= \frac{1}{\Gamma(m-\alpha)}\int_a^t (x-t)^{m-\alpha-1}u^{(m)}(t)dt, \end{aligned}$$

where Γ is a Gamma function.

Based on Definition 1, we see that the α-order fractional derivative is not only defined locally at time t, but also depends on the total effects of the commonly used m-order integer derivative on the interval $[0, t]$. Thus, it can be used to illustrate the variation of a system in which the change rate depends on the whole past state, which is called the "memory effect" in a visualized manner [18].

Motivated by the above considerations, we reconsider system (2) to include the memory effect. By assuming that the growth rate of population does not only depend on the present conditions but also on all previous condition, we replace the integer-order derivative into a fractional-order Caputo derivative:

$$\begin{aligned} {}_{t_0}^{c}D_t^{\alpha}x(t) &= x\left[\left(1-\frac{x}{\gamma}\right)-\frac{(1-c')y}{1+\theta\xi+x}\right] \\ {}_{t_0}^{c}D_t^{\alpha}y(t) &= y\left[\frac{\beta[(1-c')x+\xi]}{1+\theta\xi+x}-\delta\right], \end{aligned} \tag{2}$$

where $\alpha \in (0, 1)$. We determine the points of equilibrium, their existence, and behavior stability analysis, both locally and globally. Then, a numerical simulation is performed using Grünwald-Letnikov approximation method to represent analytical results.

2. Nonnegativity, Boundedness, Existence and Uniqueness of the Solution

Let $\mathbb{R}_+$ be the set of all non-negative real numbers and $\Omega_+ = \{(x, y) \in \Omega : x \in \mathbb{R}_+ \text{and } y \in \mathbb{R}_+\}$. In this section, we will guarantee non-negativity and boundedness of the solutions of system (2). To this aim, we need the following Lemmas.

Lemma 2 (see [19]). *Let $0 < \alpha \leq 1$, $f(t) \in C[a, b]$ and ${}_{t_0}^{c}D_t^{\alpha} \in C[a, b]$. Then, the following conditions hold:*

1. *If ${}_{t_0}^{c}D_t^{\alpha}f(t) \geq 0, \forall t \in (a, b)$, then $f(t)$ is a non-decreasing function for each $t \in [a, b]$.*

2. *If ${}_{t_0}^{c}D_t^{\alpha}f(t) \leq 0, \forall t \in (a, b)$, then $f(t)$ is a non-increasing function for each $t \in [a, b]$*

Lemma 3 (see [15]). *Let $u(t)$ be continuous function on $[t_0, +\infty]$ and satisfying*

$$\begin{cases} {}_{t_0}^{c}D_t^{\alpha}u(t) \leq -\lambda u(t) + \mu, \\ u(t_0) = u_{t_0}, \end{cases} \tag{3}$$

where $0 < \alpha < 1, (\lambda, \mu) \in \mathbb{R}^2$, $\lambda \neq 0$ *and* $t_0 \geq 0$ *is the initial time. Then*

$$u(t) \leq \left(u_{t_0} - \frac{\mu}{\lambda}\right) E_\alpha[-\lambda(t-t_0)^\alpha] + \frac{\mu}{\lambda}.$$

Lemma 4 (see [16]). *We consider a fractional differential system*

$${}_{t_0}^{c}D_t^\alpha u(t) = f(t,u), t > t_0$$

with initial condition $u(t_0) = u_0$, *where* $0 < \alpha < 1, f : [t_0, \infty) \times \Omega \to \mathbb{R}^n, \Omega \subseteq \mathbb{R}^n$. *If* $f(t,u)$ *satisfies the Lipschitz condition with respect to* u, *then there exist a unique solution of the above system on* $[t_0, \infty) \times \Omega$.

Theorem 5. *All solutions of system* (2) *with initial condition* $x(0) \geq 0$ *and* $y(0) \geq 0$ *are non-negative i.e.,* $(x(t), y(t)) \in \Omega_+$.

