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FUZZY SUBORDINATION AND SUPERORDINATION RESULTS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH SRIVASTAVA-ATTITYA OPERATOR

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Abstract: The main purpose of this present paper is to investigate the some fuzzy subordination and superordination results for subclasses of analytic functions involving Srivastava-Attitya operator.

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1. Introduction and Preliminaries

Let $\mathcal{H}$ be the class of analytic functions in U and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$, consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots .$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_2 z^2 + \ldots = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

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The convolution or Hadamard product of two functions $f, g \in \mathcal{A}$ is denoted by $f * g$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \tag{2}$$

where $f(z)$ is given by (1) and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$.

Definition 1 (Miller.S.S,and P.T.Mocanu(2003) p.817, Definition 2)**.** [3] Denote by Q, the set of all functions f that are analytic and injective on $\overline{U} - \mathcal{E}(f)$, where

$$\mathcal{E}(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - \mathcal{E}(f)$.

Definition 2. [4] Let X be a non empty set. The application $X \to [0, 1]$ is called fuzzy subset. An alternative definition, more precise, would be the following A pair (A, F_A) where $F_A : X \to [0, 1]$ and $A = \{x \in X : 0 < F_A \leq 1\} = supp\{A, F_A\}$ is called fuzzy subset.The function F_A is called the membership function of the fuzzy subset (A, F_A).

Definition 3. [5] Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x)$, $x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \leq F_N(x)$, $x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Definition 4. [4] Let $\mathcal{D} \subset C$, $z_0 \in \mathcal{D}$ be a fixed point and let the functions The functions $f, g \in \mathcal{H}(\mathcal{D})$.The function f is said to be fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied

1. $f(z_0) = g(z_0)$
2. $F_{f(D)} f(z) = F_{g(D)} g(z)$

where

$$f(D) = supp(D, F_{f(D)}) = \{z \in C \mid 0 < F_{f(D)}(z) \leq 1\}$$

$$g(D) = supp(D, F_{g(D)}) = \{z \in C \mid 0 < F_{g(D)}(z) \leq 1\}.$$

Lemma 5. [5] Let h and q be univalent in U, with $q(0) = a$, and let $h_\rho(z) = h(\rho z)$ and $q_\rho(z) = q(\rho z)$. Let $\psi : C^3 \times U \to C$ satisfy one of the following conditions:

1. $\psi \in \Psi_n(h, q_\rho)$ for some $\rho \in (0,1)$

2. there exist $\rho_0 \in (0,1)$ such that $\psi \in \Psi_n[h_\rho, q_\rho]$ for all $\rho_0 \in (0,1)$

If $p \in \mathcal{H}[a,1]$ and $\psi(p(z), zp^{'}(z), z^2p^{''}(z))$ is analytic in U, $h(0) = \psi(a,0,0;0)$ and

$$F_{\psi(C^3\times U)}[\psi(p(z), zp^{'}(z), z^2p^{''}(z))] \leq F_{h(U)}h(z)$$
$$F_{p(U)}p(z) \leq F_{q(U)}q(z) \quad (z \in U).$$

Definition 6. [5] Let Ω be a set in C, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : C^3 \times U \to C$ that satisfy the admissibility condition:

$$F_\Omega\psi(r,s,t;z) = 0, z \in U, \tag{3}$$

whenever $r = q(\zeta),\ s = m\zeta q^{'}(\zeta)$

$$\Re(\frac{t}{s}+1) \geq m\Re[\frac{\zeta q^{''}(\zeta)}{q(\zeta)}+1]$$

$\zeta \in \partial U \setminus E(Q)$ and $m \geq n$.
We write $\Psi_1(\Omega, q)$ as $\Psi(\Omega, q)$. In the special case when Ω is a simply connected domain, $\Omega \neq C$ and h is conformal mapping of U into Ω, we denote this class by $\Psi_n(h(U), q)$ or $\Psi_n(h, q)$.
If $C^2 \times U \to C$ then the admissibility condition (3) reduces to

$$F_\Omega\psi(q(\zeta), m\zeta q^{'}(\zeta); z) = 0,$$

when $z \in U \quad \zeta \in \partial U \setminus E(Q)$ and $m \geq n$.

