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PROPERTY (z), DIRECT SUMS AND A NOTE ON AN A-BROWDER TYPE THEOREM

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Abstract: We characterize the properties (z) and (az) for an operator T whose dual T^* has the SVEP on the complementary of the upper semi-Weyl spectrum of T. If S and T are Banach space operators satisfying property (z) or (az), we give conditions on S and T to ensure the preservation of these properties by the direct sum $S \oplus T$. Some results are given for multipliers and in general for (H)-operators. Also we give a correct proof of [11, Theorem 2.3] which was proved by using the equality $\sigma_p^0(S \oplus T) = \sigma_p^0(S) \cup \sigma_p^0(T)$. However this equality is not true; we give counterexamples to show that.

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1. Introduction and Preliminaries

Let X denote an infinite dimensional complex Banach space, and denote by $L(X)$ the algebra of all bounded linear operators on X. For $T \in L(X)$, we denote by $\alpha(T)$ the dimension of the kernel $N(T)$ and by $\beta(T)$ the codimension of the

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range $R(T)$. By $\sigma(T), \sigma_a(T), \sigma_s(T)$, we denote the spectrum, the approximate spectrum and the surjectivity spectrum of T, respectively.

Recall that T is said to be *upper semi-Fredholm*, if $R(T)$ is closed and $\alpha(T) < \infty$, while T is called *lower semi-Fredholm*, if $R(T)$ is closed and $\beta(T) < \infty$. $T \in L(X)$ is said to be *semi-Fredholm* if T is either an upper semi-Fredholm or a lower semi-Fredholm operator. T is *Fredholm* if T is upper semi-Fredholm and lower semi-Fredholm. If T is semi-Fredholm then the index of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. For an operator $T \in L(X)$, the *ascent* $a(T)$ and the *descent* $d(T)$ are defined by $a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and $d(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, respectively; the infimum over the empty set is taken ∞. If the ascent and the descent of T are both finite, then $a(T) = d(T) = p$, and $R(T^p)$ is closed. An operator T is said to be *Weyl* if it is Fredholm of index zero. It is called *upper semi-Weyl* (resp., *lower semi-Weyl*) if it is upper semi-Fredholm of index ≤ 0 (resp., lower semi-Fredholm of index ≥ 0). T is called *upper semi-Browder* if it is an upper semi-Fredholm operator with finite ascent and it is called *Browder* if it is Fredholm of finite ascent and descent.

If $T \in L(X)$ and $n \in \mathbb{N}$, we denote by T_n the restriction of T on $R(T^n)$. T is said to be *upper semi b-Weyl*, if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n : R(T^n) \to R(T^n)$ is upper semi-Weyl.

We recall that a complex number $\lambda \in \sigma(T)$ is a *pole* of the resolvent of T, if $T - \lambda I$ has finite ascent and finite descent and $\lambda \in \sigma_a(T)$ is a *left pole* of T if $p = a(T - \lambda I) < \infty$ and $R(T^{p+1})$ is closed.

In the following list, we summarize the notations and symbols needed later.

$\operatorname{iso} A$: isolated points of a subset $A \subset \mathbb{C}$,
$\operatorname{acc} A$: accumulations points of a subset $A \subset \mathbb{C}$,
A^C: the complementary of a subset $A \subset \mathbb{C}$,
$D(0,1)$: the closed unit disc in $\mathbb{C}$,
$C(0,1)$: the unit circle of $\mathbb{C}$,
$\mathcal{H}(\sigma(T))$: the set of all analytic functions defined on an open neighborhood of $\sigma(T)$,
$p_0(T)$: poles of T,
$p_{00}(T)$: poles of T of finite rank,
$p_0^a(T)$: left poles of T,
$p_{00}^a(T)$: left poles of T of finite rank,
$\sigma_p(T)$: eigenvalues of T,
$\sigma_p^0(T)$: eigenvalues of T of finite multiplicity,
$\pi_0(T) := \operatorname{iso} \sigma(T) \cap \sigma_p(T)$,
$\pi_{00}(T) := \operatorname{iso} \sigma(T) \cap \sigma_p^0(T)$,

$\pi_0^a(T) := \operatorname{iso}\sigma_a(T) \cap \sigma_p(T)$,
$\pi_{00}^a(T) := \operatorname{iso}\sigma_a(T) \cap \sigma_p^0(T)$,
$\sigma_{uf}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not upper semi-Fredholm$\}$: upper semi-Fredholm spectrum,
$\sigma_{lf}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not lower semi-Fredholm$\}$: lower semi-Fredholm spectrum,
$\rho(T) = \mathbb{C} \setminus \sigma(T)$; $\rho_a(T) = \mathbb{C} \setminus \sigma_a(T)$; $\rho_{uf}(T) = \mathbb{C} \setminus \sigma_{uf}(T)$,
$\sigma_b(T) = \sigma(T) \setminus p_{00}(T)$: Browder spectrum of T,
$\sigma_{ub}(T) = \sigma_a(T) \setminus p_{00}^a(T)$: upper Browder spectrum of T,
$\sigma_w(T)$: Weyl spectrum of T,
$\sigma_{uw}(T)$: upper semi-Weyl spectrum of T,
$\sigma_{lw}(T)$: lower semi-Weyl spectrum of T,
$\sigma_{ubw}(T)$: upper semi-b-Weyl spectrum of T,

Definition 1.1. [9], [18], [19] Let $T \in L(X)$. T is said to satisfy
a) a-Browder's theorem if $\sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}^a(T)$.
b) a-Weyl's theorem if $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T)$.
c) property (z) if $\sigma(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T)$.
d) property (az) if $\sigma(T) \setminus \sigma_{uw}(T) = p_{00}^a(T)$.
e) property (gaz) if $\sigma(T) \setminus \sigma_{ubw}(T) = p_0^a(T)$.
f) property (gz) if $\sigma(T) \setminus \sigma_{ubw}(T) = \pi_0^a(T)$.

The relationship between a-Browder's theorem and the properties given in the precedent definition was studied in [19], and it is summarized as follows:

$$\text{property } (gz) \Longrightarrow \text{property } (z) \Longrightarrow \text{property } (az) \Longrightarrow \text{a-Browder's theorem}$$

$$\text{property } (az) \Longleftrightarrow \text{property } (gaz)$$

Moreover, in [19] counterexamples were given to show that the reverse of each implication in the diagram is not true.

