International Journal of Pure and Applied Mathematics
Volume 118 No. 4 2018, 1083-1090
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v118i4.20

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CERTAIN CONCEPTS IN FUZZY TOPOLOGY VIA FUZZY PREOPEN SETS

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Abstract: In 1991, S., Nanda introduced fuzzy preopen set. Using this concept, here we first introduce fuzzy regular preopen set, the collection of which is strictly larger than that of fuzzy preopen sets. Also some properties of fuzzy regular preopen sets are investigated here. Afterwards, a new type of fuzzy space, viz., fuzzy p-space is introduced in which finite intersection property of fuzzy regular preopen sets are true. In Section 2, we also introduced fuzzy regular p-space in which the collection of fuzzy preopen sets and fuzzy regular preopen sets are identical. In Section 3, we introduce fuzzy semi regular preopen set, the collection of which is strictly larger than that of fuzzy regular preopen sets. Also fuzzy extremally p-disconnected space is introduced in which every fuzzy regular preopen set is fuzzy preclopen.

AMS Subject Classification: 03E72
Key Words: Fuzzy regular preopen set, fuzzy p-space, fuzzy semi regular preopen set, fuzzy regular p-space, fuzzy extremally p-disconnected space.

1. Introduction and Preliminaries

Using fuzzy preopen set as a basic tool, a new type of fuzzy set viz., fuzzy regular preopen set is introduced which is fuzzy preopen as well as the fuzzy preinterior of whose fuzzy preclosure is the set itself. For any two fuzzy sets

Received: September 29, 2016
Revised: May 27, 2018
Published: May 31, 2018

A, B, it is shown in (see [3]) that $A \leq B \Rightarrow pintA \leq pintB$ and $pclA \leq pclB$. But the reverse implications are true only if A, B are fuzzy regular preclosed and fuzzy regular preopen respectively. Here we also introduce fuzzy extremally p-disconnected space in which fuzzy regular preopen set is fuzzy preclopen.

Throughout the paper, (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang (see [2]). A fuzzy set A in an fts X is a mapping from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The support (see [4]) of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement (see [5]) of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ (see [5]) while AqB means A is quasi-coincident (q-coincident, for short) (see [4]) with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not q B$ respectively. For a fuzzy set A, clA and $intA$ stand for fuzzy closure and fuzzy interior of A in X (see [2]). $A \in I^X$ is called fuzzy preopen (see [3]) if $A \leq int(clA)$. The complement of fuzzy preopen set is called fuzzy preclosed (see [3]) set. The smallest fuzzy preclosed set containing a fuzzy set A in X is called fuzzy prelosure (see [3]) of A, denoted by $pclA$. $A \in I^X$ is fuzzy preclosed if $A = pclA$. The union of all fuzzy preopen sets contained in a fuzzy set A in X is called fuzzy preinterior of A, denoted by $pintA$ (see [3]). $A \in I^X$ is fuzzy preopen if $A = pintA$. The collection of all fuzzy preopen (resp., fuzzy preclosed) sets in X is denoted by $FPO(X)$ (resp., $FPC(X)$).

2. Fuzzy Regular Preopen Set : Some Properties

In this section fuzzy regular preopen set is introduced and studied.

Definition 1. A fuzzy preopen set A in an fts (X, τ) is said to be fuzzy regular preopen if $A = pint(pclA)$.
The complement of a fuzzy regular preopen set is called fuzzy regular preclosed. The collection of all fuzzy regular preopen (resp., fuzzy regular preclosed) sets in X is denoted by $FRPOX)$ (resp., $FRPC(X)$).

Remark 2. It is clear from definition that every fuzzy preclopen set is fuzzy regular preopen. But the converse is not true, in general follows from the following example.

Example 3. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Then $FPO(X) = \{0_X, 1_X, U, V\}$ where $U \leq A, V > 1_X \setminus A$ and $FPC(X) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\}$ where $1_X \setminus U \geq 1_X \setminus A, 1_X \setminus V < A$. Then $A \in FPO(X)$. Now $pint(pclA) = pint(1_X \setminus A) = A \Rightarrow A \in FRPO(X)$. But $A \notin FPC(X)$.

Remark 4. Intersection and union of two fuzzy regular preopen sets need not be so, follow from the following examples.

