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SOME FIXED POINT THEOREMS OF GENERALIZED $(\psi_1, \psi_2, \varphi, \phi)-$CONTRACTION IN PARTIALLY ORDERED MODULAR METRIC SPACES

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Abstract: The purpose of this paper is to establish some fixed point results for mappings, satisfying some weak contractive inequalities, in a partially ordered modular metric space.

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1. Introduction

Many generalizations of The Banach contraction principle have been established in various settings. In 2016, Azizi et al.in [5] introduced the notion of almost generalized $\mathcal{C}-$contraction as follows:

Let $(X, \preceq, d)$ be an ordered metric space. A mapping $T : X \to X$ is said to be almost generalized $\mathcal{C}-$contractive if there exist $L \geq 0$, a function $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ which is lower semi-continuous and non-decreasing with respect to both of its components and satisfying $\varphi(s,t) = 0$ if and only if

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$s = t = 0$ and a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\psi(t) = 0$ if and only if $t = 0$, such that for all comparable elements $x, y \in X$, we have:

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M_1(x, y), M_2(x, y)) + L\psi(N(x, y))$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\},$$

$$M_1(x, y) = \max\{d(x, y), d(x, Tx), d(x, Ty)\},$$

$$M_2(x, y) = \max\{d(x, y), d(y, Ty), d(y, Tx)\}$$

and

$$N(x, y) = \min\{d(x, Tx), d(y, Tx)\}$$

and proved the following results:

Theorem 1.1. *Let $(X, \preceq, d)$ be an ordered complete metric space and $T : X \to X$ be a nondecreasing, continuous and almost generalized $\mathcal{C}-$contractive. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point. In particular, if $\mathfrak{F}(T)$, the set of fixed points of T, is a totally ordered subset of X, then T has a unique fixed point.*

Theorem 1.2. *Let $(X, \preceq, d)$ be an ordered complete metric space and $S, T : X \to X$ be two weakly increasing mappings which S is almost generalized $\mathcal{C}$-contractive with respect to T. If either S or T is continuous, then the fixed point set of S is nonempty and $\mathfrak{F}(S, T) = \mathfrak{F}(S) = \mathfrak{F}(T)$. Particularly, if $\mathfrak{F}(S)$ is a totally ordered subset of X, then S and T have a unique common fixed point.*

On the other hand, in 2010, Chystyakov in [6, 7] has introduced the concept of modular metric spaces. This is a generalization of the classical modular spaces like Orlicz spaces (see [9]). Fixed point theorems in modular function spaces, generalizing the classical Banach fixed point theorem in metric spaces, have been studied extensively (see [11, 1, 10]).

In recent years, there has been a great interest in the study of the fixed point property in modular metric spaces (see [2, 3, 8]). For more on modular metric fixed point theory, the reader may consult the book [4].

In this paper we prove some fixed and common fixed point theorems for a weak contractive mapping in modular metric spaces. Our results generalize and extend the above theorems in partially ordered modular metric spaces for mappings satisfying weak contraction and mappings satisfying almost generalized contraction that involves four control functions.

2. Preliminaries

Let X be a nonempty set. For a function $\omega :]0,+\infty[\times X \times X \to [0,+\infty]$, we will use the notation:

$$\omega_\lambda(x,y) = \omega(\lambda, x, y), \quad \text{for all } \lambda > 0 \text{ and } x, y \in X.$$

Definition 2.1. [6] A function $\omega :]0,+\infty[\times X \times X \to [0,+\infty]$ is said to be modular metric on X if it satisfies the following conditions:

(i) Given $x, y \in X$, $x = y$ if and only if $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$;

(ii) For all $x, y \in X$, for all $\lambda > 0$, $\omega_\lambda(x,y) = \omega_\lambda(y,x)$;

(iii) For all $x, y, z \in X$ and for all $\lambda, \mu > 0$, $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$.

In this case, (X,ω) is called modular metric space.
The modular ω is said to be convex if for all $\lambda, \mu > 0$ and $x, y, z \in X$, we have:

$$\omega_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y).$$

Note that for a modular metric ω on a set X, and any $x, y \in X$, the function $\lambda \to \omega_\lambda(x,y)$ is non-increasing on $]0,+\infty[$. Indeed, if $0 < \mu < \lambda$, then $\omega_\lambda(x,y) \leq \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y)$.

Definition 2.2. [6] Let (X,ω) be a modular metric space. Fix $x_0 \in X$. Set

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \longrightarrow 0 \;\; as \;\; \lambda \longrightarrow \infty\},$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists \lambda > 0 \;,\; \omega_\lambda(x,x_0) < \infty\}.$$

The two linear spaces X_ω and X^*_ω are said to be modular spaces (around x_0).

