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A REMARK ON HARDY-RAMANUJAN'S APPROXIMATION OF DIVISOR FUNCTIONS

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Abstract: We obtain asymptotic estimates for the partial sums $\sum_{1<n\leq x} a_n$ for $a_n =$ (i) $\frac{(logn)^{log2}}{d(n)}$ (ii) $\frac{2^{\omega(n)}}{(logn)^{log2}}$ (iii) $\frac{1}{d(n)}$ (due to Ramanujan-Wilson). These are based on Delange's Tauberian theorem. We deduce that the normal order of Hardy-Ramanujan's approximation to $d(n)$ is more often lower than $d(n)$ as asserted by them.

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1. Introduction

In his interesting lecture on "Round Numbers" ([2], Ch3), G.H.Hardy describes how he and S.Ramanujan arrived at the approximations $\omega(n) \approx loglogn$ "usually" and $d(n) \approx (logn)^{log2}$ "usually". These were later expressed probabilistically by P.Turan([3]). Our Proposition below supports their claim that "Round numbers are rare": most numbers tend to have a small number of prime divisors.

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We first recall the notations used here. If $n = p_1^{a_1} p_2^{a_2} \dots p_\nu^{a_\nu}$ is the Prime factorization of n, then $\omega(n) = \nu$, $\Omega(n) = \sum_{i=1}^{\nu} a_i$ and $d(n) = \prod_{i=1}^{\nu}(1+a_i)$. Thus ω counts the number of prime divisors, $d(n)$ is the number of divisors and $\Omega(n) =$ number of prime divisors with multiplicity. Since $2 \leq (1 + a_i) \leq 2^{a_i}$ for all i, we have $2^{\omega(n)} \leq d(n) \leq 2^{\Omega(n)}$.

Recall Density of a subset $S = \lim_{x\to\infty} \frac{|n \leq x/n \in S|}{x}$.
$f \sim g$ means f is asymptotic to g.

Definition 1. ([3],p299) An arithmetic function $f(n)$ is said to have normal order $g(n)$ if given $\epsilon > 0$ $|\frac{f(n)}{g(n)} - 1| < \epsilon$ for n in a set of density 1.

Definition 2. A subset $A \subset \mathbb{N}$ is said to be negligible if its density is 0.

Example 1. Both $A = Squares$ and $A = Primes$ have density 0. But $\sum \frac{1}{n^2} < \infty$ and $\sum \frac{1}{p} = \infty$.

Theorem 1. *([2], Ch3) (a)* $\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} d(n) \sim logx$ *and so on average the divisor function* $d(n) \sim logn$*.(Dirichlet's Theorem, [1] Th3.3, p57)*
(b) $\omega(n)$ *has normal order* $loglogn$
(c)$\Omega(n) - \omega(n)$ *is increasing only on a negligible set*
(d) $\Omega(n)$ *has normal order* $loglogn$

On the basis of this theorem, Hardy-Ramanujan stated that $d(n)$ is "usually $2^{loglogn} = (logn)^{log2}$" ([2],(3.10.4), p55). This is suggested by the common behaviour of $\omega(n)$ and $\Omega(n)$ ((b),(d) of the theorem above): $2^{\omega(n)} \leq d(n) \leq 2^{\Omega(n)}$ gives $(log2)\omega(n) \leq logd(n) \leq (log2)\Omega(n)$

Proposition 1. *(a)* $\sum_{n \leq x} \frac{(logn)^{log2}}{d(n)} \sim cx(logx)^{log2-0.5} = cx(logx)^{0.19\dots}$
(b) $\sum_{2<n \leq x} \frac{2^{\omega(n)}}{(logn)^{log2}} \sim c_1 x(logx)^{1-log2} = c_1 x(logx)^{0.31\dots}$
Note that the order in (b) is greater than in (a).

Proof. (a) It is known that (see Appendix) the following estimate of Ramanujan-Wilson holds

$$\frac{1}{x}\sum_{n \leq x} \frac{1}{d(n)} \sim \frac{c}{\sqrt{logx}}$$

By Abel's Summation Formula (see Appendix)

$$\sum_{n\leq x}\frac{(logn)^{log2}}{d(n)}\sim cx\frac{(logx)^{log2}}{(logx)^{0.5}}=cx(logx)^{0.19...}$$

since $log2\approx 0.69\ldots$

(b) One knows that $\sum\limits_{n\leq x}2^{\omega(n)}\sim kxlogx$, applying Delange's Tauberian Theorem([3], Th15, p243-244) to $\sum\frac{2^{\omega(n)}}{n^s}=\frac{\zeta^2(s)}{\zeta(2s)}$
We use the same approach as in (a) (Abel's Summation) to obtain the result (b). □

Remark 1. We may interpret (a), (b) of Proposition 1 to mean that the normal order of $d(n)$ is below $2^{\omega(n)}$ $(\leq d(n))$ more often than it is above $d(n)$. So most n fail to have $2^{\omega(n)}$ divisors i.e.,most n are not round numbers ([2], p48).

Appendix:
Abel's Summation Formula: $\sum\limits_{y<n\leq x}a(n)f(n)=A(x)f(x)-A(y)f(y)-\int\limits_y^x A(t)f'(t)dt$ where $\sum\limits_{n\leq x}a(n)=A(x)$, $f(x)$ is a differentiable function.([1],p77, Thm 4.2)

Lemma([3],p339) Let g be a multiplicative function with $|g(n)|\leq 1$. Then $\sum\limits_{n=1}^{\infty}\frac{g(n)}{n^s}=(1+D(s))F_1(s)J(s)$ with $D(s)=\sum_{\nu=1}^{\infty}\frac{g(2^\nu)}{2^{\nu s}}$, $F_1(s)=exp\sum\limits_{p>2}\frac{g(p)}{p^s}$, $J(s)$ analytic for $Re\ s>\frac{1}{2}$
Eg1: $g(n)=\frac{1}{2^{\omega(n)}}$. Then $g(2^\nu)=\frac{1}{2}$ for all ν, since $\omega(p)=1$,$g(p)=\frac{1}{2^1}$
Eg2: $g(n)=\frac{1}{d(n)}$, $d(p)=2$, $g(p)=\frac{1}{2}$, $g(2^\nu)=\frac{1}{(\nu+1)}$
Eg3: $g(n)=\frac{1}{2^{\Omega(n)}}$, $\Omega(p)=1$, $g(p)=\frac{1}{2}$, $g(2^\nu)=\frac{1}{\nu}$

In each of these cases, $D(1)$ is a finite constant. $F_1(s)=\frac{k}{(s-1)^{\frac{-1}{2}+1}}$ in view of $\sum\limits_p p^{-s}\sim log(\frac{1}{s-1})$ near $s=1$. Hence $\frac{1}{x}\sum\limits_{n\leq x}g(n)\sim\frac{k}{\sqrt{logx}}$ in each case (Tauberian Theorem of Delange [3],Thm 15, p243-244).

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