Example 5. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.3, B(a) = 0.6, B(b) = 0.4$. Then (X, τ) is an fts. Now $FPO(X) = \{0_X, 1_X, B, U, V\}$ where $0.4 < U(a) \leq 0.5, U(b) \leq 0.3, V \not\leq 1_X \setminus A$, $FPC(X) = \{0_X, 1_X, 1_X \setminus B, 1_X \setminus U, 1_X \setminus V\}$ where $0.5 \leq 1 - U(a) < 0.6, 1 - U(b) \geq 0.7, 1_X \setminus V \not\geq A$. Consider two fuzzy sets C and D defined by $C(a) = 0.55, C(b) = 0.7, D(a) = 0.5, D(b) = 0.8$. Then $int(clC) = 1_X > C, int(clD) = 1_X > D \Rightarrow C, D \in FPO(X)$. Again, $pint(pclC) = pintC = C, pint(pclD) = pintD = D \Rightarrow C, D \in FRPO(X)$. But $C \bigwedge D = 1_X \setminus A \notin FRPO(X)$ as $1_X \setminus A \notin FPO(X)$.

Example 6. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.7, B(b) = 0.5$. Then (X, τ) is an fts. Then $FPO(X) = \{0_X, 1_X, B, U, V\}$ where $0.3 < U(a) \leq 0.5, U(b) \leq 0.4, V \not\leq 1_X \setminus A$ and $FPC(X) = \{0_X, 1_X, 1_X \setminus B, 1_X \setminus U, 1_X \setminus V\}$ where $0.5 \leq 1 - U(a) < 0.7, 1 - U(b) \geq 0.6, 1_X \setminus V \not\geq A$. Consider two fuzzy sets C and D defined by $C(a) = C(b) = 0.4, D(a) = 0.6, D(b) = 0.3$. Then $C, D \in FPO(X)$. Again, $pint(pclC) = C, pint(pclD) = D \Rightarrow C, D \in FRPO(X)$. Now $E = C \bigvee D$ is the fuzzy set defined by $E(a) = 0.6, E(b) = 0.4$. Then $E \in FPO(X)$, but $pint(pclE) = pint(F)$ (where $F = pclE$ is defined by $F(a) = F(b) = 0.6$) $= F \neq E \Rightarrow E \notin FRPO(X)$.

It is clear from above discussion that the set of all fuzzy regular preopen sets in an fts (X, τ) does not form a fuzzy topology.

Remark 7. Example 6 shows that fuzzy open and fuzzy regular preopen sets are independent concepts. Here $B \in \tau$, but $pint(pclB) = 1_X \neq B \Rightarrow B \notin FRPO(X)$ and $C \in FRPO(X)$ though $C \notin \tau$.

Theorem 8. *Let (X, τ) be an fts and $A, B \in FPO(X)$. The following statements are true :*
(i) $A \leq B \Rightarrow pint(pclA) \leq pint(pclB)$,
(ii) $A \leq pint(pclA)$,
(iii) $pint(pcl(pintA)) = pint(pclA)$,
(iv) if $A \not{q} B$, then $pint(pclA) \not{q} pint(pclB)$.

Proof. (i), (ii) and (iii) are obvious.

(iv) We know from (see [1]) that if $A, B \in FPO(X)$, $A \not q B \Rightarrow A \not q pclB \Rightarrow A \not q pint(pclB) \Rightarrow pclA \not q pint(pclB)$ from (see [1]) again $\Rightarrow pint(pclA) \not q pint(pclB)$. □

Theorem 9. *Let (X, τ) be an fts and $A, B \in I^X$. Then the following statements are true:*
(i) if $A \in FPC(X)$, then $pintA \in FRPO(X)$,
(ii) if $A = pintA$, then $pclA \in FRPC(X)$,
(iii) if $A, B \in FRPC(X)$, then $A \leq B \Leftrightarrow pintA \leq pintB$,
(iv) if $A, B \in FRPO(X)$, then $A \leq B \Leftrightarrow pclA \leq pclB$.

Proof. (i) By hypothesis, $pclA = A$. Then

$$\begin{aligned} pintA &= pint(pclA) \geq pint(pcl(pintA)) = pint(pcl(pint(pclA))) \\ &\geq pint(pint(pclA)) = pint(pintA) = pintA \Rightarrow pintA = pint(pcl(pintA)). \end{aligned}$$

(ii) Similar to that of (i).
(iii) It is clear that $A \leq B \Rightarrow pintA \leq pintB$.
To prove the converse, let $pintA \leq pintB$...(1) where $A, B \in FRPC(X)$, i.e., $A = pcl(pintA), B = pcl(pintB)$. Now $A = pcl(pintA) \leq pcl(pintB)$ (by (1)) $= B$.
(iv) Similar to that of (iii). □

Now we recall a definition and a theorem from (see [1]) for ready references.