Note that X_ω is metrizable by the metric

$$d_\omega(x,y) = \inf\{t > 0 : \omega_t(x,y) \leq t\}.$$

If ω is convex, then $X^*_\omega = X_\omega$ and we can endowed these sets by the metric d^*_ω defined by:

$$d^*_\omega(x,y) = \inf\{t > 0 : \omega_t(x,y) \leq 1\}.$$

Definition 2.3. [6] Let ω be a modular metric on X.

1. We say that a sequence $\{x_n\} \subset X_\omega$ is $\omega-$convergent to some $x \in X_\omega$ if and only if $\lim\limits_{n\to+\infty} \omega_1(x_n,x) = 0$. We will call x the $\omega-$limit of $\{x_n\}$.

2. We say that a sequence $\{x_n\} \subset X_\omega$ is $\omega-$Cauchy if
$$\lim_{n,m\to+\infty} \omega_1(x_n, x_m) = 0.$$
3. We say that $M \subset X_\omega$ is ω-closed if the ω-limit of any ω-convergent sequence of M is in M.
4. We say that $M \subset X_\omega$ is ω-complete if any ω-Cauchy sequence in M is ω-convergent and its ω-limit belongs to M.
5. We say that ω satisfies Fatou property if we have
$$\omega_1(x, y) \leq \liminf_{n\to+\infty} \omega_1(x_n, y),$$
for any sequence $\{x_n\} \subset X_\omega$ which $\omega-$converges to x and for any $y \in X_\omega$.
6. We say that ω satisfies the Δ_2-condition if $\lim\limits_{n\to+\infty} \omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$ implies that $\lim\limits_{n\to+\infty} \omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$.
7. We say that ω satisfies the Δ_2- type condition if for every $\alpha > 0$, there exists a constant $K_\alpha > 0$ such that
$$\omega_{\frac{\lambda}{\alpha}}(x, y) \leq K_\alpha \omega_\lambda(x, y),$$
for all $x, y \in X_\omega$ and any $\lambda > 0$.

The following results are immediate:

Remark 2.4. If ω satisfies the Δ_2-type condition, then ω satisfies the Δ_2- condition.

Remark 2.5. Let $\{x_n\}$ be a sequence in X_ω. Let $\lambda > 0$. If ω satisfies the Δ_2- type condition, then $\{x_n\}$ is $\omega-$Cauchy if and only if $\lim\limits_{n,m\to+\infty} \omega_\lambda(x_n, x_m) = 0$.

Remark 2.6. If ω satisfies the Δ_2- type condition, then ω is regular.

In the case of Δ_2- type condition, we have:

Lemma 2.7. *If ω satisfies the Δ_2- type condition, then*
$$\omega_\lambda(x, y) < \infty, \textit{ for all } \lambda > 0 \textit{ and for all } (x, y) \in X_\omega^2.$$

Proof. If we suppose that there exist $\lambda > 0$ and $x, y \in X_\omega$ such that $\omega_\lambda(x,y) = \infty$, and since ω satisfies the Δ_2- type condition, then, for all $n \in \mathbb{N}^*$, $\omega_{2^n\lambda}(x,y) = \infty$. Since $\omega_{2^n\lambda}(x,y) \leq \omega_{2^{n-1}\lambda}(x,x_0) + \omega_{2^{n-1}\lambda}(y,x_0)$ and $x, y \in X_\omega(x_0)$, we have: $\lim\limits_{n\to+\infty} \omega_{2^{n-1}\lambda}(x,x_0) = 0$ and $\lim\limits_{n\to+\infty} \omega_{2^{n-1}\lambda}(y,x_0) = 0$. Then, $\lim\limits_{n\to+\infty} \omega_{2^n\lambda}(x,y) = 0$. Which is a contradiction. □

Lemma 2.8. *Let (X,ω) be a modular space. Let $\{x_n\}$ be a sequence in X_ω. If $\{x_n\}$ is not $\omega-$Cauchy, then there exist $\varepsilon > 0$ and two subsequences of integers $\{n_k\}$ and $\{m_k\}$ such that:*

$$n_k > m_k \geq k,\ \omega_1(x_{n_k}, x_{m_k}) \geq \varepsilon \text{ and } \omega_{\frac{1}{2}}(x_{n_k-1}, x_{m_k}) < \varepsilon.$$

Proof. If we suppose that $(x_n)_{n\in\mathbb{N}}$ is not a $\omega-$Cauchy, then there exists $\varepsilon > 0$ and for all $k \in \mathbb{N}$ there exist $n_k, m_k \in \mathbb{N}$ such that $n_k > m_k \geq k$ and $\omega_1(x_{n_k}, x_{m_k}) \geq \varepsilon$. Let us fix $k \in \mathbb{N}$, and consider the set

$$\mathcal{A}_k = \{h \in \mathbb{N}^* \ / \ h > m_k \geq k \text{ and } \omega_1(x_h, x_{m_k}) \geq \varepsilon\}.$$

