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PARADOX IN A d-DIMENSIONAL TRANSPORTATION PROBLEM

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Abstract: The d-dimensional transportation problem can be said as a generalization of classical transportation problem. Classical transportation problem has only 2 constraint vectors, those are demand and supply constraints. But d-dimensional transportation problem has d constraint vectors, 2 of the them may remain as demand and supply, while the others can be product type, transportation type, etc. A paradox in a transportation problem is when shipping more product leads to a cheaper price. In this research, a sufficient condition for the occurence of the paradox in the d-dimensional transportation problem will be discussed through its primal and dual. A method to find the upper bound of the additional shipping so that the paradox still occurs will also be discussed. And finally this research provides an algorithm combining the two previous results to find the paradox iteratively.

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1. Introduction

Transportation problem is an example of optimization problem (see [8],[10]). Classical transportation problem is first formulated by Hitchcock in 1941 [4]. This problem often occurs in everydays life, like determining a shortest path, minimizing shipping prices. Transportation problem had been researched by several researchers, one of them is a research by Wihartiko (see [10]). Generally, in a transportation problem, shipping more product will result to a higher transportation cost. But there is a case where shipping more product will reduce the transportation cost. This case is called the transportation paradox, or the more for less paradox. In a previous research, some researchers had already done some modification to the classical transportation problem and analyse the sufficient condition of the paradox. Some of them are solid transportation problem [1], and non-linear capacitated transportation problem [3]. Classical transportation problem and solid transportation problem can be said as 2 and 3-dimensional transportation problem, that is they only have 2 and 3 constraint vectors. Those transportation problems can be generalized to a d-dimensional transportation problem where this problem has d constraint vectors. Therefore, in this research, the sufficient condition for the occurence of the paradox will be analysed.

2. The d-dimensional transportation problem

The d-dimensional transportation problem is a generalization of 2 dimensional transportation problem, or classical transportation problem [2]. Classical transportation problem has only 2 constraint vectors, that is demand and supply constraints, while d-dimensional transportation problem has d constraint vectors $A_1, \dots, A_d$ where A_k has n_k elements. The formulation of this problem is as follows

$$\min \sum_{i_1,i_2,\dots,i_d} c_{i_1,i_2,\dots,i_d} x_{i_1,i_2,\dots,i_d} \quad ,$$

subject to

$$\sum_{\substack{i_1, i_2, \dots, i_d \\ i_k = I_k}} x_{i_1,i_2,\dots,i_d} = a_{I_k}(k), \quad 1 \le k \le d \; ; \quad 1 \le I_k \le n_k$$

$$x_{i_1,i_2,\dots,i_d} \geq 0,$$

$a_I(k)$: I-th element of k-th constraint vector.

3. Dual of the d-dimensional transportation problem

First we need to derive the dual problem of the d-dimensional transportation problem. Multiplying each constraint with a dual variable will have the following result

$$\sum_{\substack{i_1,i_2,\dots,i_d \\ i_k = I_k}} u_{i_k}(k) x_{i_1,i_2,\dots,i_d} = a_{i_k}(k) u_{i_k}(k), \quad 1 \leq k \leq d \ ; \quad 1 \leq I_k \leq n_k,$$

and additioning all of them will have the following result

$$\sum_{i_1,i_2,\dots,i_d} (\sum_{k=1}^{d} u_{i_k}(k)) x_{i_1,i_2,\dots,i_d} = \sum_{i_k=1}^{n_k} \sum_{k=1}^{d} a_{i_k}(k) u_{i_k}(k).$$