Definition 10. (see [1]) A fuzzy point x_α in an fts X is called a fuzzy p^*-cluster point of a fuzzy set A in X if $pclU q A$ for every fuzzy preopen set U in X with $x_\alpha q U$. The union of all fuzzy p^*-cluster points of a fuzzy set A in X is called fuzzy p^*-closure of A, denoted by $[A]_p$ $A(\in I^X)$ is fuzzy p^*-closed if $A = [A]_p$.

Theorem 11. *[1] For a fuzzy preopen set A in X, $pclA = [A]_p$.*

Theorem 12. *In an fts (X, τ), if $A(\in I^X) \in FRPO(X)$, then $A = pint([A]_p)$*

Proof. Follows from Definition 1 and Theorem 11. □

Definition 13. An fts (X, τ) is called fuzzy extremally p-disconnected if the fuzzy preclosure of every fuzzy preopen set in X is fuzzy preopen.

Theorem 14. *An fts (X, τ) is fuzzy extremally p-disconnected iff every fuzzy regular preopen set in X is fuzzy preclopen*

Proof. Let X be fuzzy extremally p-disconnected and A be fuzzy regular preopen in X. Then $A = pint(pclA) = pclA$ (by hypothesis) $\Rightarrow A$ is fuzzy preclopen.
Conversely, let $A \in FPO(X)$. By Theorem 9(ii), $pclA \in FRPC(X) \Rightarrow 1_X \setminus pclA = pint(1_X \setminus A) \in FRPO(X) \Rightarrow 1_X \setminus pclA$ is fuzzy preclopen (by hypothesis) $\Rightarrow pclA \in FPO(X) \Rightarrow X$ is fuzzy extremally p-disconnected. □

Example 5 shows that the intersection of two fuzzy preopen (resp., fuzzy regular preopen) sets need not be so. Now we define some sort of fuzzy space in which these concepts are true.

Definition 15. An fts (X, τ) is said to be fuzzy p-space if intersection of any finite collection of fuzzy preopen sets is fuzzy preopen.

Theorem 16. *In a fuzzy p-space (X, τ), intersection of any finite collection of fuzzy regular preopen sets is fuzzy regular preopen*

Proof. Let $\mathcal{U} = \{U_i : 1 \leq i \leq n\}$ be a finite family of fuzzy regular preopen sets in a fuzzy p-space (X, τ). Then $U = \bigwedge_{i=1}^{n} U_i \in FPO(X)$. Then

$$U \leq pclU \Rightarrow U = pintU \leq pint(pclU)...(1).$$

Again, $U \leq U_i$ for each $i, 1 \leq i \leq n \Rightarrow pint(pclU) \leq pint(pclU_i)$, for each $i.1 \leq i \leq n$. Again, $U_i = pint(pclU_i)$, for $1 \leq i \leq n$, then $pint(pclU) \leq pint(pclU_i) = U_i$ for $1 \leq i \leq n \Rightarrow$

$$pint(pclU) \leq \bigwedge_{i=1}^{n} U_i = U...(2).$$

Combining (1) and (2), we get $U = pint(pclU) \Rightarrow U \in FRPO(X)$. □

Note 17. Obviously every fuzzy regular preopen set is fuzzy preopen but the converse may not be true, as it seen from Example 6. Here $E \in FPO(X)$, but $E \notin FRPO(X)$.

To achieve the converse, we have to define some sort of fuzzy regular space as follows.

Definition 18. An fts (X, τ) is said to be fuzzy regular p-space if every fuzzy set in X is either fuzzy regular preopen or fuzzy regular preclosed.

The following example is an example of a fuzzy regular p-space.

Example 19. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5$. Then (X, τ) is a fuzzy regular p-space.