Since $n_k \in \mathcal{A}_k$, then $\mathcal{A}_k \neq \emptyset$. Let us consider the set:

$$\mathcal{B}_k = \{h \in \mathcal{A}_k \ / \ \omega_{\frac{1}{2}}(x_h, x_{m_k}) \geq \varepsilon\}.$$

One can see that $\mathcal{B}_k \subseteq \mathbb{N}^*$ and $\mathcal{B}_k \neq \emptyset$. Since

$$\omega_{\frac{1}{2}}(x_{n_k}, x_{m_k}) \geq \omega_1(x_{n_k}, x_{m_k}) \geq \varepsilon,$$

then $\mathcal{B}_k$ admits the least element n'_k that belongs to $\mathcal{A}_k$, and so $n'_k > m_k \geq k,\ \omega_1(x_{n'_k}, x_{m_k}) \geq \varepsilon$ and $\omega_{\frac{1}{2}}(x_{n'_k-1}, x_{m_k}) < \varepsilon$. □

Using the same argument as in the proof of Lemma 2.8 and applying Lemme 2.5, we have the following:

Lemma 2.9. *Let $s, t \in \mathbb{N}^*$. If ω satisfies the Δ_2- type condition and $\{x_n\}$ is not a $\omega-$Cauchy, then there exist $\varepsilon > 0$ and two subsequences of integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$, $\omega_{2^s}(x_{n_k}, x_{m_k}) \geq \varepsilon$ and $\omega_{\frac{1}{2^t}}(x_{n_k-1}, x_{m_k}) < \varepsilon$.*

Lemma 2.10. *Let (X,ω) be a modular space such that ω is convex and satisfies the Δ_2-condition. If $\{x_n\}$ is a sequence in X_ω such that*

$$\lim_{n\to+\infty} \omega_1(x_n, x_{n+1}) = 0,$$

then $\{x_n\}$ is $\omega-$Cauchy.

Proof. If we suppose that the sequence $\{x_n\}$ is not $\omega-$Cauchy, then according to Lemma 2.8, there exist $\varepsilon > 0$ and two subsequences of integers $\{n_k\}$ and $\{m_k\}$ such that $n_k > m_k \geq k$ and $\omega_1(x_{n_k}, x_{m_k}) \geq \varepsilon$ and $\omega_{\frac{1}{2}}(x_{n_k-1}, x_{m_k}) < \varepsilon$. Since ω is convex, we have:

$$\omega_1(x_{n_k}, x_{m_k}) \leq \frac{1}{2}\omega_{\frac{1}{2}}(x_{n_k-1}, x_{m_k}) + \frac{1}{2}\omega_{\frac{1}{2}}(x_{n_k-1}, x_{n_k}).$$

Then, for all $k \in \mathbb{N}$, $\varepsilon \leq \frac{\varepsilon}{2} + \frac{1}{2}\omega_{\frac{1}{2}}(x_{n_k-1}, x_{n_k})$. Since ω satisfies the Δ_2-type condition, then $\lim\limits_{n\to+\infty} \omega_{\frac{1}{2}}(x_{n_k-1}, x_{n_k}) = 0$. So, $\varepsilon \leq \frac{\varepsilon}{2}$. Which is a contradiction. □

Definition 2.11. Let X be a nonempty set. Then $(X, \preceq, \omega)$ is called a partially ordered modular metric space if and only if (i) (X, ω) is a modular metric space and (ii) $(X, \preceq)$ is a partially ordered set.

Definition 2.12. Let $(X, \preceq, \omega)$ a partially ordered modular metric space. We say that ω satisfies the property (P), if a non-decreasing sequence $\{x_n\}$ $\omega-$converges to some $x \in X_\omega$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Definition 2.13. Let C be a nonempty subset of X_ω. A self-mapping $T : C \to C$ is said to be $\omega-$continuous, if a sequence $\{x_n\}$ $\omega-$converges to some $x \in C$, then $\{Tx_n\}$ $\omega-$converges to Tx.

Definition 2.14. We say that a partially ordered set $(X, \preceq)$ is up-directed, if for all $(x, y) \in X^2$ there exists an element $z \in X$ such that $x \preceq z$ and $y \preceq z$.

