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THE CESÀRO χ^3 OF TENSOR PRODUCTS IN ORLICZ SEQUENCE SPACES

N. Rajagopal[1], N. Subramanian[2 §], P. Thirunavukkarasu[3], R. Srikanth[4]

[1,2]Department of Mathematics
SASTRA Deemed to be University
Thanjavur, 613 401, INDIA

[3]P.G. and Research Department of Mathematics
Periyar E.V.R. College (Autonomous)
Tiruchirappalli, 620 023, INDIA

[4]Tata realty-SASTRA
Ramunujan Research Chair for Mathematics
Thanjavur, 613 401, INDIA

Abstract: Let X be a Banach lattice and χ_f^3 be an triple chi Orlicz sequence space associated to an Orlicz function with the Δ_2- condition. In this paper we define the Cesàro χ^3 sequence space $\mathrm{Ces}_p^q\left(\chi_f^3\right)$ generated by a Orlicz sequence space and exhibit some general properties of the spaces.

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1. Introduction

A triple sequence (real or complex) can be defined as a function $x : N \times N \times N \rightarrow R\left(C\right)$, where N, R and C denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence

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[§]Correspondence author

was introduced and investigated at the initial by (see [1]),(see [11]),(see [9]),(see [10]),(see [3]),(see [4]),(see [5]),(see [2]) and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple chi sequence if

$$((m+n+k)! |x_{mnk}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

The space of all triple chi sequences are usually denoted by χ^3.

2. Definitions and Preliminaries

Definition 1. An Orlicz function (see [6]) is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called modulus function.

(see [7])used the idea of Orlicz function to construct Orlicz sequence space. An Orlicz function M is said to satisfy Δ_2- condition for all values u, if there exists $K > 0$ such that $M(2u) \le KM(u), u \ge 0$.

Corollary 2. *Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $t \ge \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v| u - (f_{mnk})(u) : u \ge 0\}, m,n,k = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, (see [8])the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \, as \, m,n,k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} \left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk} \right)$$

is an exteneded real number.

Definition 3. Let X, Y be a real vector space of dimension w, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent;

(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation;

(iii) $\|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \, \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$, $\alpha \in R$;

(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n))$
$= (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$ for $1 \leq p < \infty$; (or)

(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n))$
$:= \sup \{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$,

for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces (see [12]).

The postivity perspective, it is know that the projective tensor and the injective tensor product of two Banach lattices are, in general not Banach lattices.

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in N$, the set of positive integers.Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

Definition 4. (Positive Tensor Products) For Banach lattices X and Y, let $X \otimes Y$ denote the algebraic tensor product of X and Y. For each

$$u = \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} x_{mnk} \otimes y_{mnk} \in X \otimes Y,$$

define $T_u : X^* \to Y$ by

$$T_u(x^*) = \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} x^*(x_{mnk})\, y_{mnk}$$

for each $x^* \in X^+$.

Then injective cone on $X \otimes Y$ is defined to be

$$C_i = \left\{ u \in X \otimes Y : T_u(x^*) \in Y^*, \ \forall x^* \in X^{*+} \right\}.$$

Definition 5. Let $X\bar{\otimes}_i Y$ denote the completion of $X \otimes Y$ with respect $d(.,.)$. Then $X\bar{\otimes}_i Y$ with C_i as its positive cone is a Banach lattice called the positive injective tensor product of X and Y.

The positive cone on $x \otimes Y$ is defined to be

$$C_p = \left\{ \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} x_{mnk} \otimes y_{mnk} : r, s, t \in N^3, x_{mnk} \in X^+, y_{mnk} \in Y^+ \right\}.$$

We define the following spaces: For a Banach metric lattice X, let

$$\chi_f^3(X) = \left\{ \bar{x} = (x_{ij\ell})_{ij\ell} \in N^3 : \left(x^* \left((i+j+\ell)! \left| x_{ij\ell} \right| \right)^{1/i+j+\ell} \right)_{ij\ell} \in \chi^3, \forall x^* \in X^{*+} \right\}.$$

The metric defined to be

$$d(x,y) = sup \left\{ \left\| \left| \left(x^* \left((i+j+\ell)! \left| x_{ij\ell} - y_{ij\ell} \right| \right)^{1/i+j+\ell} \right) : x^* \in B_{X^{*+}} \right\| \right| \right\},$$
$$x = (x_{ij\ell})_{ij\ell} \in \chi_f^3(X).$$