Proof. We will prove that $x(t) \geq 0$ for all $t \geq 0$. Suppose that statement is not true, then there is a constant $t_1 > 0$ such that

$$\begin{cases} x(t) > 0, & 0 \leq t < t_1 \\ x(t_1) = 0, & \\ x(t_1^+) < 0. & \end{cases} \tag{4}$$

Substituting the second equation of system (4) into equation (2) gives

$${}_{t_0}^{c}D_{t_1}^\alpha x(t_1)|_{x(t_1)=0} \;\; = 0. \tag{5}$$

If ${}_{t_0}^{c}D_{t_1}^\alpha x(t_1) = 0$, then based on Lemma 2, it is found $x(t_1^+) = 0$ which contradicts with the fact that $x(t_1^+) < 0$. It can be concluded that $x(t) \geq 0$ for all $t \geq 0$. Using the same argument, it is shown that $y(t) \geq 0$ for all $t \geq 0$. □

Theorem 6. *All solution of system* (2) *i.e.,* $(x(t), y(t)) \in \Omega_+$ *is uniformly bounded.*

Proof. We first define a function $W(t) = x(t) + \frac{1}{\beta}y(t)$ such that

$$\begin{aligned} {}_{t_0}^{c}D_t^\alpha W(t) + \left(\frac{\beta\xi}{\sigma(x)} - \delta\right) W(t) =& x - \frac{1}{\gamma}x^2 + \left(\frac{\beta\xi}{\sigma(x)} - \delta\right) x \\ =& -\frac{1}{\gamma}\left(x - \frac{\gamma(1 + \frac{\beta\xi}{\sigma(x)} - \delta)}{2}\right)^2 \\ &+ \frac{\gamma\left(1 + \frac{\beta\xi}{\sigma(x)} - \delta\right)^2}{4}, \end{aligned} \tag{6}$$

where $\sigma(x) = 1 + \theta\xi + x$. From equation (6), we get

$$ {}_{t_0}^{c}D_t^{\alpha} W(t) \leq -\left(\frac{\beta\xi}{\sigma(x)} - \delta\right) W(t) + \frac{\gamma\left(1 + \frac{\beta\xi}{\sigma(x)} - \delta\right)^2}{4}. \tag{7} $$

Based on Lemma 3 and equation (7), we have

$$ \begin{aligned} W(t) \leq & \left(W(t_0) - \frac{\gamma\left(1 + \frac{\beta\xi}{\sigma(x)} - \delta\right)^2}{4\left(\frac{\beta\xi}{\sigma(x)} - \delta\right)} \right) E_\alpha \left[\left(\frac{\beta\xi}{\sigma(x)} - \delta \right) (t - t_0)^\alpha \right] \\ & + \frac{\gamma\left(1 + \frac{\beta\xi}{\sigma(x)} - \delta\right)^2}{4\left(\frac{\beta\xi}{\sigma(x)} - \delta\right)} \\ \leq & \left(W(t_0) - \frac{\gamma\left(1 + \frac{\beta\xi}{\sigma(x)} - \delta\right)^2}{4\left(\frac{\beta\xi}{\sigma(x)} - \delta\right)} \right) E_\alpha \left[\left(\frac{\beta\xi}{\sigma(x)} - \delta \right) (t - t_0)^\alpha \right] \\ & + \frac{\gamma(1 + \beta\xi - \delta)^2}{4(\beta\xi - \delta)} \end{aligned} \tag{8} $$

For $t \to \infty$, we have that

$$ W(t) \to \frac{\gamma(1 + \beta\xi - \delta)^2}{4(\beta\xi - \delta)}. \tag{9} $$

Hence, it is proved that all solution of system (2) is uniformly bounded:

$$ \Gamma = \left\{ (x, y) \in \Omega_+ |\ x + \frac{y}{\beta} \leq \frac{\gamma(1 + \beta\xi - \delta)^2}{4(\beta\xi - \delta)} + \epsilon, \epsilon > 0 \right\}. \tag{10} $$

□

Theorem 7. *We consider system (2) with initial value $x(t_0) = x_0$ and $y(t_0) = y_0$ in the region $[t_0, \infty) \times \Omega$ where $\Omega = \left\{(x, y) \in \mathbb{R}^2 : max\{|x|, |y|\} \leq M\right\}$ for sufficiently large M. This initial value problem has a unique solution.*