We will use the following notations:
Let X a nonempty set and S and T be a two self-mappings on X. We denote by $\mathfrak{F}(S)$ the fixed point set of S, i.e., $\mathfrak{F}(S) := \{x \in X : Sx = x\}$. Also, we denote by $\mathfrak{F}(S, T)$ the common fixed point set of S and T, i.e., $\mathfrak{F}(S, T) = \mathfrak{F}(S) \cap \mathfrak{F}(T)$

3. Main results

In this section, we obtain a common fixed point for a pair of mappings satisfying a generalized $(\psi_1, \psi_2, \varphi, \phi)-$contractive condition in the framework of a partially ordered modular metric space. We support our result by an example. Now, let us define four functions $\psi_1, \psi_2, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that:

(a) ψ_1 and ψ_2 are continuous;

(b) ψ_1 is strictly increasing;

(c) For all $i \in \{1,2\}$, $\psi_i(t) = 0$ if and only if $t = 0$;

(d) φ is non-decreasing with respect to both of its components and lower semi-continuous with respect to the second one.

(e) $\varphi(t,s) = 0$ if and only if $t = s = 0$;

(f) ϕ is continuous and, for all $t \in \mathbb{R}^+$, $\phi(t) = 0$ if and only if $t = 0$;

(g) For all $t > 0$, $\psi_1(t) - \psi_2(t) + \varphi(0,t) > 0$.

We set

$$M(x,y) = \max\{\omega_1(x,y), \omega_1(x,Sx), \omega_1(y,Ty), \frac{\omega_2(x,Ty) + \omega_2(y,Sx)}{2}\},$$

$$M_1(x,y) = \max\{\omega_1(x,y), \omega_1(x,Sx), \omega_1(x,Ty)\},$$

$$M_2(x,y) = \max\{\omega_1(x,y), \omega_1(y,Ty), \omega_1(y,Sx)\}$$

and

$$N(x,y) = \min\{\omega_1(x,Sx), \omega_1(y,Sx), \omega_1(x,Ty)\}.$$

Theorem 3.1. *Let $(X, \preceq, \omega)$ be partially ordered modular metric space where ω satisfies the Δ_2-type condition and Fatou property. Let C be an ω-complete nonempty subset of X_ω and $T, S : C \to C$ be tow self-mappings. Let $L \geq 0$. If the following conditions are verified:*

(i) for all comparable elements $x, y \in C$,

$$\psi_1(\omega_1(Sx,Ty)) \leq \psi_2(M(x,y)) - \varphi(M_1(x,y), M_2(x,y)) + L\phi(N(x,y)) \quad (1)$$

(ii) there exists an element $x_0 \in C$ such that

$$x_0 \preceq Sx_0 \preceq TSx_0 \preceq STSx_0 \preceq (TS)^2x_0 \preceq S(TS)^2x_0 \preceq ...$$

(iii) ω satisfies the property (P),

then S and T have a common fixed point in C and $\mathfrak{F}(S,T) = \mathfrak{F}(S) = \mathfrak{F}(T)$. Particulary, if $\mathfrak{F}(S,T)$ is totally ordered then T and S have a unique fixed point.

Proof. We divide this proof into five steps.

Step.1. Consider the sequence $\{x_n\}$ defined by:

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}, \text{ for all } n \in \mathbb{N}.$$

The condition (ii) insures that $\{x_n\}$ is non-decreasing. If there exists an integer n such that

$$x_{2n} = x_{2n+1} = x_{2n+2}$$

then, x_{2n} is a common fixed point of S and T. Otherwise, suppose that

$$x_{2n} \neq x_{2n+1} \text{ or } x_{2n} \neq x_{2n+2}, \text{ for all } n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$. From $x_{2n} \preceq x_{2n+1}$ and applying (9) for $x = x_{2n}$ and $y = x_{2n+1}$, we obtain

$$\begin{aligned}\psi_1(\omega_1(x_{2n+1}, x_{2n+2})) &\leq \psi_2(M(x_{2n}, x_{2n+1})) \\ &\quad - \varphi(M_1(x_{2n}, x_{2n+1}), M_2(x_{2n}, x_{2n+1})) + L\phi(N(x_{2n}, x_{2n+1})).\end{aligned}$$

From $N(x_{2n}, x_{2n+1}) = \min\{\omega_1(x_{2n}, x_{2n+1}), 0, \omega_1(x_{2n}, x_{2n+2})\} = 0$, we have

$$\psi_1(\omega_1(x_{2n+1}, x_{2n+2})) \leq \psi_2(M(x_{2n}, x_{2n+1})) - \varphi(M_1(x_{2n}, x_{2n+1}), M_2(x_{2n}, x_{2n+1})).$$

And from

$$M(x_{2n}, x_{2n+1}) = \max\{\omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2}), \frac{\omega_2(x_{2n}, x_{2n+2})}{2}\}$$

and

$$\frac{\omega_2(x_{2n}, x_{2n+2})}{2} \leq \frac{\omega_1(x_{2n}, x_{2n+1}) + \omega_1(x_{2n+1}, x_{2n+2})}{2},$$

it follows that

$$M(x_{2n}, x_{2n+1}) = \max\{\omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2})\}.$$