We see that $\sum_{k=1}^{d} u_{i_k}(k)$ is the coefficient of each $x_{i_1,i_2,\dots,i_d}$. Knowing the dual of a minimization problem is a lower bound of its primal, so the following condition must hold

$$\sum_{k=1}^{d} u_{i_k}(k) \leq c_{i_1,i_2,\dots,i_d}.$$

We have the following dual problem for the d dimentional transportation problem

$$\max \sum_{I_1}^{n_1} u_{I_1}(1) a_{I_1}(1) \ + \sum_{I_2}^{n_2} u_{I_2}(2) a_{I_2}(2) + \dots + \sum_{I_d}^{n_d} u_{I_d}(d) a_{I_d}(d),$$

subject to

$$u_{I_1}(1) + u_{I_2}(2) + \dots + u_{I_{d-1}}(d-1) + u_{I_d}(d) \leq c_{I_1,I_2,\dots,I_d},$$

$$I_1 = 1, 2, \dots, n_1$$

$$I_2 = 1, 2, \dots, n_2$$

$$\vdots$$

$$I_d = 1, 2, \dots, n_d.$$

4. *d*-dimensional transportation problem optimality sufficient condition

From the strong duality theorem in [5], it is shown that the primal and dual problem has the same optimal value, so the following must hold

$$\sum_{i_1, i_2, \dots, i_d} (\sum_{k=1}^{d} u_{i_k}(k)) x_{i_1,i_2,\dots,i_d} = \sum_{i_k=1}^{n_k} \sum_{k=1}^{d} a_{i_k}(k) u_{i_k}(k) = \sum_{i_1,i_2,\dots,i_d} c_{i_1,i_2,\dots,i_d} x_{i_1,i_2,\dots,i_d}.$$

We know from [6] that if $i_1, i_2, \dots, i_d \in B$, where B is the set of the optimal basic feasible solutions, then $x_{i_1,i_2,\dots,i_d} \geq 0$ and if $i_1, i_2, \dots, i_d \notin B$, then $x_{i_1,i_2,\dots,i_d} = 0$. This implies that if the d-dimensional transportation problem is optimal, then the following condition must hold

$$\sum_{k=1}^{d} u_{i_k}(k) = c_{i_1,i_2,\dots,i_d}, \quad \forall i_1, i_2, \dots, i_d \in B.$$

Therefore the sufficient condition for the optimality of the d-dimensional transportation problem is given below

$$\sum_{k=1}^{d} u_{i_k}(k) = c_{i_1,i_2,\dots,i_d}, \quad \forall i_1, i_2, \dots, i_d \in B$$

$$\sum_{k=1}^{d} u_{i_k}(k) \leq c_{i_1,i_2,\dots,i_d}, \quad \forall i_1, i_2, \dots, i_d \notin B.$$

5. Sufficient condition of the paradox

Let $Z(\boldsymbol{a_I}(\boldsymbol{k}), C)$ be the optimal value of the d-dimensional transportation problem with cost matrix C and constraint vectors $\boldsymbol{a_I}(\boldsymbol{k})$. Then the sufficient condition for the occurence of the transportation paradox is given by Theorem 1.

Theorem 1. *Let $p_1,\ p_2,\dots,p_d$ with $1 \le p_1 \le n_1,\ 1 \le p_2 \le n_2,\ \dots,\ 1 \le p_d \le n_d$, so that*

$$u_{p_1}(1) + u_{p_2}(2) + \cdots + u_{p_{d-1}}(d-1) + u_{p_d}(d) < 0.$$

If there is a $\theta > 0$ so that if the constraints $a_{p_1}(1),\ a_{p_2}(2),\dots,\ a_{p_d}(d)$ are replaced by $\hat{a}_{p_1}(1) = a_{p_1}(1) + \theta,\ \hat{a}_{p_2}(2) = a_{p_2}(2) + \theta, \dots,\ \hat{a}_{p_d}(d) = a_{p_d}(d) + \theta$, the set of optimal basic feasible solutions remains the same, then the paradox will occur.