The following property named SVEP has relevant role in local spectral theory. For more details see the recent monographs [1] and [16].

Definition 1.2. [16] An operator $T \in L(X)$ is said to have the *single valued extension property* (SVEP) at $\lambda_0 \in \mathbb{C}$, if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \longrightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

It follows easily that $T \in L(X)$ has the SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, T has the SVEP at every point of iso $\sigma(T)$. We also have

$$a(T-\lambda_0 I)<\infty \Longrightarrow \text{ T has the SVEP at } \lambda_0, \quad (I_1)$$

and dually

$$d(T-\lambda_0 I)<\infty \Longrightarrow T^* \text{ has the SVEP at } \lambda_0, \quad (I_2)$$

where T^* denotes the dual of T, see [1, Theorem 3.8]. Furthermore, if $T-\lambda_0 I$ is semi-Fredholm then the implications above are equivalences.

2. Properties (az), (z) and SVEP

An important class of operators is given by the multipliers on a semi-simple Banach algebra $\mathcal{A}$. Recall that an operator $T \in L(\mathcal{A})$ is a multiplier if $aT(b) = T(a)b, \ \forall\, a, b \in \mathcal{A}$.

Proposition 2.1. *If T is a multiplier on a semi-simple Banach algebra $\mathcal{A}$, then*
i) T has the SVEP, $a(T) \leq 1$ and $\alpha(T) \leq \beta(T)$.
ii) If in addition $\mathcal{A}$ is commutative regular and Tauberian then properties (z) and (az) hold for T and they hold for T^* too if T^* has the SVEP.

Proof. i) See [1, Theorem 4.32] and [1, Theorem 3.4].
ii) From [1, Corollary 5.88] we have $\sigma_a(T) = \sigma(T) = \sigma_a(T^*)$ and from [1, Theorem 5.118], a-Weyl's theorem holds for T and it holds for T^* if it has the SVEP. The conclusion follows from [19, Theorem 2.4] and since property (z) entails property (az). □

Now we give a characterization of property (az) for a linear operator T whose dual T^* has the SVEP on $\sigma_{uw}(T)^C$. By duality we give a similar result for T^*.

Theorem 2.2. *Let $T \in L(X)$, then:*
i) T^* has the SVEP on $\sigma_{uw}(T)^C$ if and only if T satisfies property (az).
ii) T has the SVEP on $\sigma_{lw}(T)^C$ if and only if T^* satisfies property (az).

Proof. T^* has the SVEP on $\sigma_{uw}(T)^C$ then from [3, Theorem 2.2], T satisfies a-Browder's theorem $\sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}^a(T)$. We only have to prove that $\sigma(T) = \sigma_a(T)$. Let $\mu_0 \notin \sigma_a(T)$ be arbitrary, then $\mu_0 \notin \sigma_{uw}(T)$, $T-\mu_0 I$ is injective and $R(T-\mu_0 I)$ is closed. So $T-\mu_0 I$ is an upper semi-Fredholm operator, and by the implication (I_2) above we conclude that $\beta(T-\mu_0 I) = 0$.

Hence $T-\mu_0 I$ is surjective and $\mu_0 \notin \sigma(T)$.
Conversely, suppose that T satisfies (az). Let $\lambda_0 \in \sigma_{uw}(T)^C$ be arbitrary. We distinguish two cases: if $\lambda_0 \notin \sigma(T)=\sigma(T^*)$ then T^* has the SVEP at λ_0. If $\lambda_0 \in \sigma(T)$ then $\lambda_0 \in p_{00}^a(T)$. Thus λ_0 is isolated in $\sigma_a(T)=\sigma_s(T^*)$ and hence T^* has the SVEP at λ_0.

ii) By the duality between T and T^* the proof goes similarly with (i). □

Remark 2.3. In Theorem 2.2, we cannot replace the SVEP for T^* on $\sigma_{uw}(T)^C$ (resp., the SVEP for T on $\sigma_{lw}(T)^C$) by the SVEP for T on $\sigma_{uw}(T)^C$ (resp., the SVEP for T^* on $\sigma_{lw}(T)^C$). Here and elsewhere R and L denote the unilateral right and left shifts operators on $\ell^2(\mathbb{N})$ defined by $R(x_1,x_2,\ldots)=(0,x_1,x_2,x_3,\ldots)$ and $L(x_1,x_2,x_3,\ldots)=(x_2,x_3,\ldots)$. Evidently $L^*=R$ has the SVEP, but R does not satisfy property (az), since $\sigma(R)=D(0,1)$, $\sigma_{uw}(R)=C(0,1)$ and $p_{00}^a(R)=\emptyset$.
Another example on $\ell^2(\mathbb{N})$ is given by $T(x_1,x_2,x_3,\ldots)=(\frac{1}{2}x_1,0,x_2,x_3,\ldots)$. We have T^* has the SVEP; since its approximate spectrum $\sigma_a(T^*)=C(0,1)\cup\{\frac{1}{2}\}$ has empty interior. But, T^* does not satisfy property (az); since $\sigma(T^*)=D(0,1)$, $\sigma_{uw}(T^*)=C(0,1)$ and $p_{00}^a(T^*)=\{\frac{1}{2}\}$.

Proposition 2.4. *Suppose that the dual T^* of $T\in L(X)$ has the SVEP, then*
i) If $Q\in L(X)$ is a quasi-nilpotent operator which commutes with T, then $f(T)+Q$ and $f(T+Q)$ satisfy property (az) for every $f\in\mathcal{H}(\sigma(T))$.
ii) If $K\in L(X)$ is an algebraic (resp., F a finite rank) operator which commutes with T, then $T+K$ (resp., $T+F$) satisfies property (az).