If we suppose that there exists an integer n such that:

$$\omega_1(x_{2n}, x_{2n+1}) \leq \omega_1(x_{2n+1}, x_{2n+2}),$$

then

$$M(x_{2n}, x_{2n+1}) = M_2(x_{2n}, x_{2n+1}) = \omega_1(x_{2n+1}, x_{2n+2}).$$

Thus

$$\begin{aligned}&\psi_1(\omega_1(x_{2n+1}, x_{2n+2})) \\ &\leq \psi_2(\omega_1(x_{2n+1}, x_{2n+2})) - \varphi(0, \omega_1(x_{2n+1}, x_{2n+2})) \\ &< \psi_1(\omega_1(x_{2n+1}, x_{2n+2})),\end{aligned}$$

a contradiction. Hence, for all $n \in \mathbb{N}$,

$$\omega_1(x_{2n+1}, x_{2n+2}) < \omega_1(x_{2n}, x_{2n+1}).$$

By the same argument, if we take, in the inequality (9), $x = x_{2n-1}$ and $y = x_{2n}$ we obtain

$$\omega_1(x_{2n}, x_{2n+1}) < \omega_1(x_{2n-1}, x_{2n}), \text{ for all } n \in \mathbb{N}^*.$$

Then $\omega_1(x_{n+1}, x_{n+2}) < \omega_1(x_n, x_{n+1})$, for all $n \in \mathbb{N}$. Thus, the sequence $\{\omega_1(x_n, x_{n+1})\}$ is decreasing and bounded below. Therefore it $\omega-$converges to some $r \geq 0$. Since

$$\lim_{n\to+\infty} M(x_{2n}, x_{2n+1}) = \lim_{n\to+\infty} \max\{\omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2})\} = r,$$

$$M_1(x_{2n}, x_{2n+1}) = \max\{\omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n}, x_{2n+2})\} \geq \omega_1(x_{2n}, x_{2n+1})$$

and

$$M_2(x_{2n}, x_{2n+1}) = \max\{\omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n+1}, x_{2n+2})\} \geq \omega_1(x_{2n}, x_{2n+1}),$$

Then, according to the inequality (10), we have

$$\begin{aligned}\psi_1(\omega_1(x_{2n+1}, x_{2n+2})) &\leq \psi_2(M(x_{2n}, x_{2n+1})) - \varphi(\omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n}, x_{2n+1})) \\ &\leq \psi_2(M(x_{2n}, x_{2n+1})) - \varphi(0, \omega_1(x_{2n}, x_{2n+1})).\end{aligned}$$

Thus

$$\begin{aligned}&\limsup_{n\to+\infty} \psi_1(\omega_1(x_{2n+1}, x_{2n+2})) \\ &\leq \limsup_{n\to+\infty} (\psi_2(M(x_{2n}, x_{2n+1})) - \varphi(0, \omega_1(x_{2n}, x_{2n+1}))) \\ &\leq \limsup_{n\to+\infty} \psi_2(M(x_{2n}, x_{2n+1})) - \liminf_{n\to+\infty} \varphi(0, \omega_1(x_{2n}, x_{2n+1}))\end{aligned}$$

From the continuity of ψ_1 and ψ_2 and the lower semi-continuity of φ with respect to the second component, we get

$$\psi_1(r) \leq \psi_2(r) - \varphi(0, r),$$

which implies $r = 0$. Thus, $\lim_{n\to+\infty} \omega_1(x_n, x_{n+1}) = 0$.

Step.2. Let us prove that the sequence $\{x_n\}$ is $\omega-$Cauchy. For this, it's sufficient to show that the subsequence $\{x_{2n}\}$ is $\omega-$Cauchy. Assume the contrary. Then according to Lemma 2.9, there exists $\varepsilon > 0$ such that we can find two subsequences $\{m_k\}$ and $\{n_k\}$ of positive integers satisfying $n_k > m_k \geq k$ such that the following inequalities hold:

$$\omega_8(x_{2n_k}, x_{2m_k}) \geq \varepsilon \quad \text{and} \quad \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2m_k}) < \varepsilon.$$