Definition 7. [5] Let $\psi : C^3 \times U \to C$ and let h be univalent in U with $h(0) = \psi(a,0,0;0)$. If p is analytic in U with $p(0) = a$ and satisfies the fuzzy differential subordination

$$F_{\psi(C3\times U)}\psi(p(z), zp^{'}(z), z^2p^{''}(z)) \leq F_{h(U)}h(z), z \in U, \tag{4}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant,if$p(0) = q(0)$ and $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, $z \in U$, for all p satisfying(4). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(0) = q(0)$ and $F_{\tilde{q}(U)}\tilde{q}(z) \in F \in_{q(U)} q(z), z \in U$, for all fuzzy dominants q of (4) is said to be the fuzzy best dominant of (4). Note that the fuzzy best dominant is unique up to a rotation of U. If we require the more restrictive condition $p \in H[a,n]$, then p will be called an (a,n)-fuzzy solution, q an (a,n)-fuzzy dominant, and $\tilde{q}$ the best (a,n)-fuzzy dominant.

Definition 8. [8] Let $\psi : C^3 \times U \to C$ and let h be univalent in U . If p and $\psi(p(z), zp^{'}(z), z^2p^{''}(z), z)$ are univalent in U and satisfy the (second-order) fuzzy differential superordination

$$F_{\psi(C3\times U)}\psi(p(z), zp^{'}(z), z^2p^{''}(z)) \leq F_{h(U)}h(z) \tag{5}$$

$$\psi(p(z), zp^{'}(z), z^2p^{''}(z)) \leq h(z) \tag{6}$$

then p is called a fuzzy solution of the fuzzy differential superordination. An analytic function q is called fuzzy subordinant of the fuzzy differential superordination, or more simply a fuzzy subordination if $q(z) \prec_F p(z), z \in U$, for all p satisfying (5). A univalent fuzzy subordination $\widetilde{q}$ that satisfies $q \prec_F \widetilde{q}$ for all fuzzy subordinate q of(5) is said to be the fuzzy best subordinate of (5).

Definition 9. [6] The function $L(z,t)$, z in U, $t \geq 0$ is a fuzzy subordination chain if $L(.,t)$ is analytic and univalent in U for all $t \geq 0$. $L(z,t)$ is continuosly differentiable in $[0,\infty)$ for all z in U and $F_{L[U\times(0,\infty)]}L(z,t_1) \leq F_{L[U\times(0,\infty)]}L(z,t_2)$ when $t_1 \leq t_2$.

Lemma 10 (Ch.Pommerenke(1975) p.159, Theorem 6.2)**.** [2] The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = +\infty$, is a subordination chain if

$$\Re\left\{ z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0, z \in U,\ t \geq 0.$$

Motivated essentially by the Srivastava-Attitya operator, Al-Shaqsi and Darus(2009)[1] introduced and investigated the integral operator

$$\mathcal{J}_{s,b}^{\lambda,\mu} = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^s \frac{\lambda!(n+\mu-2)!}{(\mu-2)!(n+\lambda-1)!} a_n z^n \ (z \in U), \tag{7}$$

where the parameters μ, b are constrained as $b \in C\backslash\{Z_0^-\}; s \in C, \mu \geq 0,\ \lambda > -1$.

Making use of the operator given in (7) and results given in [7], in this paper we find sufficient condition for certain normalized analytic functions $f(z)$ in U such that $(f * \Psi)(z) \neq 0$ and f to satisfy

$$q_1(z) \prec_F \left(\frac{\alpha\mathcal{J}_{s,b}^{\lambda,\mu+1}(f*\Phi)(z) + \beta\mathcal{J}_{s,b}^{\lambda,\mu}(f*\Psi)(z)}{(\alpha+\beta)z}\right)^{\eta} \prec_F q_2(z), \tag{8}$$

where q_1, q_2 are given univalent functions in U with $q_1(0) = 1$, $q_2(0) = 1$ and $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty} \mu_n z^n$ are analytic functions in U with $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$.

