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THE MAPS WHICH PRESERVING COAPPROXIMATION IN BANACH LATTICES

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Abstract: The aim of this paper is to introduce the concept of coapproximation preserving operators on Banach lattices with a strong unit. We show that every lattice isomorphism is an coapproximation preserving operator.

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1. Introduction and preliminaries

A vector lattice space (or a Riesz space) is an ordered vector space X with the additional property that for each pair of vectors $x, y \in X$, the $\sup\{x, y\}$ and the $\inf\{x, y\}$ both exist in X. As usual, $\sup\{x, y\}$ is denoted by $x \vee y$ and $\inf\{x, y\}$ by $x \wedge y$.

If X is an ordered vector space, then the set $X^+ = \{x \in X \ : \ x \geq 0\}$ is called a positive cone of X, and its members are called the positive elements of X. For any vector x in an ordered vector space define $x^+ := x \vee 0,\ x^- := x \wedge 0$ and $|x| := x \vee (-x)$. The element x^+ is called the positive part, x^- is called the

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negative part, and $|x|$ is called the absolute value of x. An element $\mathbf{1} \in X$ is called a strong unit if for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $x \leq \lambda \mathbf{1}$. Then for each $x \in X$ there exists $0 < \lambda \in \mathbb{R}$ such that $|x| \leq \lambda \mathbf{1}$. Using $\mathbf{1}$ we can define a norm on X by

$$\|x\| = \inf\{\lambda > 0 \ : \ |x| \leq \lambda \mathbf{1}\}. \tag{1}$$

So we have $|x| \leq \|x\|\mathbf{1}$ for all $x \in X$.

Recall that a norm $\|.\|$ on a vector lattice space is said to be a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A vector lattice space equipped with a lattice norm is known as a normed vector lattice space. If a normed vector lattice space is also norm complete, then it is referred to as a Banach lattice. It is well known that X equipped with the norm (1) is a Banach lattice which is called a Banach lattice with strong unit $\mathbf{1}$.

Let W be a non-empty subset of a normed linear space X. An element $w_0 \in W$ is called a best coapproximation to $x \in X$ from W if for every $w \in W$,

$$\|w - w_0\| \leq \|x - w\|.$$

The set of all elements of best coapproximation to $x \in X$ from W is denoted by $R_W(x)$. If each $x \in X$ has at least one best coapproximation $w_0 \in W$, then W is called a coproximinal subset of X. If for each $x \in X$ there exists a unique best coapproximation $w_0 \in W$, then W is called a coChebyshev subset of X.

In [2] the authors described the maps which preserve approximation by elements of subspaces of normed linear spaces. They proved that every isometry operator preserves best approximation by subspaces of normed linear spaces. In [3] the authors introduced some conditions for maps which preserve best approximation by downward subsets of Banach lattices. In this paper we want to extend the results of Moddares and Dehghani in [3] to obtain some conditions for maps which preserve best coapproximation by downward subsets of Banach lattices.

2. Main results

In this section we shall obtain characterization of preserving coapproximation maps on Banach lattices.

Definition 1. A linear mapping from a Banach lattice to a Banach lattice is positive if it carries positive vectors to positive vectors.

Definition 2. An operator $T : X \longrightarrow Y$ between two vector lattice spaces is said to be a lattice homomorphism whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in X$.

Note that every lattice homomorphism is necessarily a positive operator and it is equivalent to

$$|T(x)| = T(|x|) \tag{2}$$

for all $x \in X$. Also, it is important to note that the range of a lattice homomorphism is a lattice subspace. A lattice homomorphism which is in addition one-to-one is referred to as a lattice isomorphism.(for more details see [1], page 93).

The next theorem which has been proved in [1], page 94, described necessary and sufficient conditions for an operator between to vector lattice spaces, that is an lattice isomorphism.

Theorem 3. *Assume that an operator $T : X \longrightarrow Y$ between two vector lattice spaces is one-to-one and onto. Then T is a lattice isomorphism if and only if T and T^{-1} are both positive operators.*

Definition 4. A subset W of X is called downward if

$$(w \in W, x \leq w) \Longrightarrow x \in W.$$

For example if $f : X \longrightarrow \mathbb{R}$ is an increasing function, then its lower level sets $S_c(f) = \{x \in X : f(x) \leq c\}$ for all $c \in \mathbb{R}$ are downward.

The following lemma characterizes the maps which preserve downwardness of downward subsets of vector lattices.