Proof. Let $X = (x, y)$, $\bar{X} = (\bar{x}, \bar{y})$ and consider a mapping $H(X) = (H_1(X), H_2(X))$, where

$$ \begin{aligned} H_1(X) = &\ x\left(1 - \frac{x}{\gamma}\right) - \frac{(1 - c')xy}{\sigma(x)}, \\ H_2(X) = &\ \frac{\beta[(1 - c')x + \xi]y}{\sigma(x)} - \delta y. \end{aligned} \tag{11} $$

For any $X, \bar{X} \in \Omega$, we get

$$\begin{aligned}
\|H(X) - H(\bar{X})\| &= |H_1(X) - H_1(\bar{X})| + |H_2(X) - H_2(\bar{X})| \\
&= \left| x\left(1 - \frac{x}{\gamma}\right) - \frac{(1-c')xy}{\sigma(x)} - \bar{x}\left(1 - \frac{\bar{x}}{\gamma}\right) \right. \\
&\quad \left. + \frac{(1-c')\bar{x}\bar{y}}{\sigma(\bar{x})} \right| + \left| \frac{\beta[(1-c')x + \xi]y}{\sigma(x)} - \delta y \right. \\
&\quad \left. - \frac{\beta[(1-c')\bar{x} + \xi]\bar{y}}{\sigma(\bar{x})} + \delta \bar{y} \right| \\
&= \left| (x - \bar{x}) - \frac{1}{\delta}(x^2 - \bar{x}^2) \right. \\
&\quad \left. - \left[\frac{\bar{\sigma}(1-c')xy - \sigma(1-c')\bar{x}\bar{y}}{\sigma(x).\sigma(\bar{x})} \right] \right| \\
&\quad + \left| \frac{\beta\sigma(\bar{x})((1-c')x + \xi)y}{\sigma(x).\sigma(\bar{x})} \right. \\
&\quad \left. - \frac{\beta\sigma(x)((1-c')\bar{x} + \xi)\bar{y}}{\sigma(x).\sigma(\bar{x})} - \delta(y - \bar{y}) \right| \\
&\leq L_1|x - \bar{x}| + L_2|y - \bar{y}| \\
&\leq L\left[(x - \bar{x}) + (y - \bar{y})\right] = L\|X - \bar{X}\|, \qquad (12)
\end{aligned}$$

where $L_1 = 1 + 2M + \beta\xi M + (1+\beta)(1+\theta\xi)(1-c')M, L_2 = ((1+\beta)(1+\theta\xi)(1-c')M + (1-c')M^2(1+\beta) + \beta\xi(1+\theta\xi+M) + \delta)$ and $L = max\{L_1, L_2\}$. Clearly that $H(X)$ satisfies the Lipschitz condition with respect to X. Hence, using Lemma 4, there exists a unique solution of system (2) with initial value $x(t_0) = x_0$ and $y(t_0) = y_0$. □

3. Stability of Equilibrium Points

The equilibrium points of system (2) can be determined by setting

$$ {}^{c}_{t_0}D^{\alpha}_{t} x(t) = {}^{c}_{t_0}D^{\alpha}_{t} y(t) = 0. $$

In this way, we have three equilibrium points i.e., $E_0 = (0,0), E_1 = (\gamma, 0)$, $E_2 = (x^*, y^*)$ where

$$ x^* = \frac{\delta + (\delta\theta - \beta)\xi}{\beta(1-c') - \delta} $$

and

$$y^* = \left(1 - \frac{x^*}{\gamma}\right)\left(\frac{1+\theta\xi+x^*}{(1-c')}\right).$$

E_2 exist if $x^* < \gamma$, $\beta < \frac{\delta}{(1-c')}$ and $\delta < \frac{\beta\xi}{1+\theta\xi}$. We remark that all equilibrium points and their existence conditions are exactly the same as those of integer-order model, see [11]. To analyze the local stability of equilibrium points, we need the following Theorem.

Theorem 8 (see [20]). *Consider the following autonomous nonlinear fractional-order system*

$$D_t^\alpha \vec{u}(t) = \vec{f}(\vec{u}(t));\ \vec{u}(0) = \vec{u}_0;\ 0 < \alpha < 1.$$

The equilibrium points of the above system are solution to the equation $\vec{f}(\vec{u}(t)) = 0$. *An equilibrium point* $\vec{u}^*$ *is locally asymptotically stable if all eigenvalues* (λ_i) *of the Jacobian matrix* $J = \frac{\partial \vec{f}}{\partial \vec{u}}$ *evaluated at equilibrium* $\vec{u}^*$ *satisfy* $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

The local stability of each equilibrium point of system (2) is shown in Theorem 9.