2. Subordination Results

Theorem 11. *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(0)$ with $\phi(\omega) \neq 0$ when $\omega \in q(U)$. Set $Q(z) = \gamma_4 \frac{zq'(z)}{q(z)}$ and $h(z) = \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + \gamma_4 \frac{zq'(z)}{q(z)}$ and suppose that either (i) Q is starlike (or) (ii)h is convex and assume that*

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\gamma_3}{\gamma_4}q(z) + \frac{2\gamma_2}{\gamma_4}q^2(z) - \frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in U).$$

then q is the best dominant.
In addition assume that

$$\Re\{\phi[h(z)]\} > 0 \tag{9}$$

and p is analytic in U,
with $p(0) = h(0)$, $p(U) \subset D$ and $\psi : C^2 \times U \to C$

$$\psi(p(z), zp'(z)) = \theta(p(z)) + zp'(z)\phi(p(z))$$

is analytic in U, then

$$F_{\psi(C^2\times U)}[\theta(p(z)) + zp'(z)\phi(p(z))] \leq F_{h(U)}h(z)$$

implies

$$F_{p(U)}p(z) \leq F_{q(U)}q(z) \quad (z \in U),$$
$$p(z) \prec_F q(z)$$

and q is the best fuzzy dominant,
where

$$\psi_{(C^2\times U)} = supp\{C^2 \times U, F_{\psi(C^2\times U)}\psi(p(z), zp'(z))\}$$
$$= \{z \in C, 0 \leq F_{\psi(C^2\times U)}\psi(p(z), zp'(z)) \leq 1\}$$

and

$$h(U) = supp(U, F_{h(U)}h(z)) = \{z \in C, 0 \leq F_{h(U)}h(z)) \leq 1\}$$

Proof. Without loss of generality we can assume that p and h satisfy the conditions of the theorem on the closed disc $\overline{U}$. If not, then we can replace $p(z)$ by $p_\rho(z) = p(\rho z)$, and $h(z)$ by $h_\rho(z) = h(\rho z)$ where $0 < \rho < 1$. These new functions satisfy the conditions of the theorem on $\overline{U}$. We would then prove that

$$F_{p_\rho(U)}p_\rho(z) \leq F_{p(U)}p(z),\ forall\ 0 < \rho < 1.$$

By letting $\rho \to 1$, we obtain

$$F_{p(U))}p(z) \leq F_{h(U))}h(z).$$

In order to prove the theorem, we apply Lemma (5), and we show that $\psi \in \Psi_1[h_\rho, h_\rho]$
Suppose (a) and (b) are satisfied, but p is not fuzzy subordinate to h.
According to definition(6),there are points $z_0 \in U$ and $\zeta_0 \in \partial U$ and $m \geq 1$ with

$$p(z_0) = h\zeta_0, z_0 p^{'}(z_0) = m\zeta_0 q^{'}(\zeta_0)$$

such that

$$\begin{aligned}\psi_0 = \psi(p(z_0), zp^{'}(z_0)) &= \theta(p(z_0)) + z_0 p^{'}(z_0)\phi(p(z_0)) \\ &= \theta(h(\zeta_0)) + m\zeta_0 h^{'}(\zeta_0)\phi(h(\zeta_0)) \qquad (10)\end{aligned}$$