Let

$$\chi_f^3(X) = \left\{ \bar{x} \in \chi_f^3(X) : lim_{i,j,\ell} \left\| \left| \left((i+j+\ell)! \left| \bar{x}_{ij\ell}(>n) \right| \right)^{1/i+j+\ell} \right\| \right| \to 0 \ \ as \ \ i,j,\ell \to \ \infty \right\},$$

with the metric

$$d(x,y) = sup \left\{ \left\| \left| \left((i+j+\ell)! \left| \left(\bar{x}_{ij\ell} - \bar{y}_{ij\ell} \right)(>n) \right| \right)^{1/i+j+\ell} \right\| \right| \forall \left(\bar{x}_{ij\ell} \right) \in \chi_f^3\left(X^{*+} \right) \right\}.$$

3. Notations

For a vector space X, a vector $\bar{x} = (x_{ij\ell})_{ij\ell} \in X^{N^3}$ and $n \in N$, we write $\bar{x}(\leq n)$ is a three dimensional matrix from first term to nth term and remaining term zero. and $\bar{x}(>n)$ is a three dimensional matrix from first term to nth term zero and start with (n+1)th term.

If X is an ordered set, the usual order on X^{N^3} is defined by $\bar{x} = (x_{ij\ell})_{ij\ell} \geq 0 \Longleftrightarrow (x_{ij\ell}) \geq 0$ for each $i, j, \ell \in N^3$. for Banach lattice X, X^* denotes its dual space, B_X denotes its closed unit ball, and X^+ denotes its positive cone.

4. Some New Cesàro Orlicz sequence space of Tensor products in Musielak

The main aim of this article is to introduce the following sequence spaces and examine the topological and algebraic properties of the resulting sequence spaces. Let $f = (f_{mnk})$ be a sequence Orlicz function,

$$\left(\bar{X}, \|(d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0))\|_p\right)$$

be a $p-$ metric space, and consider

$$\mu_{mn}(\bar{x}) = \left\|\left|((i+j+\ell)!\,|\bar{x}_{ij\ell}(>n)|)^{1/i+j+\ell}\right|\right\|$$

and

$$\eta_{mnk}(\bar{x}) = \left\|\left||\bar{x}_{ij\ell}(>n)|^{1/i+j+\ell}\right|\right\|.$$

We define the following sequence spaces as follows:

$$\left[\chi^3_f, \|(d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0))\|_p\right] = \lim_{mnk\to\infty}$$
$$\left\{\sum_m\sum_n\sum_k \left[f_{mnk}\left(\|\mu_{mnk}(\bar{x}), (d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0))\|_p\right)\right] = 0\right\},$$

and

$$\left[\Lambda^3_f, \|(d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0))\|_p\right] = sup$$
$$\left\{\sum_m\sum_n\sum_k \left[f_{mnk}\left(\|\eta_{mnk}(\bar{x}), (d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0))\|_p\right)\right] < \infty\right\}$$

indent Let $p \in [1, \infty)$ and q be a triple chi sequence of positive real numbers such that

$$Q_{ij} = \sum_{m=0}^{i}\sum_{n=0}^{j}\sum_{k=0}^{\ell} q_{mnk}, i, j, \ell \in N^3$$

$$Ces^q_p\left(\chi^3_f\right) = d(x, 0)$$
$$= x \in \chi^3 := lim_{m,n,k\to\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}(\bar{x}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0)\right\|_p\right)\right]^{p_{mnk}}\Bigg)^{1/p_{mnk}}=0.$$

If $q_{mnk}=1$ for all $m,n,k\in N^3$, then $Ces_p^q\left(\chi_f^3\right)$ reduces to $Ces_p\left(\chi_f^3\right)$ and if $f(x)=x$ then $Ces_p^q\left(\chi_f^3\right)$ reduces to $Ces_p^q\left(\chi^3\right)$.

