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A VARIANT OF BLACK-SCHOLES EQUATION WHICH HAS A FORM OF HEAT EQUATION AND, ITS SOLUTION BY LAPLACE TRANSFORM

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Abstract: We have investigated a variant of Black-Scholes equation which has the form of heat equation with a special initial condition. The proposed method is done by Laplace transform without using formulas of heat equation.

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1. Introduction

The Black-Scholes equation gives a theoretical estimate of the price of European call and put options which are exercised only at expiration. It is a known fact that the equation is fairly closed to the observed price by many empirical tests, although there are some discrepancies with real option price, and lately, the equation is used in predicting the stock price as well.

To begin with, let us see the form of the equation. The form is

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 v}{\partial s^2}+rs\frac{\partial v}{\partial s}-rv=0$$

where v is the value of option as a function of stock price s and time t, r is the risk-free interest rate, and σ is the volatility of the stock. If the conditions

$$v(0,t)=0,\ \lim_{S\to\infty} v(S,t)=0$$

and $v(S,t)=\max(S-K,\ 0)$ are given for the strike price K, then we call the

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equation with the conditions as the Black-Scholes terminal value problem.

The [8] is dealt with Black-Scholes equation by Laplace homotopy perturbation method, and the [10] is dealt with the modified Laplace transformation method. These previous researches are very impressive and exquisite, and while, this proposed method is easier and simpler than the existing researches. The used tool is an integral transform, and an outline with respect to integral transforms appears well in [1-6, 9].

Normally, Black-Scholes equation can be changed to

$$v_\tau = v_{xx} + (K-1)v_x - Kv$$

by a proper replacement as a variant of the equation, and if we put $v = e^{ax+b\tau}u(x,\tau)$ for unknown a and b, then we obtain a form of heat equation as

$$u_\tau = u_{xx}$$

with the condition of

$$u(x,0) = \max(e^{\frac{k+1}{2}}x - e^{\frac{k-1}{2}}x\ ,0)$$

for K is the strike price.

In this article, we have checked the above form of Black-Scholes equation by using Laplace transform.

2. A variant of Black-Scholes equation which has a form of heat equation, and its solution by Laplace transform

We are trying to do a check about a variant of Black-Scholes equation and its solution, and the tool used here is Laplace transform. We recall that the form of the equation is

$$v_t + \frac{1}{2}\sigma^2 s^2\ v_{SS} + rs\ v_s - rv = 0 \tag{1}$$

where v is the value of option as a function of stock price s and time t, r is the risk-free interest rate, and σ is the volatility of the stock.

To begin with, let us see the conversion of Black-Scholes equation to heat equation.

Let us put τ and x as new time and new price, and put

$$t = T - \frac{2\tau}{\sigma^2},\ S = Ke^x$$

provided $v(S,t) = Kv(s,\tau)$ for K is the strike price. Then we can rewrite the above equality as

$$\tau = \frac{\sigma^2}{2}(T-t),\ x = \ln(\frac{S}{K}).$$

Since $v_t = Kv_\tau \cdot (-\frac{\sigma^2}{2})$, $v_S = Kv_x \cdot\ 1/S$ and

$$v_{SS} = (v_S)_S = Kv_x(-\frac{1}{S^2}) + Kv_{xx}\frac{1}{S^2},$$

substituting v_t, v_s and v_{ss} into Black-Scholes equation, we have

$$v_\tau = v_{xx} + (k-1)v_x - kv \tag{2}$$

for $2r/\sigma^2 = k$. In the above equation, we note that $v(S,t) = Kv(x,\tau)$ gives

$$Kv(x,0) = v(S,T) = \max(S-K,\ 0) = \max(Ke^x - K,\ 0).$$

Thus $v(x,0) = \max(e^x - 1,\ 0)$. The above equation (2) can be simplified by the

$$v = e^{ax+b\tau}u(x,\tau)$$

for unknown a and b. Substituting v_τ, v_x and v_{xx} into the equation (2), and organizing the equation, we have

$$u_\tau = u_{xx} + (2a+k-1)u_x + \{a^2 + (k-1)a - k - b\}u.$$

Choosing

$$a = \frac{1-k}{2},\ b = -\frac{(k+1)^2}{4},$$

the equation become

$$u_\tau = u_{xx}. \tag{3}$$

Let us change the initial condition as

$$u(x,0) = e^{-\frac{1-k}{2}x} \cdot v(x,0) = e^{\frac{k-1}{2}} \max(e^x - 1, 0)$$

$$\max(e^{\frac{k+1}{2}}x - e^{\frac{k-1}{2}}x\ ,0) = e^{\frac{k+1}{2}}x - e^{\frac{k-1}{2}}x. \tag{4}$$

From now on, we would like to solve the equation (3) with the condition (4) by Laplace transform without using formulas of heat equation.