From (10) we have $\psi_0 = \theta(h(\zeta_0)) + m\zeta_0 h^{'}(\zeta_0)\phi(h(\zeta_0)),\ \zeta_0 \in \partial U, |\zeta_0| = 1,\ m \geq 1$ which gives

$$\frac{\rho_0 - \theta(h(\zeta_0))}{\zeta_0 h'(\zeta_0)} = m\phi(h(\zeta_0))$$

Using the conditions from the hypothesis of the theorem, we have:

$$\Re(\frac{\rho_0 - \theta(h(\zeta_0))}{\zeta_0 h'(\zeta_0)}) = \Re(m\phi(h(\zeta_0))) > 0$$

which implies

$$|arg\frac{\rho_0 - \theta(h(\zeta_0))}{\zeta_0 h'(\zeta_0)}| < \frac{\pi}{2}$$

which is equivalent to

$$|arg[\rho_0 - \theta(h(\zeta_0))] - \zeta_0 h^{'}(\zeta_0)| < \frac{\pi}{2} \qquad (11)$$

Since $\zeta_0 h^{'}(\zeta_0)$ is the outer normal at the border of the convex domain $h_\rho(U)$ at $h_\rho(\zeta_0)$ from (11) we get ρ_0 does not belongs to $h_\rho(U)$ which means

$$F_{h(U)}\psi(p(z_0), zp^{'}(z_0), z_0) = F_{h(U)}\psi(h(\zeta_0), m\zeta_0 h^{'}(\zeta_0), z_0) \qquad (12)$$

Using Definition (6), from (12) we have $\psi \in \Psi[h(U), h]$.
Using condition (1) from Lemma (5) we have

$$F_{p(U)}p(z) \leq F_{q(U)}q(z) \quad (z \in U)$$

□

3. Superordination Results

Theorem 12. *Let q be univalent in$U$ and ϑ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(\omega) \neq 0$ when $\omega \in q(U)$ .Set $\phi(z) = \frac{\gamma_4}{\omega}$ and $\vartheta(z) = \gamma_1 + \gamma_2\omega^2 + \gamma_3\omega$*

$$\Re\left\{\frac{\gamma_3}{\gamma_4}q(z) + \frac{2\gamma_2}{\gamma_4}q^2(z)\right\} \geq 0. \tag{13}$$

If $p \in H[q(0),1] \cap Q$ with $p(0) = q(0)$, $p(U) \subset D$ and $\varphi : U \times [0,\infty) \to C$ and

$$\varphi(p(z), zp^{'}(z)) = \upsilon(p(z)) + zp^{'}(z)\varphi(p(z))$$

is analytic in U then

$$F_{h(U)}h(z) \leq F_{\varphi(U\times[0,\infty))}[\upsilon(p(z)) + zp^{'}(z)\psi(p(z))]$$

with corresponding differential equation

$$h(z) = \upsilon(q(z)) + zq^{'}(z)\psi(q(z)) \tag{14}$$

implies that

$$F_{q(U)}q(z) \leq F_{p(U)}p(z) \quad (z \in U)$$

,

$$q(z) \prec_F p(z)$$

and q is the best fuzzy subordinant. where

$$\varphi_{(U\times[0,\infty))} = supp\{U \times [0,\infty), F_{\varphi(U\times[0,\infty))}\varphi(p(z), zp^{'}(z))\}$$

$$= \{z \in C, 0 \leq F_{\varphi(U\times[0,\infty))}\varphi(p(z), zp^{'}(z)) \leq 1\}$$

and

$$h(U) = supp(U, F_{h(U)}h(z)) = \{z \in C, 0 \leq F_{h(U)}h(z)) \leq 1\}$$

Proof. Define the function p by

$$p(z) := \left(\frac{\alpha\mathcal{J}_{s,b}^{\lambda,\mu+1}(f * \Phi)(z) + \beta\mathcal{J}_{s,b}^{\lambda,\mu}(f * \Psi)(z)}{(\alpha+\beta)z}\right)^{\eta}$$