Proof. i) We know from [1, Theorem 2.40] that if T^* has the SVEP, then $(f(T))^*=f(T^*)$ has the SVEP. Since Q is quasi-nilpotent and commutes with T then from [1, Corollary 2.12], T^*+Q^* has the SVEP. It follows that $(f(T+Q))^*=f((T+Q)^*)$ and $(f(T)+Q)^*$ have the SVEP. From Theorem 2.2, $f(T)+Q$ and $f(T+Q)$ satisfy property (az).
ii) If K is algebraic and commutes with T, then K^* is also algebraic and commutes with T^*. From [5, Theorem 2.14], $T^*+K^*=(T+K)^*$ has the SVEP. Hence $T+K$ satisfies property (az). If F is a finite rank operator and commutes with T, then $T^*+F^*=(T+F)^*$ has the SVEP, see the proof of [2, Lemma 2.8]. Hence $T+F$ satisfies property (az). □

From Theorem 2.2 and [19, Theorem 3.6] and [19, Corollary 3.7], we obtain immediately the following characterizations for properties (z) and (gz). We recall that $T\in L(X)$ is said to be *finitely a-polaroid* if every isolated point of $\sigma_a(T)$ is a left pole of T of finite rank and is said to be *a-polaroid* if every isolated point of $\sigma_a(T)$ is a left pole of T. Note that every finitely a-polaroid operator is a-polaroid, but the converse is not true. For this, consider the operator P defined on $\ell^2(\mathbb{N})$ by: $P(x_1, x_2, \ldots) = (0, x_2, x_3, \ldots)$. Then $\operatorname{iso}\sigma_a(T) = \{0, 1\} = p_0^a(P)$ and $p_{00}^a(P) = \{0\}$.

Corollary 2.5. *Let $T \in L(X)$, then:*
i) T satisfies property (z) if and only if T^* has the SVEP on $\sigma_{uw}(T)^C$ and $\pi_{00}^a(T) = p_{00}^a(T)$. In particular, if T is finitely a-polaroid, then T satisfies property (z) if and only if T^* has the SVEP on $\sigma_{uw}(T)^C$.
ii) T satisfies property (gz) if and only if T^* has the SVEP on $\sigma_{uw}(T)^C$ and $\pi_0^a(T) = p_0^a(T)$. In particular, if T is a-polaroid, then T satisfies property (gz) if and only if T^* has the SVEP on $\sigma_{uw}(T)^C$.
iii) T^* satisfies property (z) if and only if T has the SVEP on $\sigma_{lw}(T)^C$ and $\pi_{00}^a(T^*) = p_{00}^a(T^*)$.
iv) T^* satisfies property (gz) if and only if T has the SVEP on $\sigma_{lw}(T)^C$ and $\pi_0^a(T^*) = p_0^a(T^*)$.

Remark 2.6. In Corollary 2.5, we do not expect neither property (z) nor property (gz) for an operator T with only (as hypothesis) the SVEP of its dual on $\sigma_{uw}(T)^C$. Indeed, let T be the operator defined on $\ell^2(\mathbb{N})$ by $T(x_1, x_2, \ldots) = (\frac{x_2}{2}, \frac{x_3}{3}, \ldots)$ then T^* has the SVEP, but T does not satisfy property (z) and then it does not satisfy property (gz) too. Note that here $\pi_{00}^a(T) = \pi_0^a(T) = \{0\}$ and $p_{00}^a(T) = p_0^a(T) = \emptyset$.

3. Properties (az), (z) and Direct Sums

In the following, Y denotes an infinite dimensional complex Banach space.

Definition 3.1. Let $T \in L(X)$ and $S \in L(Y)$. We say that T and S have a shared stable sign index if for each $\lambda \in \rho_{uf}(T)$ and each $\mu \in \rho_{uf}(S)$, $\operatorname{ind}(T - \lambda I)$ and $\operatorname{ind}(S - \mu I)$ have the same sign.

Examples 3.2. We give some examples of operators with shared stable sign index.

(a) For an hyponormal operator T on a Hilbert space we always have $\operatorname{ind}(T-\lambda I) \leq 0$, for each $\lambda \in \rho_{uf}(T)$. Hence two hyponormal operators acting on Hilbert spaces have a shared stable sign index.

(b) By Proposition 2.1, two multipliers on semi-simple Banach algebras have a shared stable sign index. Moreover, according to [6, Theorem 4.5], any multiplier T on a commutative semi-simple algebra is Fredholm if and only it is upper semi-Fredholm and in this case $\operatorname{ind}(T) = 0$.

c) If S and T are two operators having the SVEP on the complementary of their upper semi-Fredholm spectra respectively, then they have a shared stable sign index. The same occurs when S^* and T^* have the SVEP on the complementary of their lower semi-Fredholm spectra respectively.

Suppose for instance that S has the SVEP on $\rho_{uf}(S)$ and T has the SVEP on $\rho_{uf}(T)$. If $\lambda \in \rho_{uf}(S)$ and $\mu \in \rho_{uf}(T)$ then $S - \lambda I$ and $T - \mu I$ are upper semi-Fredholm and since S and T have the SVEP at λ and μ respectively, then from the implication (I_1) above we have $a(S - \lambda I)$ and $a(T - \mu I)$ are finite. Hence from [1, Theorem 3.4], S and T have a shared stable index.

Note that the definition of *shared stable sign index* used here is weaker and slightly different from [7, Definition 1.3].

Lemma 3.3. Let $S \in L(X)$ and $T \in L(Y)$, then the following properties hold:
i) $\sigma_{uw}(S \oplus T) \subseteq \sigma_{uw}(S) \cup \sigma_{uw}(T)$.
ii) If $S \oplus T$ satisfies a-Browder's theorem, then $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$.
iii) If S and T have a shared stable sign index, then $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$.
In particular, this equality holds if S and T have the SVEP .