Let $p\in[1,\infty)$ and q be a triple analytic sequence of positive real numbers such that

$$Q_{ij}=\sum_{m=0}^{i}\sum_{n=0}^{j}\sum_{k=0}^{\ell}q_{mnk},i,j,\ell\in N^3$$

$Ces_p^q\left(\Lambda_f^3\right)=d(x,0)=$
$x\in\Lambda^3:=sup\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}$
$\left[f_{mnk}\left(\left\|\eta_{mnk}(\bar{x}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{p_{mnk}/p_{mnk}}$
$<\infty$

If $q_{mnk}=1$ for all $m,n,k\in N^3$, then $Ces_p^q\left(\Lambda_f^3\right)$ reduces to $Ces_p\left(\Lambda_f^3\right)$ and if $f(x)=x$ then $Ces_p^q\left(\Lambda_f^3\right)$ reduces to $Ces_p^q\left(\Lambda^3\right)$.

The space $Ces_p^q\left(\chi_f^3\right)$ is a metric space with the metric
$d(x,y)=inf$
$sup\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}$
$\left[f_{mnk}\left\|\mu_{mnk}(\bar{x})-0,(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0)\right\|_p\right]^{p_{mnk}}\Bigg)^{1/p_{mnk}}\leq 1$

The space $Ces_p^q\left(\Lambda_f^3\right)$ is a metric space with the metric
$d(x,y)=inf$
$sup\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}$
$\left[f_{mnk}\left(\left\|\eta_{mnk}(\bar{x})-0,(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0)\right\|_p\right]\right)^{p_{mnk}}\Bigg)^{1/p_{mnk}}\leq 1.$

5. Main Results

Theorem 6. *Let, the tensor produce of Orlicz sequence space $x,y\in Ces_p^q\left(\chi_f^3\right)$. Then for any $\epsilon>0$ and $L>0$, there exists $\delta>0$ such that $(d(x+y,0),0)^{p_{mnk}}=d(x,0)^{p_{mnk}}+\epsilon$, whenever $d(x,0)^{p_{mnk}}\leq L$ and $d(y,0)^{p_{mnk}}$ $\leq\delta$*

Proof. For any fix $\epsilon > 0$, we have

$$d\left(x+y,0\right)^{p_{mnk}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}+\bar{y}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}$$

$$\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}$$

$$+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{y}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}$$

$$\leq (1-\beta)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}+\beta$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}+\beta$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\frac{\mu_{mnk}\left(\bar{y}\right)}{\beta},\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}$$

$$\leq (1-\beta)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}+\frac{\beta}{2}$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} 2q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}+\frac{\beta}{2}$$
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}2q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\frac{\mu_{mnk}\left(\bar{y}\right)}{\beta},\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}$$
$$\leq d\left(x,0\right)^{p_{mnk}}+\frac{\epsilon}{2}+\left(\frac{2}{\beta}\right)^{p_{mnk}-1}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}$$
$$\leq d\left(x,0\right)^{p_{mnk}}+\frac{\epsilon}{2}+\frac{\epsilon}{2}$$
$$\leq d\left(x,0\right)^{p_{mnk}}+\epsilon. \qquad \square$$

Theorem 7. *For every* $p=(p_{mnk})$,

$$\left[Ces_p^q\left(\Lambda_f^3\right)\right]^{\beta}=\left[Ces_p^q\left(\Lambda_f^3\right)\right]^{\alpha}=\left[Ces_p^q\left(\Lambda_f^3\right)\right]^{\gamma}=\left[Ces_p^q\left(\eta_f^3\right)\right]^{\beta},$$

where

$$\left[Ces_p^q\left(\eta_f^3\right)\right]=\bigcap_{N\in N-\{1\}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right)N^{m+n+k/p_{mnk}},\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{p_{mnk}}$$
$$<\infty.$$

Proof. First we will show that

$$\left[Ces_p^q\left(\eta_f^3\right)\right]\subset\left[Ces_p^q\left(\Lambda_f^3\right)\right]^{\beta}.$$

Let $x\in\left[Ces_p^q\left(\eta_f^3\right)\right]$ and $y\in\left[Ces_p^q\left(\Lambda_f^3\right)\right]^{\beta}$. Then we can find a positive integer N such that

$$\left(\left|\mu\left(\bar{y}\right)\right|\right)<max1,sup_{m,n,k\geq 1}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right\|_p\right]\right)^{p_{mnk}}\right)^{1/p_{mnk}}<N,$$

for all m, n, k.