If we take $x = x_{2n_k}$ and $y = x_{2m_k-1}$, then $x \preceq y$ and the inequality (9) becomes

$$\begin{aligned}\psi_1(\omega_1(x_{2n_k+1}, x_{2m_k})) \leq \psi_2(M(x_{2n_k}, x_{2m_k-1}))-\\ \varphi(M_1(x_{2n_k}, x_{2m_k-1}), M_2(x_{2n_k}, x_{2m_k-1})) + L\phi(N(x_{2n_k}, x_{2m_k-1}))\end{aligned}$$

where

$$\begin{aligned}M(x_{2n_k}, x_{2m_k-1}) = \max\{\omega_1(x_{2n_k}, x_{2m_k-1}), \omega_1(x_{2n_k}, x_{2n_k+1}), \omega_1(x_{2m_k-1}, x_{2m_k}),\\ \frac{\omega_2(x_{2n_k}, x_{2m_k}) + \omega_2(x_{2m_k-1}, x_{2n_k+1})}{2}\},\end{aligned}$$

$$M_1(x_{2n_k}, x_{2m_k-1}) = \max\{\omega_1(x_{2n_k}, x_{2m_k-1}), \omega_1(x_{2n_k}, x_{2n_k+1}), \omega_1(x_{2n_k}, x_{2m_k})\},$$

$$\begin{aligned}&M_2(x_{2n_k}, x_{2m_k-1})\\ &= \max\{\omega_1(x_{2n_k}, x_{2m_k-1}), \omega_1(x_{2m_k-1}, x_{2m_k}), \omega_1(x_{2m_k-1}, x_{2n_k+1})\}\end{aligned}$$

and

$$N(x_{2n_k}, x_{2m_k-1}) = \min\{\omega_1(x_{2n_k}, x_{2n_k+1}), \omega_1(x_{2m_k-1}, x_{2n_k+1}), \omega_1(x_{2n_k}, x_{2m_k})\}.$$

Since

$$\begin{aligned}\varepsilon \leq \omega_8(x_{2n_k}, x_{2m_k}) &\leq \omega_2(x_{2n_k}, x_{2m_k})\\ &\leq \omega_1(x_{2n_k}, x_{2m_k})\\ &\leq \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2m_k}) + \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2n_k})\\ &\leq \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2m_k}) + \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2n_k})\\ &\leq \varepsilon + \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2n_k}),\end{aligned}$$

it follows that $\lim_{k\to+\infty} \omega_2(x_{2n_k}, x_{2m_k}) = \lim_{k\to+\infty} \omega_1(x_{2n_k}, x_{2m_k}) = \varepsilon$.

From

$$\varepsilon \leq \omega_2(x_{2n_k}, x_{2m_k}) \leq \omega_1(x_{2n_k}, x_{2n_k+1}) + \omega_1(x_{2n_k+1}, x_{2m_k}),$$

we get

$$\begin{aligned}\varepsilon - \omega_1(x_{2n_k}, x_{2n_k+1}) &\leq \omega_1(x_{2n_k+1}, x_{2m_k})\\ &\leq \omega_{\frac{1}{2}}(x_{2n_k-1}, x_{2m_k}) + \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2n_k}) +\\ &\quad \omega_{\frac{1}{4}}(x_{2n_k}, x_{2n_k+1})\end{aligned}$$

$$\leq \varepsilon + \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2n_k}) + \omega_{\frac{1}{4}}(x_{2n_k}, x_{2n_k+1}).$$

Thus $\lim\limits_{k\to+\infty} \omega_1(x_{2n_k+1}, x_{2m_k}) = \varepsilon$.
Similarly, using

$$\varepsilon \leq \omega_2(x_{2n_k}, x_{2m_k}) \leq \omega_1(x_{2n_k}, x_{2m_k-1}) + \omega_1(x_{2m_k-1}, x_{2m_k}),$$

we get

$$\begin{aligned}\varepsilon - \omega_1(x_{2m_k-1}, x_{2m_k}) &\leq \omega_1(x_{2n_k}, x_{2m_k-1}) \\ &\leq \omega_{\frac{1}{2}}(x_{2n_k}, x_{2n_k-1}) + \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2m_k}) + \\ &\quad \omega_{\frac{1}{4}}(x_{2m_k}, x_{2m_k-1}) \\ &\leq \omega_{\frac{1}{2}}(x_{2n_k}, x_{2n_k-1}) + \varepsilon + \omega_{\frac{1}{4}}(x_{2m_k}, x_{2m_k-1}).\end{aligned}$$

Therefore $\lim\limits_{k\to+\infty} \omega_1(x_{2n_k}, x_{2m_k-1}) = \varepsilon$.
From

$$\begin{aligned}&\omega_8(x_{2n_k}, x_{2m_k}) - \omega_4(x_{2n_k}, x_{2n_k+1}) - \omega_2(x_{2m_k-1}, x_{2m_k}) \\ &\leq \omega_2(x_{2m_k-1}, x_{2n_k+1}) \\ &\leq \omega_1(x_{2m_k-1}, x_{2n_k}) + \omega_1(x_{2n_k}, x_{2n_k+1}),\end{aligned}$$

we get $\lim\limits_{k\to+\infty} \omega_2(x_{2m_k-1}, x_{2n_k+1}) = \varepsilon$. Since