Proof. i) If $\lambda \notin \sigma_{uw}(S) \cup \sigma_{uw}(T)$, then $S - \lambda I$ and $T - \lambda I$ are upper semi-Weyl operators. Hence $(S \oplus T) - \lambda I$ is an upper semi-Fredholm operator with $\text{ind}((S \oplus T) - \lambda I) = \text{ind}(S - \lambda I) + \text{ind}(T - \lambda I) \leq 0$. So $\lambda \notin \sigma_{uw}(S \oplus T)$.

ii) If $S \oplus T$ satisfies a-Browder's theorem, then $\sigma_{uw}(S \oplus T) = \sigma_{ub}(S \oplus T)$. As $\sigma_{ub}(S \oplus T) = \sigma_{ub}(S) \cup \sigma_{ub}(T)$, then $\sigma_{uw}(S \oplus T) = \sigma_{ub}(S) \cup \sigma_{ub}(T)$. Since the inclusion
$\sigma_{uw}(S) \cup \sigma_{uw}(T) \subset \sigma_{ub}(S) \cup \sigma_{ub}(T)$ is always true, we then have $\sigma_{uw}(S) \cup \sigma_{uw}(T) \subset \sigma_{uw}(S \oplus T)$. We conclude by i) that $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$.

iii) If $\lambda \notin \sigma_{uw}(S \oplus T)$, then $(S \oplus T) - \lambda I$ is an upper semi-Weyl operator. It follows that both $S - \lambda I$ and $T - \lambda I$ are upper semi-Fredholm . Since $\text{ind}((S \oplus T) - \lambda I) = \text{ind}(S - \lambda I) + \text{ind}(T - \lambda I) \leq 0$ and S and T have a shared stable sign index, we have $\text{ind}(S - \lambda I) \leq 0$ and $\text{ind}(T - \lambda I) \leq 0$. Hence $\lambda \notin \sigma_{uw}(S) \cup \sigma_{uw}(T)$. $\square$

Remark 3.4. The inclusion showed in the first statement is proper: for this let R and L be the operators defined on $\ell^2(\mathbb{N})$ in Remark 2.3. We then have $\sigma_{uw}(R \oplus L) = C(0,1)$ and $\sigma_{uw}(R) \cup \sigma_{uw}(L) = D(0,1)$. Observe that L and R are upper semi-Fredholm operators with $\text{ind}(R) = -1$ and $\text{ind}(L) = 1$.

In the next theorem, we characterize the stability of property (az) under diagonal operator matrices in terms of upper semi-Weyl spectra of its components.

Theorem 3.5. *Suppose that $S \in L(X)$ and $T \in L(Y)$ satisfy property (az), then $S \oplus T$ satisfies property (az) if and only if $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$.*

Proof. ($\Leftarrow$) Since S and T satisfy property (az) then by [19, Theorem 3.2], S and T satisfy a-Browder theorem, $\sigma(S) = \sigma_a(S)$ and $\sigma(T) = \sigma_a(T)$. It follows that $\sigma_{uw}(S) = \sigma_{ub}(S)$, $\sigma_{uw}(T) = \sigma_{ub}(T)$ and $\sigma(S \oplus T) = \sigma_a(S \oplus T)$. Moreover as $\sigma_{ub}(S \oplus T) = \sigma_{ub}(S) \cup \sigma_{ub}(T)$ we have : $\sigma_{ub}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T) = \sigma_{uw}(S \oplus T)$. This implies that $S \oplus T$ satisfies a-Browder's theorem. According to [19, Theorem 3.2] we deduce that $S \oplus T$ satisfies property (az).

($\Rightarrow$) Suppose that property (az) holds for $S \oplus T$ then $S \oplus T$ satisfies a-Browder's theorem, and by Lemma 3.3, it follows that $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$. □

As a consequence of Lemma 3.3 and Theorem 3.5 we have the next corollary.

Corollary 3.6. *Suppose that $S \in L(X)$ and $T \in L(Y)$ satisfy property (az), and have a shared stable sign index then $S \oplus T$ satisfies property (az).*

We recall that $\sigma_p(S \oplus T) = \sigma_p(S) \cup \sigma_p(T)$ and $\alpha(S \oplus T) = \alpha(S) + \alpha(T)$ for every pair of operators so that $\sigma_p^0(S \oplus T) = \{\lambda \in \sigma_p^0(S) \cup \sigma_p^0(T) \mid \alpha(S - \lambda I) + \alpha(T - \lambda I) < \infty\}$. Moreover, if A and B are bounded subsets of the complex plane $\mathbb{C}$ then $\operatorname{acc}(A \cup B) = \operatorname{acc}(A) \cup \operatorname{acc}(B)$.

Lemma 3.7. If $S \in L(X)$ and $T \in L(Y)$ satisfy $\sigma_p^0(S) = \sigma_p^0(T)$, then
i) $\pi_{00}^a(S \oplus T) = \pi_{00}^a(S) \cap \pi_{00}^a(T)$,
ii) $\pi_{00}(S \oplus T) = \pi_{00}(S) \cap \pi_{00}(T)$,
iii) $p_{00}(S \oplus T) = p_{00}(S) \cap p_{00}(T)$,
iv) $p_{00}^a(S \oplus T) = p_{00}^a(S) \cap p_{00}^a(T)$.
And if $\sigma_p(S) = \sigma_p(T)$, then
v) $\pi_0^a(S \oplus T) = \pi_0^a(S) \cap \pi_0^a(T)$.

Proof. i) As $\sigma_p^0(T) = \sigma_p^0(S)$
then $\sigma_p^0(S \oplus T) = \sigma_p^0(S) = \sigma_p^0(T)$ and so $\pi_{00}^a(S) \cap \rho_a(T) = \pi_{00}^a(T) \cap \rho_a(S) = \emptyset$. Thus we have:

$$\begin{aligned}
\pi_{00}^a(S \oplus T) &= \{\operatorname{iso} \sigma_a(S \oplus T)\} \cap \sigma_p^0(S \oplus T) \\
&= \{\operatorname{iso}[\sigma_a(S) \cup \sigma_a(T)]\} \cap \sigma_p^0(S) \\
&= \{[\sigma_a(S) \cup \sigma_a(T)] \setminus \operatorname{acc}[\sigma_a(S) \cup \sigma_a(T)]\} \cap \sigma_p^0(S)
\end{aligned}$$

$$\begin{aligned}
&= \{[\sigma_a(S) \cup \sigma_a(T)] \setminus [\operatorname{acc}\sigma_a(S) \cup \operatorname{acc}\sigma_a(T)]\} \cap \sigma_p^0(S) \\
&= \{[\operatorname{iso}\sigma_a(S) \cap \rho_a(T)] \cup [\operatorname{iso}\sigma_a(T) \cap \rho_a(S)] \cup [\operatorname{iso}\sigma_a(S) \\
&\quad \cap \operatorname{iso}\sigma_a(T)]\} \cap \sigma_p^0(S) \\
&= [\pi_{00}^a(S) \cap \rho_a(T)] \cup [\pi_{00}^a(T) \cap \rho_a(S)] \cup [\pi_{00}^a(S) \cap \pi_{00}^a(T)] \\
&= \pi_{00}^a(S) \cap \pi_{00}^a(T).
\end{aligned}$$