Hence we may write
$\left|\sum_{m,n,k} \mu_{mnk}(\bar{x}\bar{y})\right| \leq \sum_{m,n,k} |\mu_{mnk}(\bar{x}\bar{y})| \leq \sum_{m,n,k} \mu_{mnk}(\bar{x}) N^{m+n+k}$.
Since $x \in Ces_p^q\left(\eta_f^3\right)$. the series on the right side of the above inequality is convergent, whence $x \in Ces_p^q\left(\Lambda_f^3\right)$. Hence $\left[Ces_p^q\left(\eta_f^3\right)\right] \subset \left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta$

indent Now we show that $\left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta \subset \left[Ces_p^q\left(\eta_f^3\right)\right]$

For this, let $x \in \left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta$ and suppose that $x \notin \left[Ces_p^q\left(\Lambda_f^3\right)\right]$. Then there exists a positive integer $N > 1$ such that $\mu_{mnk}(\bar{x}) N^{m+n+k} = \infty$.

If we define $\mu_{mnk}(\bar{y}) = N^{m+n+k} Sgn \mu_{mnk}(\bar{x})$ $m, n, k = 1, 2, \cdots$, then

$$\mu_{mnk}(\bar{y}) \in \left[Ces_p^q\left(\Lambda_f^3\right)\right].$$

But, since

$$\left|\sum_{m,n,k} \mu_{mnk}(\bar{x}\bar{y})\right| = \sum_{mnk} |\mu_{mnk}(\bar{x}\bar{y})| = \sum_{m,n,k} |\mu_{mnk}(\bar{x})| N^{m+n+k} = \infty,$$

we get $x \notin \left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta$, which contradicts to the assumption

$$x \in \left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta.$$

Therefore $x \in \left[Ces_p^q\left(\eta_f^3\right)\right]$. Therefore $\left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta = \left[Ces_p^q\left(\eta_f^3\right)\right]$. (ii) and (iii) can be shown in a similar way of (i). Therefore we omit it. □

Theorem 8. *Let $p = (p_{mnk})$ be a tensor products of Cesàro space of triple analytic Orlicz sequence of strictly positive real numbers p_{mnk}. Then:*

(i) $Ces_p^q\left(\Lambda_f^3\right)$ is a paranormed space with

$$g(x) = sup \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{1}{Q_{ij\ell}} \sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$
$$\left[f_{mnk}\left\|\mu_{mnk}(\bar{x}), (d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0)\right\|_p^{1/M}\right]\Big) \cdots\cdots *$$

if and only if $h = inf p_{mnk} > 0$*, where* $M = max(1, H)$ *and* $H = supp_{mnk}$.

(ii) $Ces_p^q\left(\Lambda_f^3\right)$ *is a complete paranormed linear metric space if the condition* p *in* $*$ *is satisfied.*

Proof. (i) Sufficiency. Let $h > 0$. It is trivial that $g(\theta) = 0$ and $g(-x) = g(x)$.

The inequality $g(x+y) \leq g(x) + g(y)$ follows from the inequality $*$, since $p_{mnk}/M \leq 1$ for all positive integers m, n, k. We also may write $g(\lambda x) \leq max\left(|\lambda|, |\lambda|^{h/M}\right) g(x)$, since $|\lambda|^{p_{mnk}} \leq max\left(|\lambda|^h, |\lambda|^M\right)$ for all positive integers m, n, k and for any $\lambda \in C$, the set of complex numbers. Using this inequality, it can be proved that $\lambda x \to \theta$, when x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

Necessity. Let $Ces_p^q\left(\Lambda_f^3\right)$ be a paranormed space with the paranorm

$$g(x) = sup \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{1}{Q_{ij\ell}} \sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$
$$\left.\left[f_{mnk}\left(\left\|\mu_{mnk}(\bar{x}), (d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0)\right\|_p\right)\right]^{1/M}\right)$$

and suppose that $h = 0$. Since $|\lambda|^{p_{mnk}/M} \leq |\lambda|^{h/M} = 1$ for all positive integers m, n, k and $\lambda \in C$ such that $0 < |\lambda| \leq 1$, we have

$$sup \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{1}{Q_{ij\ell}} \sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$
$$\left.\left[f_{mnk}\left(\left\|\mu_{mnk}(\bar{x}), (d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0)\right\|_p\right)\right]^{1/M}\right) = 1.$$

Hence it follows that

$$g(\lambda x) = sup \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{1}{Q_{ij\ell}} \sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$
$$\left.\left[f_{mnk}\left(\left\|\mu_{mnk}(\bar{x}), (d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0)\right\|_p\right)\right]^{1/M}\right) = 1$$

for $x = (\alpha) \in Ces_p^q\left(\Lambda_f^3\right)$ as $\lambda \to 0$. But this contradicts the assumption $Ces_p^q\left(\Lambda_f^3\right)$ is a paranormed space with $g(x)$. □