Proof. Let $p_1,\ p_2,\dots,p_d$ with $1 \le p_1 \le n_1,\ 1 \le p_2 \le n_2,\ \dots,\ 1 \le p_d \le n_d$ with

$$u_{p_1}(1) + u_{p_2}(2) + \cdots + u_{p_{d-1}}(d-1) + u_{p_d}(d) < 0,$$

and each constraints is replaced by $\hat{a}_{p_1}(1) = a_{p_1}(1) + \theta,\ \hat{a}_{p_2}(2) = a_{p_2}(2) + \theta, \dots,\ \hat{a}_{p_d}(d) = a_{p_d}(d) + \theta$. Suppose after adding θ to each constraints, the set of optimal basic feasible solutions remains the same. Based on the sufficient condition of the optimality of the d-dimensional transportation problem, the dual variables may remain the same. So the change of the dual optimal value may be written as follows

$$Z(\widehat{\boldsymbol{a}}_{\boldsymbol{I}}(\boldsymbol{k}), C) = \sum_{I_1}^{n_1} u_{I_1}(1) a_{I_1}(1) + u_{p_1}(1)\theta + \sum_{I_2}^{n_2} u_{I_2}(2) a_{I_2}(2) + u_{p_2}(2)\theta$$

$$+ \cdots + \sum_{I_d}^{n_d} u_{I_d}(d) a_{I_d}(d) + u_{p_d}(d)\theta = \sum_{I_1}^{n_1} u_{I_1}(1) a_{I_1}(1) + \sum_{I_2}^{n_2} u_{I_2}(2) a_{I_2}(2)$$

$$+ \cdots + \sum_{I_d}^{n_d} u_{I_d}(d) a_{I_d}(d) + \big(u_{p_1}(1) + u_{p_2}(2) + \cdots + u_{p_{d-1}}(d-1) + u_{p_d}(d)\big)\theta$$

$$= Z(\boldsymbol{a_I}(\boldsymbol{k}), C) + \big(u_{p_1}(1) + u_{p_2}(2) + \cdots + u_{p_{d-1}}(d-1) + u_{p_d}(d)\big)\theta.$$

We have that $u_{p_1}(1) + u_{p_2}(2) + \cdots + u_{p_{d-1}}(d-1) + u_{p_d}(d) < 0$ and $\theta > 0$, so $\big(u_{p_1}(1) + u_{p_2}(2) + \cdots + u_{p_{d-1}}(d-1) + u_{p_d}(d)\big)\theta < 0$,which implies $Z(\widehat{\boldsymbol{a}}_{\boldsymbol{I}}(\boldsymbol{k}), C) < Z(\boldsymbol{a_I}(\boldsymbol{k}), C)$. □

6. Upper Bound for θ

From that theorem, the value of θ is still unknown. There is a method from [7] to find the upper bound of θ called path method. In this research, the method to find the path will not be exactly the same as [7] and [4], but by looking at the change of value by adding $\theta = 1$. Let a 4-dimensional transportation problem given below

$$c_{I_1,I_2,1,1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 9 & 2 \\ 1 & 9 & 2 \end{bmatrix} \quad c_{I_1,I_2,1,2} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 9 & 2 \\ 1 & 9 & 2 \end{bmatrix} \quad c_{I_1,I_2,1,3} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 9 & 2 \\ 1 & 9 & 2 \end{bmatrix}$$

$$c_{I_1,I_2,2,1} = \begin{bmatrix} 7 & 9 & 8 \\ 7 & 3 & 8 \\ 7 & 3 & 8 \end{bmatrix} \quad c_{I_1,I_2,2,2} = \begin{bmatrix} 7 & 9 & 8 \\ 7 & 3 & 8 \\ 7 & 3 & 8 \end{bmatrix} \quad c_{I_1,I_2,2,3} = \begin{bmatrix} 7 & 9 & 8 \\ 7 & 3 & 8 \\ 7 & 3 & 8 \end{bmatrix}$$

$$c_{I_1,I_2,3,1} = \begin{bmatrix} 8 & 3 & 3 \\ 1 & 7 & 4 \\ 1 & 9 & 2 \end{bmatrix} \quad c_{I_1,I_2,3,2} = \begin{bmatrix} 8 & 3 & 3 \\ 1 & 7 & 4 \\ 1 & 9 & 2 \end{bmatrix} \quad c_{I_1,I_2,3,3} = \begin{bmatrix} 8 & 3 & 3 \\ 1 & 7 & 4 \\ 1 & 7 & 4 \end{bmatrix}$$