The proof of ii) goes similarly with i).
iii) Since $p_{00}(S) \cap \rho(T) = p_{00}(T) \cap \rho(S) = \emptyset$ then we have

$$\begin{aligned}
p_{00}(S \oplus T) &= \sigma(S \oplus T) \setminus \sigma_b(S \oplus T) \\
&= [\sigma(S) \cup \sigma(T)] \setminus [\sigma_b(S) \cup \sigma_b(T)] \\
&= \{[\sigma(S) \setminus \sigma_b(S)] \cap \rho(T)\} \cup \{[\sigma(T) \setminus \sigma_b(T)] \cap \rho(S)\} \\
&\quad \cup\{[\sigma(S) \setminus \sigma_b(S)] \cap [\sigma(T) \setminus \sigma_b(T)]\} \\
&= [p_{00}(S) \cap \rho(T)] \cup [p_{00}(T) \cap \rho(S)] \cup [p_{00}(S) \cap p_{00}(T)] \\
&= p_{00}(S) \cap p_{00}(T).
\end{aligned}$$

The proof of iv) goes similarly with iii) and the proof of v) goes similarly with i). □

Example 3.8. Generally the equalities in Lemma 3.7 are not true and the hypothesis assumed on the point spectra are essential as we can see in the following examples:

a) Let $T \in L(\mathbb{C}^n)$ be a non trivial nilpotent operator and consider $R \in L(\ell^2(\mathbb{N}))$ the unilateral right shift. We have $\pi_{00}^a(R \oplus T) = \pi_0^a(R \oplus T) = \{0\}$, but $\pi_{00}^a(R) \cap \pi_{00}^a(T) = \pi_0^a(R) \cap \pi_0^a(T) = \emptyset$. Here $\{0\} = \sigma_p(T) = \sigma_p^0(T) \neq \sigma_p(R) = \sigma_p^0(R) = \emptyset$. On the other hand, if we consider the operator A defined on $L(\ell^2(\mathbb{N}))$ by $A(x_1, x_2, x_3, \ldots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \ldots)$, then it is easily seen that $\pi_{00}(A \oplus T) = \{0\}$, but $\pi_{00}(A) \cap \pi_{00}(T) = \emptyset$. Note also that $\sigma_p^0(T) \neq \sigma_p^0(A) = \emptyset$.

b) Consider the operator K defined on $\ell^2(\mathbb{N})$ by

$$K(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_3}{4}, \ldots),$$

and $T \in L(\mathbb{C}^n)$ a non trivial nilpotent operator. Then $\sigma_p^0(K) = p_{00}(K) = p_{00}^a(K) = \{\frac{1}{n} \,|\, n \in \mathbb{N}^*\}$; $\sigma_p^0(T) = p_{00}(T) = p_{00}^a(T) = \{0\}$ and $p_{00}(K \oplus T) = p_{00}^a(K \oplus T) = \{\frac{1}{n} \,|\, n \in \mathbb{N}^*\}$, but $p_{00}(K) \cap p_{00}(T = p_{00}^a(K) \cap p_{00}^a(T) = \emptyset$.

In the next, we give conditions to ensure the transmission of property (z) from $S \in L(X)$ and $T \in L(Y)$ to their direct sum $S \oplus T$.

Theorem 3.9. *Suppose $S \in L(X)$ and $T \in L(Y)$ satisfy $\sigma_p^0(S) = \sigma_p^0(T)$. If property (z) holds for S and T, then it holds for $S \oplus T$ if and only if $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$.*

Proof. $(\Rightarrow)$ If $S \oplus T$ satisfies property (z), then by [19, Theorem 3.6], it satisfies property (az). As seen in the proof of Theorem 3.5, we conclude that $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$.

$(\Leftarrow)$ Since S and T satisfy property (z), then

$$\begin{aligned}
\sigma(S \oplus T) \setminus \sigma_{uw}(S \oplus T) &= [\sigma(S) \cup \sigma(T)] \setminus [\sigma_{uw}(S) \cup \sigma_{uw}(T)] \\
&= [(\sigma(S) \setminus \sigma_{uw}(S)) \cap \rho(T)] \cup [(\sigma(T) \setminus \sigma_{uw}(T)) \cap \rho(S)] \\
&\quad \cup [(\sigma(S) \setminus \sigma_{uw}(S)) \cap (\sigma(T) \setminus \sigma_{uw}(T))] \\
&= [\pi_{00}^a(S) \cap \rho(T)] \cup [\pi_{00}^a(T) \cap \rho(S)] \cup [\pi_{00}^a(S) \cap \pi_{00}^a(T)].
\end{aligned}$$

As $\pi_{00}^a(T) \cap \rho(S) = \pi_{00}^a(S) \cap \rho(T) = \emptyset$, then $\sigma(S \oplus T) \setminus \sigma_{uw}(S \oplus T) = \pi_{00}^a(S) \cap \pi_{00}^a(T)$. We conclude by Lemma 3.7 that property (z) holds for $S \oplus T$. $\square$

We recall that $T \in L(X)$ is said to be *a-isoloid* if every isolated point of $\sigma_a(T)$ is an eigenvalue of T, and is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T. Clearly, if T is a-isoloid then it is isoloid. However the converse is not true. Consider the following example: let $T = R \oplus Q$ the operator on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, where R is the unilateral right shift and Q is an injective quasi-nilpotent operator. Then $\sigma(T) = D(0,1)$ and $\sigma_a(T) = C(0,1) \cup \{0\}$. Therefore T is isoloid but not a-isoloid.