Theorem 1. *The Laplace transform of the equation (3) with the condition (4) can be represented as*

$$u(x,\tau) = \pounds^{-1}(U_h + U_p)$$

where,

$$U_h(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x}$$

and

$$U_p(x,s) = \frac{4}{(k-1)^2 - 2s(k-1)} e^{\frac{k-1}{2}} x - \frac{4}{(k+1)^2 - 2s(k+1)} e^{\frac{k+1}{2}} x$$

for $\pounds(u) = U$. *Alternately,* U_p *can be represented by the integral form of*

$$U_p(x,s) = \frac{e^{\sqrt{s}x}}{2\sqrt{s}} \int e^{(\frac{k-1}{2} - \sqrt{s})x} - e^{(\frac{k+1}{2} - \sqrt{s})x}\, dx$$

$$-\frac{e^{-\sqrt{s}x}}{2\sqrt{s}} \int e^{(\frac{k-1}{2} + \sqrt{s})x} - e^{(\frac{k+1}{2} + \sqrt{s})x}\, dx.$$

Proof. Let us take Laplace transform on both sides, and put $\pounds(u) = U$. Then

$$\pounds(u_\tau) = \int_0^\infty u_\tau e^{-s\tau} d\tau = ue^{-s\tau}]_0^\infty + s\int_0^\infty ue^{-s\tau} d\tau$$
$$= -u(x,0) + sU$$

and $\pounds(u_{xx}) = U_{xx}$. Hence the equation (3) becomes

$$-u(x,0) + sU = U_{xx}.$$

Organizing this equality, we have

$$U_{xx} - sU = -u(x,0) = e^{\frac{k-1}{2}} x - e^{\frac{k+1}{2}} x. \tag{5}$$

First, let us find a general solution of homogeneous ODE $U_{xx} - sU = 0$. Since this equation has a constant coefficients, its solution $U_h(x,s)$ is

$$U_h(x,s) = A(s)e^{\sqrt{s}x} + B(s)e^{-\sqrt{s}x}.$$

Next, let us find a particular solution $U_p(x,s)$. By the method of undetermined coefficients[7, p78], let us take U_p as

$$U = U_p = c_1 e^{\frac{k-1}{2}} x + c_2 e^{\frac{k+1}{2}} x.$$

Then we have

$$U' = c_1 \frac{k-1}{2}\ e^{\frac{k-1}{2}} x + c_2 \frac{k+1}{2}\ e^{\frac{k+1}{2}} x$$

and

$$U'' = c_1 (\frac{k-1}{2})^2\ e^{\frac{k-1}{2}} x + c_2 (\frac{k+1}{2})^2\ e^{\frac{k+1}{2}} x.$$

Substituting U-terms into the equation (5), and collecting $e^{\frac{k-1}{2}} x$ and $e^{\frac{k+1}{2}} x$, we would have

$$\{(\frac{k-1}{2})^2 c_1 - \frac{k-1}{2} c_1 s - 1\} e^{\frac{k-1}{2}} x + \{(\frac{k+1}{2})^2 c_2 - \frac{k+1}{2} c_2 s + 1\} e^{\frac{k+1}{2}} x = 0$$

for all $s > 0$. Since the exponential functions are positive,

$$\{(\frac{k-1}{2})^2 c_1 - \frac{k-1}{2} c_1 s - 1\} = 0$$

and

$$\{(\frac{k+1}{2})^2 c_2 - \frac{k+1}{2} c_2 s + 1\} = 0.$$

Thus

$$c_1 = \frac{4}{(k-1)^2 - 2s(k-1)}$$

and

$$c_2 = -\frac{4}{(k+1)^2 - 2s(k+1)},$$

and so, we have

$$U_p(x,s) = \frac{4}{(k-1)^2 - 2s(k-1)} e^{\frac{k-1}{2}} x - \frac{4}{(k+1)^2 - 2s(k+1)} e^{\frac{k+1}{2}} x.$$

This gives the solution

$$u(x,\tau) = \pounds^{-1}(U_h + U_p)$$

where,

$$U_h(x,s) = A(s) e^{\sqrt{s}x} + B(s) e^{-\sqrt{s}x}$$

and

$$U_p(x,s) = \frac{4}{(k-1)^2 - 2s(k-1)} e^{\frac{k-1}{2}} x - \frac{4}{(k+1)^2 - 2s(k+1)} e^{\frac{k+1}{2}} x.$$

Alternately, U_p has the integral form as the following; Note that $e^{\sqrt{s}x}$ and $e^{-\sqrt{s}x}$ are bases of solutions of the corresponding homogeneous ODE

$$U_{xx} - sU = 0.$$

The Wronskian of $e^{\sqrt{s}x}$ and $e^{-\sqrt{s}x}$ is $W = -2\sqrt{s}$, and so, by the method of variation of parameter of Lagrange[5, p98], a particular solution U_p has the integral form of

$$U_p(x,s) = -e^{\sqrt{s}x}\int \frac{e^{-\sqrt{s}x}(e^{\frac{k-1}{2}x} - e^{\frac{k+1}{2}x})}{-2\sqrt{s}}\,dx$$

$$+e^{-\sqrt{s}x}\int \frac{e^{\sqrt{s}x}(e^{\frac{k-1}{2}x} - e^{\frac{k+1}{2}x})}{-2\sqrt{s}}\,dx.$$

Organizing this equality, we obtain the result

$$U_p(x,s) = \frac{e^{\sqrt{s}x}}{2\sqrt{s}}\int e^{(\frac{k-1}{2}-\sqrt{s})x} - e^{(\frac{k+1}{2}-\sqrt{s})x}\,dx$$

$$-\frac{e^{-\sqrt{s}x}}{2\sqrt{s}}\int e^{(\frac{k-1}{2}+\sqrt{s})x} - e^{(\frac{k+1}{2}+\sqrt{s})x}\,dx.$$

□

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