Proof. (ii) The proof is clear. □

Corollary 9. *$Ces_p^q\left(\Lambda_f^3\right)$ is a complete paranormed space with the natural paranorm if and only if $Ces_p^q\left(\Lambda_f^3\right) = Ces^q\left(\Lambda_f^3\right)$.*

Theorem 10. *For every $p = (p_{mnk})$, then $Ces_p^q\left(\eta_f^3\right) \subset \left[Ces_p^q\left(\chi_f^3\right)\right]^\beta \underset{\neq}{\subset} Ces_p^q\left(\Lambda_f^3\right)$*

Proof. **Case (i):** First we show that $Ces_p^q\left(\eta_f^3\right) \subset \left[Ces_p^q\left(\chi_f^3\right)\right]^\beta$. We know that $\left[Ces_p^q\left(\chi_f^3\right)\right] \subset Ces_p^q\left(\Lambda_f^3\right)$.
$\left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta \subset \left[Ces_p^q\left(\chi_f^3\right)\right]^\beta$. But $\left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta = Ces_p^q\left(\eta_f^3\right)$, by Proposition 5.2. Therefore

$$Ces_p^q\left(\eta_f^3\right) \subset \left[Ces_p^q\left(\chi_f^3\right)\right]^\beta \tag{1}$$

Case (2): Now we show that $\left[Ces_p^q\left(\chi_f^3\right)\right]^\beta \underset{\neq}{\subset} Ces_p^q\left(\Lambda_f^3\right)$.
Let $y = \mu_{mnk}(\bar{y})$ be an arbitrary point $\left[Ces_p^q\left(\chi_f^3\right)\right]^\beta$. If y is not $Ces_p^q\left(\Lambda_f^3\right)$, then for each natural number d, we can find an index $m_d n_d k_d$ such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{m_d n_d k_d k_d} \left[f_{mn}\left(\left\|\mu_{m_d n_d k_d}(\bar{x}), \left(d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0)\right\|_p\right)\right]^{1/M}\right) > d,$$

$$(1, 2, 3, \cdots).$$

Define $x = \{x_{mnk}\}$ by

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{m_d n_d k_d} \left[f_{mnk}\left(\left\|\mu_{m_d n_d k_d}(\bar{x}), \left(d(\bar{x}_1, 0), d(\bar{x}_2, 0), \cdots, d(\bar{x}_{n-1}, 0)\right\|_p\right)\right]^{1/M}\right) > d,$$

for $(mnk) = (m_d n_d k_d)$ for some $d \in N^3$; and

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{m_dn_dk_d}$$
$$\left[f_{mnk}\left(\left\|\mu_{m_dn_dk_d}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right\|_p\right)\right]^{1/M}\right)=0,$$

otherwise.

Then x is $Ces_p^q\left(\chi_f^3\right)$, but for infinitely mnk,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M}>1**,$$

Consdier the sequence $z=\{z_{mnk}\}$, where $\mu_{mnk}\left(\bar{z}\right)=\mu_{mnk}\left(\bar{x}\right)-s$ with

$$\begin{aligned}s=&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}\\&\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M}\\&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}\\&\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{z}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M}\\=&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}\\&\left[f_{mnk}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M}.\end{aligned}$$

The z is a point of $Ces_p^q\left(\chi_f^3\right)$. Also

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{z}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M}=0.$$

Hence z is in $Ces_p^q\left(\chi_f^3\right)$. But, by the equation $**$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\bar{z}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M}$$

does not converge. Hence $\mu_{mnk}\left(\bar{x}\bar{y}\right)$ diverges. Thus, the sequence $\mu_{mnk}\left(\bar{y}\right)$ would not be $\left[Ces_p^q\left(\chi_f^3\right)\right]^{\beta}$.
This contradiction proves that

$$\left[Ces_p^q\left(\chi_f^3\right)\right]^{\beta}\subset Ces_p^q\left(\Lambda_f^3\right) \tag{2}$$