Theorem 9. *1. The equilibrium point* E_0 *is unstable.*

2. *The equilibrium point* E_1 *is asymptotically stable if* $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+\gamma} < \delta$.

3. *If the equilibrium point* E_2 *exists, then* E_2 *is asymptotically stable.*

Proof. 1. The Jacobian matrix of system (2) at E_0 has eigenvalues $\lambda_1 = 1 > 0$ and $\lambda_2 = \frac{\beta\xi}{1+\theta\xi} - \delta$, which implies $|arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$. Hence, E_0 is unstable.

2. The eigenvalues of Jacobian matrix of system (2) at E_1 are $\lambda_1 = -1$ and $\lambda_2 = \frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+\gamma} - \delta$. Hence $|arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ and $|arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$ when $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+\gamma} < \delta$. This completes the proof.

3. The characteristic equations of $J(E_2)$ is

$$P(\lambda) = \lambda^2 - a_1\lambda + a_2 = 0 \tag{13}$$

where

$$a_1 = \left[\frac{x^*}{\gamma}\left(\frac{\gamma - x^*}{1+\theta\xi+x^*} - 1\right)\right],$$

$$a_2 = \frac{\beta(1-c')[(1-c')(1+\theta\xi)-\xi]x^*y^*}{(1+\theta\xi+x^*)^3}.$$

The eigenvalues corresponding to equation (13) are

$$\begin{aligned}\lambda_1 &= \frac{1}{2}\left(a_1+\sqrt{D}\right),\\ \lambda_2 &= \frac{1}{2}\left(a_1-\sqrt{D}\right),\end{aligned}$$

where $D = (a_1)^2 - 4a_2$. Hence, E_2 is asymptotically stable if one of the following conditions are satisfied

(a) If $D > 0, a_1 < 0, a_2 > 0$ and $\sqrt{D} < |a_1|$, then $\lambda_1, \lambda_2 < 0$. Obviously, $|arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$.

(b) If $D = 0$ and $a_1 < 0$, then $\lambda_2, \lambda_3 < 0$, such that $|arg(\lambda_{1,2})| > \frac{\alpha\pi}{2}$.

(c) If $D < 0$, then $|arg(\lambda_{1,2})| = \left|\frac{\sqrt{D}}{a_1}\right| = \pi > \frac{\alpha\pi}{2}$.

□

To prove the global stability, we need following Lemma.

Lemma 10 (see [21])**.** *Let $x(t) \in \mathbb{R}_+$ be continuous and derivable function. Then, for any time instant $t \geq t_0$ and $\alpha \in (0,1)$*

$$ {}_{t_0}^{c}D_t^{\alpha}\left[x(t) - x^* - x^*\ln\frac{x(t)}{x^*}\right] \leq \left(1 - \frac{x^*}{x(t)}\right) {}_{t_0}^{c}D_t^{\alpha}x(t),\ x^* \in \mathbb{R}.$$

Lemma 11 (see [22])**.** *Suppose D is a bounded closed set. Every solution of $D^{\alpha}x(t) = f(x)$ starts from a point in D and remains in D for all time. If $\exists V(x) : D \to \mathbb{R}$ with continuous first partial derivatives satisfies following condition:*

$$D^{\alpha}V|_{D^{\alpha}x(t)=f(x)} \leq 0. \tag{14}$$

If $E = \{x|D^{\alpha}V|_{D^{\alpha}x(t)=f(x)} = 0\}$ and M be the largest invariant set of E. Then every solution $x(t)$ originating in D tends to M as $t \to \infty$. Particularly, when $M = 0$, then $x \to 0$, $t \to \infty$.

Now, we study the global stability of the equilibrium points $E_1 = (\gamma, 0)$ and $E_2(x^*, y^*)$ as follows

Theorem 12. *1. The equilibrium point E_1 is globally asymptotically stable if $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+x} \leq \delta$.*

2. *The equilibrium point E_2 is globally asymptotically stable in the region* $\Omega = \left\{(x,y) : \frac{y}{y^*} > \frac{x}{x^*} > 1\right\}$.