Lemma 3.10. If $S \in L(X)$ and $T \in L(Y)$ are a-isoloid then

$$\pi_{00}^a(S \oplus T) = [\pi_{00}^a(S) \cap \rho_a(T)] \cup [\pi_{00}^a(T) \cap \rho_a(S)] \cup [\pi_{00}^a(S) \cap \pi_{00}^a(T)].$$

Proof. It is easy to see that the inclusion $\supset$ is always true without condition on S and T.
Suppose now that $\lambda \in \pi_{00}^a(S \oplus T)$, then λ is isolated in $\sigma_a(S \oplus T) = \sigma_a(S) \cup \sigma_a(T)$.
*Case*1: $\lambda \in \sigma_a(S) \setminus \sigma_a(T)$. As $\sigma_p^0(S \oplus T) \subset \sigma_p^0(S) \cup \sigma_p^0(T)$ is always true then $\lambda \in \pi_{00}^a(S) \cap \rho_a(T)$.
*Case*2: $\lambda \in \sigma_a(T) \setminus \sigma_a(S)$. Similarly with case1 we conclude that $\lambda \in \pi_{00}^a(T) \cap \rho_a(S)$.
*Case*3: $\lambda \in \sigma_a(T) \cap \sigma_a(S)$. Then $\lambda \in \operatorname{iso} \sigma_a(S) \cap \operatorname{iso} \sigma_a(T)$ and since S and T are a-isoloid and λ is an eigenvalue of finite multiplicity of $S \oplus T$, then $\lambda \in \pi_{00}^a(S) \cap \pi_{00}^a(T)$. $\square$

In the next theorem, we give a similar characterization of the property (z) for $S \oplus T$ under the hypothesis that S and T are a-isoloid. Notice that the condition "$\sigma_p^0(S) = \sigma_p^0(T)$" assumed in Theorem 3.9, and the condition "S and T being a-isoloid" of Theorem 3.11 below are independent: indeed, the operators T and R defined in Example 3.8 are a-isoloid, but $\sigma_p^0(R) = \emptyset$ and $\sigma_p^0(T) = \{0\}$. Conversely, if we consider the operator A defined on $\ell^2(\mathbb{N})$ by $A(x_1, x_2, x_3, \ldots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \ldots)$, and the Volterra operator on the Banach space $C[0,1]$ defined by $V(f)(x) = \int_0^x f(t)dt$ for all $f \in C[0,1]$. Then $\sigma_p^0(A) = \sigma_p^0(V) = \emptyset$, but A and V are not a-isoloid.

Theorem 3.11. *Suppose that $S \in L(X)$ and $T \in L(Y)$ satisfy property (z) and are a-isoloid, then $S \oplus T$ satisfies property (z) if and only if $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$.*

Proof. $(\Rightarrow)$ See the proof of necessity condition of the precedent theorem.

$(\Leftarrow)$ Since S and T are a-isoloid, then

$$\pi_{00}^a(S \oplus T) = [\pi_{00}^a(S) \cap \rho_a(T)] \cup [\pi_{00}^a(T) \cap \rho_a(S)] \cup [\pi_{00}^a(S) \cap \pi_{00}^a(T)], \text{ see Lemma 3.10.}$$

On the other hand, as S and T satisfy property (z) then $\rho(S) = \rho_a(S)$ and $\rho(T) = \rho_a(T)$. Therefore

$$\begin{aligned}
\sigma(S \oplus T) \setminus \sigma_{uw}(S \oplus T) &= [\sigma(S) \cup \sigma(T)] \setminus [\sigma_{uw}(S) \cup \sigma_{uw}(T)] \\
&= [(\sigma(S) \setminus \sigma_{uw}(S)) \cap \rho(T)] \cup [(\sigma(T) \setminus \sigma_{uw}(T)) \cap \rho(S)] \\
&\quad \cup [(\sigma(S) \setminus \sigma_{uw}(S)) \cap (\sigma(T) \setminus \sigma_{uw}(T))] \\
&= [\pi_{00}^a(S) \cap \rho_a(T)] \cup [\pi_{00}^a(T) \cap \rho_a(S)] \\
&\quad \cup [\pi_{00}^a(S) \cap \pi_{00}^a(T)].
\end{aligned}$$

Hence $\pi_{00}^a(S \oplus T) = \sigma(S \oplus T) \setminus \sigma_{uw}(S \oplus T)$ and $S \oplus T$ satisfies property (z). $\square$

To give the reader a good overview of the subject, we present here another proof of the sufficient condition of Theorem 3.11:

Proof. Since S and T are a-isoloid, then

$$\pi_{00}^a(S \oplus T) = [\pi_{00}^a(S) \cap \rho_a(T)] \cup [\pi_{00}^a(T) \cap \rho_a(S)] \cup [\pi_{00}^a(S) \cap \pi_{00}^a(T)],$$

and since S and T satisfy (z) then from [19, Theorem 3.6], $\pi_{00}^a(T) = p_{00}^a(T)$ and $\pi_{00}^a(S) = p_{00}^a(S)$. So

$$\begin{aligned}
\pi_{00}^a(S \oplus T) &= [p_{00}^a(S) \cap \rho_a(T)] \cup [p_{00}^a(T) \cap \rho_a(S)] \cup [p_{00}^a(S) \cap p_{00}^a(T)] \\
&= [\sigma_a(S) \cup \sigma_a(T)] \setminus [\sigma_{ub}(S) \cup \sigma_{ub}(T)]
\end{aligned}$$

$$= \quad p_{00}^a(S \oplus T).$$

On the other hand, S and T satisfy also property (az) and since by hypothesis $\sigma_{uw}(S\oplus T) = \sigma_{uw}(S)\cup\sigma_{uw}(T)$ then from Theorem 3.5, $S\oplus T$ satisfies property (az). Hence $S \oplus T$ satisfies property (z). □

Example 3.12. The hypothesis "$\sigma_p^0(S) = \sigma_p^0(T)$" in Theorem 3.9 and the hypothesis "S and T are a-isoloid" in Theorem 3.11 are essential. Indeed, let S be the operator defined on $\ell^2(\mathbb{N})$ by $S(x_1, x_2, x_3, \ldots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \ldots)$ and T be a non trivial nilpotent operator on $\mathbb{C}^n$. S and T satisfy property (z) since $\sigma(S) \setminus \sigma_{uw}(S) = \emptyset = \pi_{00}^a(S)$, and $\sigma(T) \setminus \sigma_{uw}(T) = \{0\} = \pi_{00}^a(T)$. But $S \oplus T$ does not satisfy property (z); $\sigma(S\oplus T)\setminus\sigma_{uw}(S\oplus T) = \emptyset$ and $\pi_{00}^a(S\oplus T) = \{0\}$. Note that here $\sigma_p^0(S) = \emptyset$, $\sigma_p^0(T) = \{0\}$ and T is a-isoloid but S isn't.