$$\begin{aligned}&\omega_2(x_{2m_k-1}, x_{2n_k+1}) \leq \omega_1(x_{2m_k-1}, x_{2n_k+1}) \\ &\leq \omega_{\frac{1}{2}}(x_{2m_k-1}, x_{2m_k}) + \omega_{\frac{1}{4}}(x_{2n_k-1}, x_{2m_k}) + \omega_{\frac{1}{8}}(x_{2n_k-1}, x_{2n_k}) + \\ &\quad \omega_{\frac{1}{8}}(x_{2n_k}, x_{2n_k+1}) \\ &\leq \omega_{\frac{1}{2}}(x_{2m_k-1}, x_{2m_k}) + \varepsilon + \omega_{\frac{1}{8}}(x_{2n_k-1}, x_{2n_k}) + \omega_{\frac{1}{8}}(x_{2n_k}, x_{2n_k+1})\end{aligned}$$

and by letting $k \to +\infty$, we obtain $\lim\limits_{k\to+\infty} \omega_1(x_{2m_k-1}, x_{2n_k+1}) = \varepsilon$. Therefore

$$\begin{aligned}\lim_{k\to+\infty} M(x_{2n_k}, x_{2m_k-1}) &= \lim_{k\to+\infty} M_1(x_{2n_k}, x_{2m_k-1}) \\ &= \lim_{k\to+\infty} M_2(x_{2n_k}, x_{2m_k-1}) = \varepsilon\end{aligned}$$

and

$$\lim_{k\to+\infty} N(x_{2n_k}, x_{2m_k-1}) = 0.$$

From the continuity of ψ_1, ψ_2 and ϕ and the semi-continuity of φ with respect to the second component, we have $\psi_1(\varepsilon) \leq \psi_2(\varepsilon) - \varphi(0, \varepsilon)$, a contradiction. Therefore the sequence $\{x_n\}$ is $\omega-$Cauchy. Using the $\omega-$completeness of C, there exists $x \in C$ such that $\lim_{n\to+\infty} \omega_1(x_n, x) = 0$.

Step.3. We show that $x \in \mathfrak{F}(T)$. Since $x_{2n} \preceq x$, then according to the inequality (9) we have

$$\psi_1(\omega_1(x_{2n+1}, Tx)) \leq \psi_2(M(x_{2n}, x)) - \varphi(M_1(x_{2n}, x), M_2(x_{2n}, x)) + L\phi(N(x_{2n}, x))$$

where

$$M(x_{2n}, x) = \max\{\omega_1(x_{2n}, x), \omega_1(x_{2n}, x_{2n+1}), \omega_1(x, Tx), \frac{\omega_2(x_{2n}, Tx) + \omega_2(x, x_{2n+1})}{2}\}.$$

From $\omega_2(x_{2n}, Tx) \leq \omega_1(x_{2n}, x) + \omega_1(x, Tx)$, we obtain

$$\lim_{n\to+\infty} M(x_{2n}, x) = \omega_1(x, Tx)),$$

$$M_1(x_{2n}, x) = \max\{\omega_1(x_{2n}, x), \omega_1(x_{2n}, x_{2n+1}), \omega_1(x_{2n}, Tx)\},$$

$$M_2(x_{2n}, x) = \max\{\omega_1(x_{2n}, x), \omega_1(x, Tx), \omega_1(x, x_{2n+1})\} \geq \omega_1(x, Tx)$$

and

$$\lim_{n\to+\infty} N(x_{2n}, x) = \lim_{n\to+\infty} \min\{\omega_1(x_{2n}, x_{2n+1}), \omega_1(x, x_{2n+1}), \omega_1(x_{2n}, Tx)\} = 0.$$

Since ω satisfies Fatou property, we have

$$\begin{aligned}\psi_1(\omega_1(x, Tx)) &\leq \psi_1(\liminf_{n\to+\infty} \omega_1(x_{2n+1}, Tx)) \\ &\leq \limsup_{n\to+\infty} \psi_1(\omega_1(x_{2n+1}, Tx)) \\ &\leq \limsup_{n\to+\infty}(\psi_2(M(x_{2n}, x)) - \varphi(0, \omega_1(x, Tx)) \\ &+ L\phi(N(x_{2n}, x))) \\ &\leq \psi_2(\omega_1(x, Tx)) - \varphi(0, \omega_1(x, Tx)).\end{aligned}$$

Which implies that $\omega_1(x, Tx) = 0$. The regularity insures that $Tx = x$.