Proof. 1. Define a Lyapunov function

$$U(x,y) = \left(x - \gamma - \gamma \ln \frac{x}{\gamma} + \frac{y}{\beta}\right).$$

Calculating the α-order derivative of $U(x,y)$ and applying Lemma 10 give

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\alpha} U(x,y) &\leq \frac{x-\gamma}{x}\, {}_{t_0}^{c}D_t^{\alpha} x(t) + \frac{1}{\beta}\, {}_{t_0}^{c}D_t^{\alpha} y(t), \\
&= \frac{x-\gamma}{x}\left[x\left(1 - \frac{x}{\gamma}\right) - \frac{(1-c')xy}{1+\theta\xi+x}\right] \\
&+ \frac{1}{\beta}\left[\frac{\beta[(1-c')x+\xi]y}{1+\theta\xi+x} - \delta y\right], \\
&= -\frac{1}{\gamma}(x-\gamma)^2 + \left[\frac{(1-c')\gamma+\xi}{1+\theta\xi+x} - \frac{\delta}{\beta}\right] y. \qquad (15)
\end{aligned}$$

If $\frac{\beta[(1-c')\gamma+\xi]}{1+\theta\xi+x} \leq \delta$, then ${}_{t_0}^{c}D_t^{\alpha} U(x,y) \leq 0$ for all$(x,y) \in \mathbb{R}_+^2$ and ${}_{t_0}^{c}D_t^{\alpha} U(x,y) = 0$ at $E_1 = (\gamma, 0)$. Therefore, the only invariant set on which ${}_{t_0}^{c}D_t^{\alpha} U(x,y) = 0$ is singleton $\{E_1\}$. Based on Lemma 11, the proof is completed.

2. If (x^*, y^*) is an equilibrium point of system (2), then we have

$$x^*\left[\left(1 - \frac{x^*}{\gamma}\right) - \frac{(1-c')y^*}{1+\theta\xi+x^*}\right] = 0,$$

$$y^*\left[\frac{\beta[(1-c')x^*+\xi]}{1+\theta\xi+x^*} - \delta\right] = 0.$$

Define a Lyapunov function

$$V(x,y) = x - x^* - x^* \ln \frac{x}{x^*} + \frac{1}{\beta}\left(y - y^* - y^* \ln \frac{y}{y^*}\right). \qquad (16)$$

By taking the α-order derivative of $V(x,y)$ and using Lemma 10, we get

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\alpha} V(x,y) &\leq \left(\frac{x-x^*}{x}\right) {}_{t_0}^{c}D_t^{\alpha} x(t) + \frac{1}{\beta}\left(\frac{y-y^*}{y}\right) {}_{t_0}^{c}D_t^{\alpha} y(t), \\
&= \left(\frac{x-x^*}{x}\right)\left[x\left(\left(1 - \frac{x}{\gamma}\right) - \frac{(1-c')y}{1+\theta\xi+x}\right)\right]
\end{aligned}$$

$$+\frac{1}{\beta}\left(\frac{y-y^*}{y}\right)\left[y\left(\frac{\beta[(1-c')x+\xi]}{1+\theta\xi+x}-\delta\right)\right],$$
$$=-\frac{(x-x^*)^2}{\gamma}-\frac{\xi(x-x^*)(y-y^*)}{(1+\theta\xi+x)(1+\theta\xi+x^*)}$$
$$-(1-c')\left(\frac{(x^*y-xy^*)(x-x^*)}{(1+\theta\xi+x)(1+\theta\xi+x^*)}\right). \quad (17)$$

If $\Omega=\{(x,y):\frac{y}{y^*}>\frac{x}{x^*}>1\}$, then ${}^{c}_{t_0}D^{\alpha}_{t}V(x,y)\leq 0$ and since ${}^{c}_{t_0}D^{\alpha}_{t}V(x,y)=0$, we have $(x,y)=(x^*,y^*)$. Based on (17), ${}^{c}_{t_0}D^{\alpha}_{t}V(x,y)=0$ implies $x=x^*$. Substituting $x=x^*$ into the first equation of system (2) gives

$$0={}^{c}_{t_0}D^{\alpha}_{t}x^*(t)=x^*\left[\left(1-\frac{x^*}{\gamma}\right)-\frac{(1-c')y^*}{1+\theta\xi+x^*}\right]. \quad (18)$$

From (16) and (18), we obtain $y=y^*$. Hence, ${}^{c}_{t_0}D^{\alpha}_{t}V(x,y)=0$ which implies $(x,y)=(x^*,y^*)$. We conclude that the invariant sets on which ${}^{c}_{t_0}D^{\alpha}_{t}V(x,y)=0$ is singleton $\{E_2\}$. Hence, by using Lemma 11, this completes the proof.