If we now choose $f=id$, where id is the identity and
$\frac{1}{Q_{1j\ell}}\left(q_{1nk}\mu_{1nk}\left(\bar{y}\right)\right)=\frac{1}{Q_{1j\ell}}\left(q_{1nk}\mu_{1nk}\left(\bar{x}\right)\right)$ and
$\frac{1}{Q_{1j\ell}}\left(q_{1nk}\mu_{mnk}\left(\bar{y}\right)\right)=\frac{1}{Q_{1j\ell}}\left(q_{1n\ell}\mu_{mnk}\left(\bar{x}\right)\right)=0\ (m,i>1)$ for all n,j,k,ℓ
then obviously $x\in Ces_p^q\left(\chi_f^3\right)$ and $y\in Ces_p^q\left(\Lambda_f^3\right)$, but

$$\sum\sum\sum\mu_{mnk}\left(\bar{x}\bar{y}\right)=\infty. Hence\,\mu_{mnk}\left(\bar{y}\right)\notin\left[Ces_p^q\left(\chi_f^3\right)\right]^{\beta} \tag{3}$$

From (2) and (3), we are granted $\left[Ces_p^q\left(\chi_f^3\right)\right]^{\beta}\overset{\subset}{\neq}Ces_p^q\left(\Lambda_f^3\right)$. □

Theorem 11. *In tensor product of Orlicz sequence space of $Ces_p^q\left(\chi_f^3\right)$ weak convergence does not imply strongly convergence.*

Proof. Assume that weak convergence implies strong convergence $Ces_p^q\left(\chi_f^3\right)$. Then we would have $\left[Ces_p^q\left(\chi_f^3\right)\right]^{\beta\beta}=Ces_p^q\left(\chi_f^3\right)$ [see Wilansky]. But

$$\left[Ces_p^q\left(\chi_f^3\right)\right]^{\beta\beta}\overset{\subset}{\neq}\left[Ces_p^q\left(\Lambda_f^3\right)\right]^{\beta}=Ces_p^q\left(\eta_f^3\right).$$

Thus $\left[Ces_p^q\left(\chi_f^3\right)\right]^{\beta\beta}\neq Ces_p^q\left(\chi_f^3\right)$. Hence tensor product of Orlicz sequence space is weak convergence does not imply strong convergence in $Ces_p^q\left(\chi_f^3\right)$. □

Theorem 12. *Let tensor product of Orlicz sequence space of f satisfies the Δ_2- condition. Then $Ces_p^q\left(\chi^3\right)\subset Ces_p^q\left(\chi_f^3\right)$.*

Proof. Let

$$x \in Ces_p^q \left(\chi^3\right) \tag{4}$$

Then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\bar{z}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M} \leq \epsilon$$

for sufficiently large m, n, k and every $\epsilon > 0$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\bar{z}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M} \leq f\left(\epsilon\right)$$

(because the tensor product of Orlicz sequence space of f is non-decreasing)

$$\Rightarrow \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\bar{z}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M} \leq Kf\left(\epsilon\right) < \epsilon$$

(by the tensor product of Orlicz sequence space of Δ_2- condition, for some $K > 0$ and by defining $f\left(\epsilon\right) < \frac{\epsilon}{K}$)

$$lim_{m,n\to\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$
$$\left[f_{mnk}\left(\left\|\mu_{mn}\left(\bar{x}\bar{z}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M} = 0.$$

Hence

$$x \in Ces_p^q \left(\chi_f^3\right). \tag{5}$$

From (4) and (5) we get $Ces_p^q \left(\chi^3\right) \subset Ces_p^q \left(\chi_f^3\right)$. □

Theorem 13. $\left[Ces_p^q\left(\Lambda_f^3\right)\right]^{\beta} \overset{\subset}{\neq} Ces_p^q\left(\chi_f^3\right)$.

Proof. Let $\mu_{mn}(\bar{x}) \in \left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta$

$$\sum\sum\sum \mu_{mnk}(\bar{x}\bar{y}) < \infty \forall \mu_{mnk}(\bar{y}) \in \left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta \tag{6}$$

Assume that $\mu_{mnk}(\bar{x}) \notin Ces_p^q\left(\chi_f^3\right)$. Then there exist a sequence positive integers

$$\mu_{m_r n_r k_r}(\bar{x}) > \frac{1}{\left((m_r+n_r+k_r!\,3)^{(m_r+n_r+k_r)}\right)}, (r = 1, 2, 3, \cdots)$$