$$\boldsymbol{a_{I_1}}(\mathbf{1}) = \begin{bmatrix} 27 \\ 12 \\ 10 \end{bmatrix}, \boldsymbol{a_{I_2}}(\mathbf{2}) = \begin{bmatrix} 13 \\ 18 \\ 18 \end{bmatrix}, \boldsymbol{a_{I_3}}(\mathbf{3}) = \begin{bmatrix} 15 \\ 25 \\ 9 \end{bmatrix}, \boldsymbol{a_{I_4}}(\mathbf{4}) = \begin{bmatrix} 15 \\ 25 \\ 9 \end{bmatrix},$$

with its optimal solutions given below

Primal:

$$x_{1,1,1,1} = 6,\ x_{1,1,2,2} = 7,\ x_{1,3,1,1} = 4,\ x_{1,3,1,2} = 5,\ x_{1,3,3,1} = 5,\ x_{2,2,2,2} = 3,$$

$$x_{2,2,2,3} = 9,\ x_{3,2,2,2} = 6,\ x_{3,3,3,2} = 4,$$

Dual:

$$u_1(1) = 1,\ u_2(1) = 0,\ u_3(1) = 0,\ u_1(2) = 0,\ u_2(2) = -3,\ u_3(2) = 1,$$

$$u_1(3) = 0,\ u_2(3) = 6,\ u_3(3) = 2,\ u_1(4) = 0,\ u_2(4) = 0,\ u_3(4) = 0,$$

$$Z(\boldsymbol{a_I}(\boldsymbol{k}), C) = 150.$$

From this result we have $u_1(1)$, $u_2(2) = -3$, $u_1(3) = 0$, $u_1(4) = 0$ with $u_1(1) + u_2(2) + u_1(3) + u_1(4) = -2 < 0$. By adding each constraints by $\theta = 1$ we have the following result

$$\boldsymbol{a_{I_1}}(\mathbf{1}) = \begin{bmatrix} 27+1 \\ 12 \\ 10 \end{bmatrix}, \boldsymbol{a_{I_1}}(\mathbf{2}) = \begin{bmatrix} 13 \\ 18+1 \\ 18 \end{bmatrix}, \boldsymbol{a_{I_1}}(\mathbf{3}) = \begin{bmatrix} 15+1 \\ 25 \\ 9 \end{bmatrix}$$

$$, \boldsymbol{a_{I_1}}(\mathbf{4}) = \begin{bmatrix} 15+1 \\ 25 \\ 9 \end{bmatrix},$$

and the new optimal solutions

$$x_{1,1,1,1} = 7,\ x_{1,1,2,2} = 6,\ x_{1,3,1,1} = 3,\ x_{1,3,1,2} = 6,\ x_{1,3,3,1} = 6,$$

$$x_{2,2,2,2} = 3,\ x_{2,2,2,3} = 9,\ x_{3,2,2,2} = 7,\ x_{3,3,3,2} = 3,$$

with $Z(\widehat{\boldsymbol{a}}_{\boldsymbol{I}}(\boldsymbol{k}), C) = 148$. The new optimal value is lower by 2 units than before adding θ, so a paradox occurs. We see that $(x_{1,1,1,1},\ x_{1,1,2,2},\ x_{1,3,1,1},\ x_{1,3,1,2},\ x_{1,3,3,1},\ x_{2,2,2,2},\ x_{3,3,3,2})$ have a change of value. Let DS be the set of indices of optimal basic feasible solutions as introduced in [4]. Following [4] we have the path DS given below

$$DS = \{(1,1,1,1), (1,1,2,2), (1,3,1,1), (1,3,1,2), (1,3,3,1),$$

$$(3,2,2,2), (3,3,3,2)\}.$$

Then the DS will be divided into 2 sets, one is a set of index where the solutions is added by θ, and the other is a set of index where the solutions is substracted by θ. Then we have

$$DS^+ = \{(1,1,1,1), (1,3,1,2), (1,3,3,1), (3,2,2,2)\}$$

$$DS^- = \{(1,1,2,2), (1,3,1,1), (3,3,3,2)\}.$$

We want an additional shipment by θ that doesn't change the set of optimal basic feasible solutions. Once a solution which is substracted by θ reaches 0, it may change the optimal basic feasible solution if we keep increasing θ. So the bound of increasing θ can be said as the smallest value of solutions which have the index in DS^-. Then we have