The following theorem gives similar results to Theorem 3.9 and Theorem 3.11 for property (gz). The proofs go similarly.

Theorem 3.13. *Suppose $S \in L(X)$ and $T \in L(Y)$ satisfy property (gz). If $\sigma_p(S) = \sigma_p(T)$, or S and T are a-isoloid, then $S \oplus T$ satisfies property (gz) if and only if $\sigma_{ubw}(S \oplus T) = \sigma_{ubw}(S) \cup \sigma_{ubw}(T)$.*

Remark 3.14. Note that the hypothesis "$\sigma_p(S) = \sigma_p(T)$ or S and T a-isoloid" assumed in Theorem 3.13 is essential. For example, the operators S and T defined in Remark 3.12 satisfy property (gz), since $\sigma(S)\setminus\sigma_{ubw}(S) = \emptyset = \pi_0^a(S)$ and $\sigma(T)\setminus\sigma_{ubw}(T) = \{0\} = \pi_{00}^a(T)$. But, since $S\oplus T$ does not satisfy property (z), then it does not satisfy property (gz) too. We note also that the conditions "S and T a-isoloid" and "$\sigma_p(S) = \sigma_p(T)$" are independent as seen in the case of Theorem 3.11.

4. Applications

We begin by recalling the definition of the class of (H)-operators, and definitions of some classes of operators which are contained in the class (H).
According to [1], the *quasinilpotent* part $H_0(T)$ of $T \in L(X)$ is defined as the set $H_0(T) = \{x \in X : \lim\limits_{n\to\infty} \|T^n(x)\|^{\frac{1}{n}} = 0\}$. Note that generally, $H_0(T)$ is not closed and from [1, Theorem 2.31], if $H_0(T - \lambda I)$ is closed then T has SVEP at λ. We also recall that T is said to belong to the class (H) if for all $\lambda \in \mathbb{C}$ there exists $p := p(\lambda) \in \mathbb{N}$ such that $H_0(T - \lambda I) = N((T - \lambda I)^p)$, see [1] for more details about this class of operators. Of course, every operator T which belongs the class (H) has SVEP, since $H_0(T - \lambda I)$ is closed, observe also that

$a(T-\lambda I) \leq p(\lambda)$, for every $\lambda \in \mathbb{C}$. The class of operators having the property (H) is large. Obviously, it contains every operator having the property (H_1). Recall that an operator $T \in L(X)$ is said to have the property (H_1) if $H_0(T-\lambda I) = N(T-\lambda I)$ for all $\lambda \in \mathbb{C}$. Although the property (H_1) seems to be strong, the class of operators having the property (H_1) is considerably large. Every *totally paranormal* operator has property (H_1), and in particular every hyponormal operator has property (H_1). Also every *transaloid* operator or *log-hyponormal* has the property (H_1). Multipliers on a semi-simple Banach algebra belong to the class (H_1). Some other operators satisfy property (H); for example *M-hyponormal* operators, *p-hyponormal* operators, *algebraically p-hyponormal* operators, *algebraically M-hyponormal* operators, *subscalar* operators and *generalized scalar* operators. For more details about the definitions and comments about these classes of operators, we refer the reader to [1], [12], [16].

Now, we give an example of an operator of the class (H) which does not satisfy the properties (az) and (z).

Example 4.1. Let T be the hyponormal operator given by the direct sum of the null operator on $\ell^2(\mathbb{N})$ and the unilateral right shift R on $\ell^2(\mathbb{N})$. Then $\sigma(T) = D(0,1)$; $\sigma_a(T) = C(0,1) \cup \{0\}$; $\sigma_{uw}(T) = C(0,1) \cup \{0\}$ and $\pi^a_{00}(T) = p^a_{00}(T) = \emptyset$. It follows that T does not satisfy the properties (z) and (az).

In the following proposition we establish the stability of properties (az) and (z) by the direct sum of two (H)-operators.

Proposition 4.2. *If $S \in L(X)$ and $T \in L(Y)$ are (H)-operators satisfying property (az) (resp., property (z)) then $S \oplus T$ satisfies property (az) (resp., property (z)).*

Proof. Since S and T are (H)-operators, then they have SVEP and so have a shared stable sign index. From Lemma 3.3, we have $\sigma_{uw}(S \oplus T) = \sigma_{uw}(S) \cup \sigma_{uw}(T)$. Thus, if S and T satisfy (az) then from Theorem 3.5, $S \oplus T$ satisfies property (az). If S and T satisfy property (z), then $\sigma(S) = \sigma_a(S)$ and $\sigma(T) = \sigma_a(T)$. This implies (since every (H)-operator is isoloid) that S and T are a-isoloid. Then we conclude by Theorem 3.11 that $S \oplus T$ satisfies property (z). $\square$

In the next proposition, we give a similar result for the class of paranormal operators on Hilbert spaces. We notice that a paranormal operator may not be in the class of (H)-operators, for instance see [4, Example 2.3]. Recall that a bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be paranormal if $||Tx||^2 \leq ||T^2x||\,||x||$, for all $x \in \mathcal{H}$.

Proposition 4.3. *If $S \in L(\mathcal{H})$ and $T \in L(\mathcal{H})$ are paranormal operators satisfying property (az) (resp., property (z)) then $S \oplus T$ satisfies property (az) (resp., property (z)).*

Proof. According to [4], every paranormal operator has the SVEP. Moreover, paranormal operators are isoloid, see [13, Lemma 2.3]. We conclude as seen in the proof of last proposition. □

A bounded linear operator $A \in L(X,Y)$ is said to be *quasi-invertible* if it is injective and has dense range. Two bounded linear operators $T \in L(X)$ and $S \in L(Y)$ on complex Banach spaces X and Y are *quasisimilar* provided there exist quasi-invertible operators $A \in L(X,Y)$ and $B \in L(Y,X)$ such that $AT = SA$ and $BS = TB$.