Lemma 5. *[3] (1) Let X and Y be two vector lattices and $T : X \longrightarrow Y$ be an injective positive operator, such that T^{-1} is a positive operator. Then W is a downward subset of X if and only if $T(W)$ is a downward subset of Y.*

(2) If $T : X \longrightarrow X$ is a positive operator and $f : X \longrightarrow \mathbb{R}$ is an increasing function, then $S_c(f \circ T) = \{x \in X : f \circ T(x) \leq c\}$ for all $c \in \mathbb{R}$ are downward.

The following proposition that is a consequence of theorem 3 and relations (1), (2), plays a key role in proving our results.

Proposition 6. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively and $T : X \longrightarrow Y$ be an injective positive operator such that T^{-1} is positive and $T(\mathbf{1}_X) = \mathbf{1}_Y$. Then T is a norm isometry, i.e. $\|T(x)\| = \|x\|$ for all $x \in X$.*

Definition 7. Let X and Y be Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. A linear operator $T : X \longrightarrow Y$ is called an coapproximation preserving operator if for all downward sets W in X and all $x \in X$:

(i) W is a downward subset of X if and only if $T(W)$ is a downward subset of Y.

(ii) $T(R_W(x)) = R_{T(W)}(T(x))$.

Theorem 8. *Let X and Y be Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. Let $T : X \longrightarrow Y$ be an injective positive operator which T^{-1} is a positive operator and $T(\mathbf{1}_X) = \mathbf{1}_Y$. Then T is an coapproximation persevering operator.*

Proof. In view of the proposition 6, we have $\|T(x)\| = \|x\|$, for all $x \in X$. Then for all $x \in X$ and all downward sets W of X and $w, w_0 \in W$,

$$\|w - w_0\| \leq \|x - w\| \Leftrightarrow \|T(w) - T(w_0)\| \leq \|T(x) - T(w)\|.$$

Therefore $T(R_W(x)) = R_{T(W)}(T(x))$. Also by lemma 5, W is downward subset of X if and only if $T(W)$ is a downward subset of Y. □

Corollary 9. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$, respectively. Every lattice isomorphism between X and Y is an coapproximation preserving operator.*

Proof. From theorems 3 and 8, the result obtained. □

Example 10. Let X be a Banach lattice with strong unit $\mathbf{1}$ and $a \in X$. Define

$$T_a : X \longrightarrow X, \quad T_a(x) = a + x.$$

Then $T^{-1} = T_{-a}$. Suppose that W is a downward subset of X; then $T_a(W) = a + W$ is a downward subset of X, and if $T(W)$ is downward, then $W = T_{-a}(T(W))$ is downward. Also for all $w, w_0 \in W$ and $x \in X$ we have

$$\|T_a(w_0) - T_a(w)\| \leq \|T_a(x) - T_a(w)\| \Longleftrightarrow \|w_0 - w\| \leq \|x - w\|.$$

Then $T(R_W(x)) = R_{T(W)}(T(x))$ and therefore T_a is an coapproximation preserving operator on X, for all $a \in X$.

Theorem 11. *Let X and Y be two Banach lattices with strong units $\mathbf{1}_X$ and $\mathbf{1}_Y$ respectively. If $T : X \longrightarrow Y$ is an coapproximation preserving operator, then:*

(1) W is a downward and coproximinal subset of X if and only if $T(W)$ is a downward and coproximinal subset of Y.

(2) W is a downward and coChebyshev subset of X if and only if $T(W)$ is a downward and coChebyshev subset of Y.

Definition 12. A subset G of the cone X^+ is called normal if

$$(g \in G, x \in X^+, x \leq g) \Longrightarrow x \in G.$$

For example, if f is an increasing function defined on X^+, then its lower level sets $S_c^+(f) = \{x \in X^+ : f(x) \leq c\}$ for all c are normal.

If $T : X \longrightarrow Y$ is a map between vector lattices X and Y, we assume that $T^+ = T|_{X^+}$. Then $T(x) = T^+(x)$ for all $x \in X^+$.

Lemma 13. *(1) Let X and Y be vector lattices and $T : X \longrightarrow Y$ be an injective positive operator such that T^{-1} is positive. Then G is closed normal subset of X^+ if and only if $T^+(G)$ is a closed normal subset of Y^+.*

(2) Let $T : X \longrightarrow X$ be a positive operator; then $S_c^+(f \circ T^+)$ are normal for all $c \in \mathbb{R}$.

Theorem 14. *Let X and Y be two Banach lattices with strong units 1_X and 1_Y , respectively. Let $T : X \longrightarrow Y$ be an injective positive operator such that T^{-1} is positive and $T(\boldsymbol{1}_X) = \boldsymbol{1}_Y$. Then for all normal subset G of $X^+$and for all $x \in X^+$,*

$$T^+(R_G(x)) = R_{T^+(G)}(T^+(x)).$$

That is, T^+ preserves coapproximation.

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