□

4. Numerical Method and Simulations

In this section, we will show some numerical simulations to illustrate our analytical results. We here implement the Grünwald-Letnikov approximation method [23] which is obtained from the nonstandard explicit scheme for system (2) as follows

$$\begin{aligned}
x_{n+1} &= h^{\alpha}f(x_n,y_n)-\sum_{j=1}^{n+1}(-1)^{j-1}\binom{\alpha}{j}x_{n+1-j}+\frac{(n+1)^{-\alpha}}{\Gamma(1-\alpha)}x_0\\
&= \frac{(n+1)^{-\alpha}}{\Gamma(1-\alpha)}x_0+h^{\alpha}x_n\left[\left(1-\frac{x_n}{\gamma}\right)-\frac{(1-c')y_n}{1+\theta\xi+x_n}\right]\\
&\quad -\sum_{j=1}^{n+1}(-1)^{j-1}\binom{\alpha}{j}x_{n+1-j},\\
y_{n+1} &= h^{\alpha}g(x_n,y_n)-\sum_{j=1}^{n+1}(-1)^{j-1}\binom{\alpha}{j}y_{n+1-j}+\frac{(n+1)^{-\alpha}}{\Gamma(1-\alpha)}y_0,\\
&= \frac{(n+1)^{-\alpha}}{\Gamma(1-\alpha)}y_0+h^{\alpha}y_n\left[\frac{\beta[(1-c')x_n+\xi]}{1+\theta\xi+x_n}-\delta\right]
\end{aligned}$$

$$-\sum_{j=1}^{n+1}(-1)^{j-1}\binom{\alpha}{j}y_{n+1-j}.$$

Here we implement the nonstandard scheme because in the finite difference method version, it can maintain the dynamical properties compared to the standard version, see e.g. [24], [25] and [26].

The parameters used in our first simulation are $\gamma = 2.6, c' = 0.6, \theta = 0.6, \xi = 0.2, \beta = 0.21$ and $\delta = 0.08$. Here, there are two equilibrium points, namely $E_0(0,0)$ and $E_1(2.6,0)$. It is found that $\frac{\beta[(1-c')\gamma+\xi)]}{1+\theta\xi+\gamma} = 0.07 < \delta = 0.08$. According to Theorem 9, E_1 is asymptotically stable while E_0 is unstable.

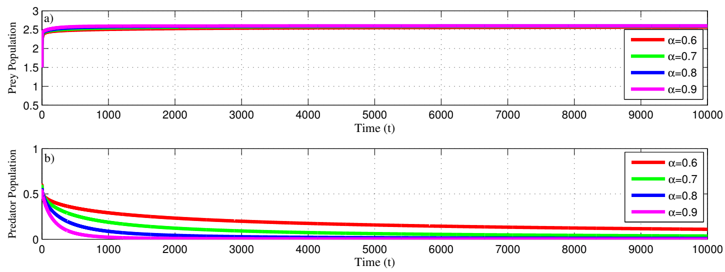

Figure 1: Numerical solution of system (2) with $\gamma = 2.6, c' = 0.6, \theta = 0.6, \xi = 0.2, \beta = 0.21$, $\delta = 0.08$ and $\alpha = \{0.6, 0.7, 0.8, 0.9\}$: (a) Prey population $x(t)$, (b) Predator population $y(t)$.

Figure 1 shows that solutions with initial value $(1.5, 0.5)$ and some different values of α are convergent to $E_1(2.6,0)$. This shows that the stability of equilibrium point E_1 does not depend on α. However, the order derivative (α) affects the convergence speed of equilibrium point E_1. This simulation indicates the situation where prey population will survive and predator population will extinct.