Corollary 2. *$\theta > 0$ exists if $x_{I_1,I_2,\ldots,I_d} > 0$, $\forall I_1, I_2, \ldots, I_d \in DS^-$ and $\theta \leq \min x_{I_1,I_2,\ldots,I_d}$, $\forall I_1, I_2, \ldots, I_d \in DS^-$.*

In the previous ilustration, the upper bound of θ is given below

$$\theta \leq \min\ x_{1,1,2,2},\ x_{1,3,1,1},\ x_{3,3,3,2}\ = \min\ 7,\ 4,\ 4\ = 4$$

$$\theta \leq 4.$$

Exeeding $\theta = 4$ doesn't imply that the paradox will not ocur anymore, because there is a possibility that the DS path is not unique. Combining all the previous results, an algorithm to find paradox can be constructed.

7. Algorithm

1. Find the initial optimal solutions X^0 to the d-dimensional transportation problem, and write them as a pair of optimal value and the total product (Z^0, F^0).

2. $i = 1$.

3. Find index $(p_1, p_2, \ldots, p_d) \notin B$ so that $u_{p_1}(1) + u_{p_2}(2) + \cdots + u_{p_{d-1}}(d-1) + u_{p_d}(d) < 0$ without changing the set of basic feasible solutions, if there aren't any, go to step 11.

4. Increase the total shipment by 1 unit at each constraints corresponding to step 3.

5. Construct a path DS, DS^+, and DS^-.

6. Find an upper bound for θ.

7. Add each constraints corresponding to step 3 by θ.

8. Find the new X^i and Z^i, write (Z^i, F^i).

9. $i = i + 1$.

10. Return to step 3.

11. Write the paradoxical pair $(Z^*, F^*) = (Z^i, F^i)$ for an optimal solution $X^* = X^i$.

The previous illustration has done steps 1-6, so we need to continue to the next steps. Adding $\theta = 4$ to each constraints will result the following results

$$\boldsymbol{a}_{\boldsymbol{I}_1}(\mathbf{1}) = \begin{bmatrix} 27+4 \\ 12 \\ 10 \end{bmatrix}, \boldsymbol{a}_{\boldsymbol{I}_1}(\mathbf{2}) = \begin{bmatrix} 13 \\ 18+4 \\ 18 \end{bmatrix}, \boldsymbol{a}_{\boldsymbol{I}_1}(\mathbf{3}) = \begin{bmatrix} 15+4 \\ 25 \\ 9 \end{bmatrix},$$

$$\boldsymbol{a}_{\boldsymbol{I}_1}(\mathbf{4}) = \begin{bmatrix} 15+4 \\ 25 \\ 9 \end{bmatrix},$$

with its solutions below
Primal:

$$x_{1,1,1,1} = 10,\ x_{1,1,2,2} = 3,\ x_{1,3,1,2} = 9,\ x_{1,3,3,1} = 9,\ x_{2,2,2,2} = 3,\ x_{2,2,2,3} = 9,$$

$$x_{3,2,2,2} = 10$$

Dual:

$$u_1(1) = 1,\ u_2(1) = -4,\ u_3(1) = -4,\ u_1(2) = 0,\ u_2(2) = 1,\ u_3(2) = 1,$$

$$u_1(3) = 0,\ u_2(3) = 6,\ u_3(3) = 1,\ u_1(4) = 0,\ u_2(4) = 0,\ u_3(4) = 0,$$

$$Z^1(\widehat{\boldsymbol{a}}_{\boldsymbol{I}}(\boldsymbol{k}), C) = 142, \left(Z^1, F^1\right) = (142, 53).$$