Take

$$\mu_{m_r n_r k_r}(\bar{y}) = (3\ (m_r + n_r + k_r)!)^{m_r+n_r+k_r} \text{ for } r = 1, 2, 3, \cdots$$

$$\mu_{m_r n_r k_r}(\bar{y}) = 0 \text{ otherwise.}$$

Then $\mu_{mnk}(\bar{y}) \in \left[Ces_p^q\left(\Lambda_f^3\right)\right]$. But
$\sum\sum\sum \mu_{mnk}(\bar{x}\bar{y}) > 1 + 1 + 1 + \cdots$.
We know that the infinite series $1+1+1+\cdots$ diverges. Hence $\sum\sum\sum \mu_{mnk}(\bar{x}\bar{y})$ diverges. This contradicts (6). Hence $\mu_{mnk}(\bar{x}) \in Ces_p^q\left(\chi_f^3\right)$. Therefore

$$\left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta \subset Ces_p^q\left(\chi_f^3\right) \tag{7}$$

If we now choose $p = (p_{mnk})$ is a constant $f = id$, where id is the identity and $\frac{1}{Q_{1j\ell}}\left(q_{1nk}\mu_{1nk}(\bar{y})\right) = \frac{1}{Q_{1j\ell}}\left(q_{1nk}\mu_{1nk}(\bar{x})\right)$ and $\frac{1}{Q_{1j\ell}}\left(q_{1nk}\mu_{mnk}(\bar{y})\right) = \frac{1}{Q_{1j\ell}}\left(q_{1nk}\mu_{mnk}(\bar{x})\right) = 0\ (m, i > 1)$ for all n, j, k, ℓ then obviously $\mu_{mnk}(\bar{x}) \in Ces_p^q\left(\chi_f^3\right)$ and $\mu_{mnk}(\bar{y}) \in Ces_p^q\left(\Lambda_f^3\right)$, but

$$\sum\sum\sum \mu_{mnk}(\bar{x}\bar{y}) = \infty. Hence\, \mu_{mnk}(\bar{y}) \notin \left[Ces_p^q\left(\chi_f^3\right)\right]^\beta \tag{8}$$

From (7) and (8) we are granted $\left[Ces_p^q\left(\Lambda_f^3\right)\right]^\beta \overset{\subset}{\neq} Ces_p^q\left(\chi_f^3\right)$. □

Theorem 14. *Let* $\left(Ces_p^q\left(\chi_f^3\right)\right)^*$ *denote the dual space of* $Ces_p^q\left(\chi_f^3\right)$. *Then we have* $\left(Ces_p^q\left(\chi_f^3\right)\right)^* = Ces_p^q\left(\Lambda_f^3\right)$.

Proof. We recall that

$$x = \Im_{mnk} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...\frac{1}{(m+n+k)!} & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix}$$

with $\frac{1}{(m+n+k)!}$ in the $(m,n,k)^{th}$ position and zero other wise, with $x = \Im_{mnk}$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{Q_{ij\ell}}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} q_{mnk}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}\left(\bar{x}\bar{z}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]^{1/M}$$

$$= \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...\left(\frac{(m+n+k)!}{(m+n+k)!}\right) & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...1^{1/m+n+k} & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix},$$

which is a $Ces_p^q\left(\chi_f^3\right)$ sequence . Hence $\Im_{mnk} \in Ces_p^q\left(\chi_f^3\right)$. Let us take $f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{mnk}\left(\bar{x}\bar{y}\right)$ with $\mu_{mnk}\left(\bar{x}\right) \in Ces_p^q\left(\chi_f^3\right)$. and $f \in \left(Ces_p^q\left(\chi_f^3\right)\right)^*$. Take $x = \mu_{mnk}\left(\bar{x}\right) = \Im_{mnk} \in Ces_p^q\left(\chi_f^3\right)$. Then

$$\mu_{mnk}\left(\bar{y}\right) \leq \|f\|\, d\left(\Im_{mnk},0\right) < \infty \text{ for each } m,n,k.$$

Thus $\mu_{mnk}\left(\bar{y}\right)$ is a bounded sequence and hence an Cesàro triple analytic sequence of modulus. In other words $\mu_{mnk}\left(\bar{y}\right) \in Ces_p^q\left(\Lambda_f^3\right)$. Therefore

$$\left(Ces_p^q\left(\chi_f^3\right)\right)^* = Ces_p^q\left(\Lambda_f^3\right).$$

□

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