The second simulation illustrates the stability of E_2. We set the same parameters as before, except $\gamma = 4, \beta = 0.15$ and $c' = 0.16$. In this case, we have three equilibrium points: $E_0 = (0,0), E_1 = (4,0)$ and $E_2 = (1.295, 1.944)$. According to Theorem 9, E_0 and E_1 are unstable. Moreover, our analysis shows that E_2 is asymptotically stable if $\alpha < \alpha^*$ and it is unstable whenever $\alpha > \alpha^*$, where $\alpha^* = 0.90$. In Figure 2 we plot our numerical solutions for $\alpha = 0.88$ and $\alpha = 0.91$. It is found that for $\alpha = 0.88 < \alpha^*$, the solution converges to $E_2 = (1.295, 1.944)$; showing that E_2 is a stable equilibrium point. However, if we take $\alpha = 0.91 > \alpha^*$, then the solution is not convergent to any point. The solution is indeed oscillating which shows that there exists a stable limit cycle.

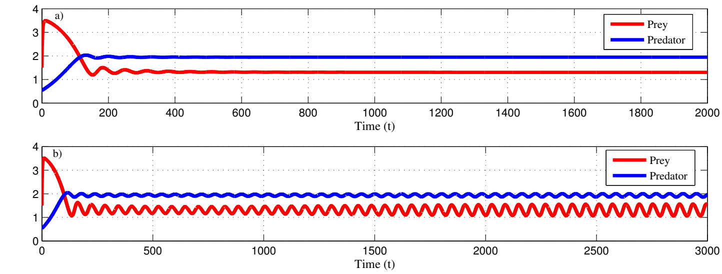

Figure 2: Numerical solution of system (2) with $\gamma = 4, c' = 0.16, \theta = 0.6, \xi = 0.2, \beta = 0.15,\ \delta = 0.08$ and (a) $\alpha = 0.88$, (b) $\alpha = 0.91$.

Finally, we perform simulation using parameters: $\gamma = 1.5, c' = 0.22, \theta = 0.6, \xi = 0.1$ and $\delta = 0.08$. Using these parameters, system (2) has three equilibrium points: $E_0 = (0, 0),\ E_1 = (1.5, 0)$ and $E_2 = (0.852, 1.058)$. Theorem 9 says that E_0 and E_1 are unstable. However, the equilibrium point $E_2(0.852, 1.058)$ is asymptotically stable for $0 < \alpha < 1$. Such behavior is clearly seen in Figure 3 where all solutions are convergent to E_2.

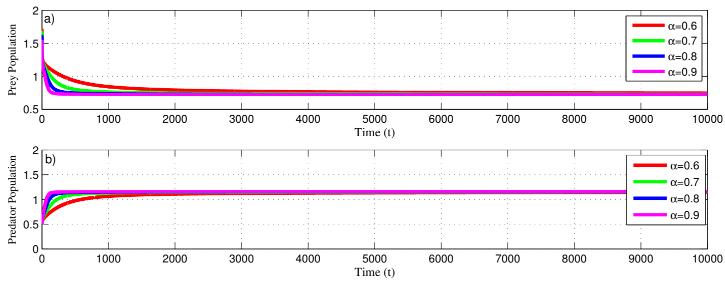

Figure 3: Numerical solution of system (2) with $\gamma = 1.5, c' = 0.22, \theta = 0.6, \xi = 0.1, \beta = 0.21,\ \delta = 0.08$ and $\alpha = \{0.6, 0.7, 0.8, 0.9\}$: (a) Prey population $x(t)$ and (b) Predator population $y(t)$.

5. Conclusion

In this work, a fractional-order predator-prey model with prey refuge and additional food has been discussed. It is shown that the model has non-negative and bounded solutions. The model has three type of equilibrium points, i.e. the extinction of both prey and predator population, the extinction of predator point and the co-existence point. The extinction of both prey and predator population is found to be unstable, while two other equilibrium points are conditionally stable. The conditions for global stability of the extinction of predator point or the co-existence point has also been derived. Our numerical simulations are in accordance with our analytical results.

6. Acknowledgments

This research is financially supported by the Directorate of Research and Community Service, The Directorate General of Strengthening Research and Development, the Ministry of Research, Technology and Higher Education, (Brawijaya University); Indonesia: No. 063/SP2H/LT/DRPM/IV/2017 and Contract No. 460.18/UN10.C10/PN/2017 dated April 18, 2107.

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