Repeating step 3, let $u_3(1)+u_2(2)+u_1(3)+u_1(4) = -4+1+0+0 = -3 < 0$. Then step 4,5,6 will be given below

$$\boldsymbol{a}_{\boldsymbol{I}_1}(\mathbf{1}) = \begin{bmatrix} 31 \\ 12 \\ 10+1 \end{bmatrix}, \boldsymbol{a}_{\boldsymbol{I}_1}(\mathbf{2}) = \begin{bmatrix} 13 \\ 22+1 \\ 18 \end{bmatrix}, \boldsymbol{a}_{\boldsymbol{I}_1}(\mathbf{3}) = \begin{bmatrix} 19+1 \\ 25 \\ 9 \end{bmatrix},$$

$$\boldsymbol{a}_{\boldsymbol{I}_1}(\mathbf{4}) = \begin{bmatrix} 19+1 \\ 25 \\ 9 \end{bmatrix},$$

with its optimal solutions

$$x_{1,1,1,1} = 11,\ x_{1,1,2,2} = 2,\ x_{1,3,1,2} = 9,\ x_{1,3,3,1} = 9,\ x_{2,2,2,2} = 3,\ x_{2,2,2,3} = 9,$$

$$x_{3,2,2,2} = 11$$

then

$$DS = \{(1,1,1,1), (1,1,2,2), (3,2,2,2)\}$$

$$DS^{+} = \{(1,1,1,1), (3,2,2,2)\}$$

$$DS^{-} = \{(1,1,2,2)\}$$

$$\theta \leq x_{1,1,2,2} = 3.$$

By doing steps 7,8, we have the following results

$$\boldsymbol{a_{I_1}}(\mathbf{1}) = \begin{bmatrix} 31 \\ 12 \\ 10+3 \end{bmatrix}, \boldsymbol{a_{I_1}}(\mathbf{2}) = \begin{bmatrix} 13 \\ 22+3 \\ 18 \end{bmatrix}, \boldsymbol{a_{I_1}}(\mathbf{3}) = \begin{bmatrix} 19+3 \\ 25 \\ 9 \end{bmatrix},$$

$$\boldsymbol{a_{I_1}}(\mathbf{4}) = \begin{bmatrix} 19+3 \\ 25 \\ 9 \end{bmatrix}$$

Primal:

$$x_{1,1,1,1} = 13,, \ x_{1,3,1,2} = 9, \ x_{1,3,3,1} = 9, \ x_{2,2,2,2} = 3, \ x_{2,2,2,3} = 9,$$

$$x_{3,2,2,2} = 13$$

Dual:

$$u_1(1) = 1, \ u_2(1) = 0, \ u_3(1) = 0, \ u_1(2) = 0, \ u_2(2) = 1, \ u_3(2) = 1,$$

$$u_1(3) = 0, \ u_2(3) = 2, \ u_3(3) = 1, \ u_1(4) = 0, \ u_2(4) = 0, \ u_3(4) = 0,$$

$$Z^2\left(\widehat{\boldsymbol{a}}_{\boldsymbol{I}}(\boldsymbol{k}), C\right) = 133, \left(Z^2, F^2\right) = (133, 56).$$

Returning to step 3 there aren't any dual variables that satisfy $u_{p_1}(1) + u_{p_2}(2) + \cdots + u_{p_{d-1}}(d-1) + u_{p_d}(d) < 0$, so the iteration stops with $(Z^*, F^*) = \left(Z^2, F^2\right) = (133, \ 56)$.

8. Conclusions

The sufficient condition for the occurence of the d-dimensional transportation problem has a more general form than the classical transportation problem. The main idea finding this sufficient condition hasn't changed, that is by using its dual variables.

The upper bound can be found by the DS path. Although the method to find DS is slightly different from [7] or [4], this research's method is quite easy to use because it only needs to find solutions which have a change of value. The upper bound still uses the smallest solution with index in DS^- like in [4].

The algorithm to find the paradoxical pair doesn't increase θ by 1 unit per iteration. This research provide an increase of θ to its upper bound so that the total iteration becomes faster. With this algorithm, the total iterations of the algorithm becomes much faster if θ has a large upper bound.

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