Proposition 4.4. *If $S \in L(X)$ and $T \in L(Y)$ are quasisimilar operators satisfying property (z) and one of them has the SVEP, then $S \oplus T$ satisfies property (z).*

Proof. Quasisimilarity implies the SVEP for both operators, and it implies that $\sigma_p^0(S) = \sigma_p^0(T)$. We conclude from Theorem 3.9. □

Remark 4.5. It is well known that if $S \in L(X)$ and $T \in L(Y)$ have the SVEP, then from [1, Theorem 2.9] the direct sum $S \oplus T$ has the SVEP. This implies that $\sigma_{ubw}(S \oplus T) = \sigma_{ubw}(S) \cup \sigma_{ubw}(T)$. From Theorem 3.13, we obtain analogous preservation results established in the three last propositions for property (gz).

5. An A-Browder Type Theorem Proof and Counterexamples

In this section we will give a correct proof of [11, Theorem 2.3]. In the original proof in [11], the equality

$$\sigma_p^0(S \oplus T) = \sigma_p^0(S) \cup \sigma_p^0(T)$$

was used and consequently gave the equality:

$$\pi_{00}^a(S \oplus T) = \mathrm{iso}[\sigma_a(S) \cup \sigma_a(T)] \cap [\sigma_p^0(S) \cup \sigma_p^0(T)],$$

(see line 8 of the proof of [11, Theorem 2.3]).

But these last 2 equalities are false, see examples 5.2 and 5.3.

We recall that an operator $T \in L(X)$ satisfies property $(sbaw)$ if $\sigma_a(T) \setminus \sigma_{ubw}(T) = \pi_{00}^a(T)$. In the following theorem, we give the same version of [11, Theorem 2.3] followed by a correct proof.

Theorem 5.1. *Let $S \in L(X)$ and $T \in L(Y)$. If S and T have property (sbaw) and are a-isoloid, then the following assertions are equivalent:*
(i) $S \oplus T$ has property $(sbaw)$;
(ii) $\sigma_{ubw}(S \oplus T) = \sigma_{ubw}(S) \cup \sigma_{ubw}(T)$.

Proof. (i) $\Longrightarrow$ (ii) The property $(sbaw)$ for $S \oplus T$ implies (ii) with no other restriction, since from [10], $S \oplus T$ satisfies generalized a-Browder's theorem, and hence by [11, Lemma 2.1], $\sigma_{ubw}(S \oplus T) = \sigma_{ubw}(S) \cup \sigma_{ubw}(T)$.
(ii) $\Longrightarrow$ (i) Suppose that $\sigma_{ubw}(S \oplus T) = \sigma_{ubw}(S) \cup \sigma_{ubw}(T)$. Since S and T are a-isoloid then

$$\pi_{00}^a(S \oplus T) = [\pi_{00}^a(S) \cap \rho_a(T)] \cup [\pi_{00}^a(T) \cap \rho_a(S)] \cup [\pi_{00}^a(S) \cap \pi_{00}^a(T)],$$

see Lemma 3.10.

As S and T satisfy property $(sbaw)$, then

$$\begin{aligned} \sigma_a(S \oplus T) \setminus \sigma_{ubw}(S \oplus T) &= [\sigma_a(S) \cup \sigma_a(T)] \setminus [\sigma_{ubw}(S) \cup \sigma_{ubw}(T)] \\ &= [\pi_{00}^a(S) \setminus \sigma_a(T)] \cup [\pi_{00}^a(T) \setminus \sigma_a(S)] \qquad (1) \\ &\quad \cup [\pi_{00}^a(S) \cap \pi_{00}^a(T)]. \end{aligned}$$

Hence $\sigma_a(S \oplus T) \setminus \sigma_{ubw}(S \oplus T) = \pi_{00}^a(S \oplus T)$ and so property $(sbaw)$ is satisfied by $S \oplus T$. ☐

We recall that $\sigma_p(S \oplus T) = \sigma_p(S) \cup \sigma_p(T)$ is always true. On the other hand we have always $\sigma_p^0(S \oplus T) \subset \sigma_p^0(S) \cup \sigma_p^0(T)$, but this inclusion may be proper as we can see in the following examples.

Example 5.2. Let $P \in L(\ell^2(\mathbb{N}))$ be defined by

$$P(x_1, x_2, \ldots) = (0, x_2, x_3, x_4, \ldots).$$

Take $S = P$ and $T = I - P$. Then $\sigma_p^0(S) = \{0\}$ and $\sigma_p^0(T) = \{1\}$ but $\sigma_p^0(S \oplus T) = \emptyset$. Note that S and T are a-isoloid.

Example 5.3. Let R be the unilateral right shift on $\ell^2(\mathbb{N})$. We define $S = R \oplus P$ and $T = U \oplus 0$ where P is the projection defined in Example 5.2

and U is defined as follows (see [15]):
$U : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$;

$$U(x) = (0, a_1x_1, a_2x_2, ..., a_kx_k, ...) \ \forall \ x = (x_i) \in \ell^1(\mathbb{N})$$

where (a_i) is a sequence of complex numbers such that $0 < |a_i| \leq 1$ and $\sum_{i=1}^{\infty} |a_i| < \infty$.
Then $S \in L(X)$ and $T \in L(Y)$ where X and Y are the Banach spaces $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ and $\ell^1(\mathbb{N}) \oplus \ell^1(\mathbb{N})$ respectively. And we have:

$$\sigma_a(S) = C(0,1) \cup \{0\}, \ \ \sigma_{ubw}(S) = C(0,1) \text{ and } \sigma_p^0(S) = \{0\} = \pi_{00}^a(S).$$

$$\sigma_a(T) = \{0\} = \sigma_{ubw}(T) \text{ and } \sigma_p^0(T) = \emptyset = \pi_{00}^a(T).$$

It follows that both S and T satisfy property $(sbaw)$ and $\sigma_p^0(S \oplus T) \neq \sigma_p^0(S) \cup \sigma_p^0(T)$, since $\sigma_p^0(S \oplus T) = \emptyset$ and $\sigma_p^0(S) \cup \sigma_p^0(T) = \{0\}$. Note that S and T are also a-isoloid and

$$\emptyset = \pi_{00}^a(S \oplus T) \neq \text{iso}[\sigma_a(S) \cup \sigma_a(T)] \cap [\sigma_p^0(S) \cup \sigma_p^0(T)] = \{0\}.$$

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