Theorem 20. *In a fuzzy regular p-space (X, τ), $A \in FPO(X) \Rightarrow A \in FRPO(X)$*

Proof. $A \in FPO(X) \Rightarrow$

$$A = pintA \Rightarrow A \leq pint(pclA) = pint(pcl(pintA))...(1).$$

By hypothesis, either $A \in FRPO(X)$ or $A \in FRPC(X)$. If $A \in FRPO(X)$, then we are done. Let $A \in FRPC(X)$. Then

$$A = pcl(pintA)...(2).$$

Then $pintA = pint(pcl(pintA)) \geq A$ (by (1)) $\Rightarrow A = pintA$. By (2), $A = pclA$. Consequently, A is fuzzy preclopen. By Remark 2, $A \in FRPO(X)$. □

3. Fuzzy Semi Regular Preopen Set : Some Properties

In this section the concept of fuzzy semi regular preopen set is introduced and shown the interrelationship between this set with fuzzy regular preopen set.

Definition 21. A fuzzy set A in an fts (X, τ) is said to be fuzzy semi regular preopen if there exists $U \in FRPO(X)$ such that $U \leq A \leq pclU$. The family of all fuzzy semi regular preopen sets in X is denoted by $FSRPO(X)$.

It is clear from Definition 1 and Definition 21 that

Proposition 22. If a fuzzy set A in an fts (X, τ) is fuzzy regular preopen, then it is fuzzy semi regular preopen. But the converse need not be true follows from the following example.

Example 23. Consider Example 5. Here $C \bigwedge D \notin FRPO(X)$. Now $A \in FRPO(X)$ such that $A \leq C \bigwedge D \leq pclA = C \bigwedge D \Rightarrow C \bigwedge D \in FSRPO(X)$.

Remark 24. The intersection of two fuzzy semi regular preopen sets need not be so follows from the next example.

Example 25. Consider Example 5 and the fuzzy sets C, D defined by $C(a) = 0.55, C(b) = 0.7, D(a) = 0.54, D(b) = 0.8$. Then clearly $C, D \in FRPO(X) \Rightarrow C, D \in FSRPO(X)$ (by Proposition 22). Now $E = C \bigwedge D$ is given by $E(a) = 0.54, E(b) = 0.7$. But there does not exist any $U \in FRPO(X)$ such that $U \leq E \leq pclU$.

Theorem 26. *Let (X, τ) be an fts and $A \in FSRPO(X)$. Then the following statements are true:*
(i) $pcl(1_X \setminus A) \in FRPC(X)$,
(ii) $pintA \in FRPO(X)$.

Proof. (i) $A \in FSRPO(X) \Rightarrow$ there exists $U \in FRPO(X)$ such that $U \leq A \leq pclU$. Now $1_X \setminus U = pcl(1_X \setminus U)$ (as $1_X \setminus U \in FPC(X)) \geq pcl(1_X \setminus A) \geq pcl(1_X \setminus pclU) = 1_X \setminus pint(pclU) = 1_X \setminus U \Rightarrow pcl(1_X \setminus A) = 1_X \setminus U \in FRPC(X)$.

(ii) Follows from (i). □

Theorem 27. *Let (X, τ) be a fuzzy p-space and $A, B \in FSRPO(X)$. Then $pint(A \bigwedge B) \in FSRPO(X)$.*

Proof. By Theorem 26(ii), $pintA, pintB \in FRPO(X) \Rightarrow pint(A \bigwedge B) = pintA \bigwedge pintB \in FRPO(X)$ as (X, τ) is a fuzzy p-space $\Rightarrow pint(A \bigwedge B) \in FSRPO(X)$ (by Proposition 22). □

Theorem 28. *Let (X, τ) be an fts and $A \in FSRPO(X)$ and $B \in I^X$ with $A \leq B \leq pclA$. Then $B \in FSRPO(X)$*

Proof. As $A \in FSRPO(X)$, there exists $U \in FRPO(X)$ such that $U \leq A \leq pclU$. Then $U \leq A \leq B \leq pclA \leq pcl(pclU) = pclU \Rightarrow B \in FSRPO(X)$. □

Note 29. Intersection of a fuzzy open set and a fuzzy semi regular preopen set is not necessarily fuzzy semi regular preopen follows from Example 5. Here $B \in \tau, C \in FSRPO(X)$. But $B \bigwedge C = E$ (say) is not fuzzy semi regular preopen.

Result 30. Let (X, τ) be an fts and $A \in I^X$. Then $A \in FRPO(X) \Leftrightarrow A \in FPO(X)$ and $A \in FSRPO(X)$

Proof. Necessary part follows from Proposition 22 and sufficient part follows from Theorem 26. □

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