If we let

$$L(z,t) = \upsilon(q(z)) + tzq^{'}(z)\psi(q(z)) = \gamma_1 + \gamma_2 q^2(z) + \gamma_3 q(z) + t\frac{\gamma_4 zq^{'}(z)}{q(z)}$$

$$\frac{\partial L(z,t)}{\partial z} = 2\gamma_2 q(z)q^{'}(z) + \gamma_3 q^{'}(z) + t\gamma_4[\frac{zq^{''}(z)}{q(z)} + \frac{q^{'}(z)}{q(z)} - z(\frac{q^{'}(z)}{q(z)})^2]$$
$$= a_1(t) + ...$$

and

$$\frac{\partial L(z,t)}{\partial t} = \gamma_4 \frac{zq^{'}(z)}{q(z)}$$

$$\frac{\partial L(0,t)}{\partial z} = \gamma_4 q^{'}(0)\left(\frac{\gamma_3}{\gamma_4} + \frac{2\gamma_2}{\gamma_4}q(0) + t\frac{1}{q(0)}\right)$$

From the univalence of q we have $q^{'}(0) \neq 0$ and $q(0) = 1$, it follows that $a_1(t) \neq 0$ for $t \geq 0$ and A simple computation yields

$$\Re\left(\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right) > 0$$

By applying lemma (10), $L(z,t)$ is a subordination chain.

If $t_1 \leq t_2$ and $L(z,t) : U \times [0,\infty) \to C$ then

$$L(z,t_1) \leq L(z,t_2)$$

this implies

$$L(z,t_1)_{(U\times[0,\infty))} = supp\{U \times [0,\infty), F_{L(U\times[0,\infty))}L(z,t_1))\}$$

and

$$L(z,t_2)_{(U\times[0,\infty))} = supp\{U \times [0,\infty), F_{L(U\times[0,\infty))}L(z,t_2))\}$$

then

$$F_{L[U\times(0,\infty)]}L(z,t_1) \leq F_{L[U\times(0,\infty)]}L(z,t_2)$$

Due to the definition (9),$L(z,t)$ is a fuzzy subordination chain.

we conclude that q is fuzzy subordinant of fuzzy differential superordination

$$F_{h(U)}h(z) \leq F_{\varphi(U\times[0,\infty))}[\upsilon(p(z)) + zp^{'}(z)\psi(p(z))]$$

Furthermore, since q is a univalent solution of (14), it also is the fuzzy best subordinant of

$$F_{h(U)}h(z) \leq F_{\varphi(U\times[0,\infty))}[\upsilon(p(z)) + zp^{'}(z)\psi(p(z))]$$

this gives the required result. □

There is a complete analog of Theorem (11) for fuzzy differential subordination and Theorem (12) for fuzzy differential superordination. We can combine the results of Theorem (11) with Theorem (12) and obtain the following sandwich theorem.

4. Sandwich Results

Theorem 13. *Let q_1 and q_2 be convex univalent in U $\gamma_i \in \mathcal{C}$ $(i = 1, \ldots, 4)(\gamma_4 \neq 0)$, $\eta, \alpha, \beta \in \mathcal{C}$ such that $\eta \neq 0$ and $\alpha + \beta \neq 0$, and let q_2 satisfy (9) and q_1 satisfy (13) . For $f, \Phi, \Psi \in \mathcal{A}$, let $p(z) \in H[1,1] \cap Q$ and satisfying*

$$F_{h_1(U)}h_1(z) \leq F_{\psi(C^2 \times U)}[\upsilon(p(z)) + zp^{'}(z)\psi(p(z))] \leq F_{h_2(U)}h_2(z)$$

which implies $q_1 \prec_F p(z) \prec_F q_2$
that is

$$q_1(z) \prec_F \left(\frac{\alpha \mathcal{J}_{s,b}^{\lambda;\mu+1}(f * \Phi)(z) + \beta \mathcal{J}_{s,b}^{\lambda;\mu}(f * \Psi)(z)}{(\alpha + \beta)z} \right)^{\eta} \prec_F q_2(z)$$

and q_1 , q_2 are respectively the best Fuzzy subordinant and best Fuzzy dominant.

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