Step.4. We show that $\mathfrak{F}(T) = \mathfrak{F}(S)$. Let $z \in \mathfrak{F}(T)$ and suppose that $Sz \neq z$. Applying the inequality (9) for $x = y = z$, we get

$$\psi_1(\omega_1(Sz, Tz)) \leq \psi_2(M(z,z)) - \varphi(M_1(z,z), M_2(z,z)) + L\phi(N(z,z)).$$

It's easy to cheek that $M(z,z) = M_1(z,z) = M_2(z,z) = \omega_1(z, Sz)$ and $N(z,z) = 0$ Thus

$$\begin{aligned}\psi_1(\omega_1(Sz,z)) &\leq \psi_2(\omega_1(Sz,z)) - \varphi(\omega_1(Sz,z), \omega_1(Sz,z)) \\ &< \psi_1(\omega_1(Sz,z)),\end{aligned}$$

a contradiction. Then $Sz = z$. Therefore $\mathfrak{F}(T) \subset \mathfrak{F}(S)$
By the same argument, we prove that $\mathfrak{F}(S) \subset \mathfrak{F}(T)$. Hence,
$x \in \mathfrak{F}(S)$.

Step.5. Let us prove that x is unique. For that, assume that there exists another common fixed point y of S and T. Since $\mathfrak{F}(S,T)$ is totally ordered, x and y are comparable. According to (9), we have

$$\psi_1(\omega_1(x,y)) \leq \psi_2(M(x,y)) - \varphi(M_1(x,y), M_2(x,y)) + L\phi(N(x,y))$$

where

$$M(x,y) = \max\{\omega_1(x,y), \omega_2(x,y)\} = \omega_1(x,y),$$
$$M_1(x,y) = M_2(x,y) = \omega_1(x,y)$$

and

$$N(x,y) = 0.$$

From $\psi_1(\omega_1(x,y)) \leq \psi_2(\omega_1(x,y)) - \varphi(\omega_1(x,y), \omega_1(x,y))$, we conclude that $\omega_1(x,y) = 0$. The regularity of ω insures that $x = y$. $\square$

Now, if we take $S = T$ and suppose that T is non-decreasing and there exists $x_0 \in C$ such that $x_0 \preceq Tx_0$, one can find the condition (ii) of the above theorem and prove the following corollary

Corollary 3.2. *Let $(X, \preceq, \omega)$ be partially ordered modular metric space . Let C be an ω-complete nonempty subset of X_ω and $T : C \to C$ be a non-decreasing mapping. Let $L \geq 0$. If the following conditions are verified:*

(i) ω satisfies the Δ_2-type condition and the Fatou property;

(ii) for all comparable elements $x, y \in C$

$$\psi_1(\omega_1(Tx, Ty)) \leq \psi_2(M(x,y)) - \varphi(M_1(x,y), M_2(x,y))$$

$$+ L\phi(N(x,y)) \quad (2)$$

where

$$M(x,y) = \max\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \frac{\omega_2(x,Ty)+\omega_2(y,Tx)}{2}\};$$

(iii) there exists an element $x_0 \in C$ such that $x_0 \preceq Tx_0$;

(iv) ω satisfies the property (P),

then T has a unique fixed point in C provided that $\mathfrak{F}(T)$ is totally ordered.

Now, if we suppose that ω is convex and T is $\omega-$continuous, we obtain the following result:

Theorem 3.3. *Let $(X, \preceq, \omega)$ be a partially ordered modular metric space where ω is convex. Let C be an ω-complete nonempty subset of X_ω and $T : C \to C$ be a continuous non-decreasing mapping. Let $L \geq 0$. If the following conditions are verified*

(i) ω satisfies the Δ_2-type condition;

(ii) for all comparable elements $x, y \in C$

$$\psi_1(\omega_1(Tx,Ty)) \leq \psi_2(M'(x,y)) - \varphi(M_1(x,y), M_2(x,y)) + L\phi(N(x,y)) \quad (3)$$

where

$$M'(x,y) = \max\{\omega_1(x,y), \omega_1(x,Tx), \omega_1(y,Ty), \omega_2(x,Ty)+\omega_2(y,Tx)\};$$

(iii) there exists an element $x_0 \in C$ such that $x_0 \preceq Tx_0$;

(iv) ω satisfies the property (P),

then T has a unique fixed point in C provided that $\mathfrak{F}(T)$ is totally ordered.

Proof. We reproduce the same proof as in Theorem 3.3, applying Lemma 2.10 we assert that the sequence $\{x_n\}$ is $\omega-$Cauchy. Let x be the $\omega-$limit of $\{x_n\}$. Since $\lim_{n\to+\infty} \omega_1(x, x_n) = 0$ and T is $\omega-$continuous, then

$$\lim_{n\to+\infty} \omega_1(Tx, Tx_n) = 0.$$

Using the inequality $\omega_2(x, Tx) \le \omega_1(x, x_{n+1}) + \omega_1(Tx, x_{n+1})$ and letting $n \to +\infty$, we obtain $\omega_2(x, Tx) = 0$. Since ω is regular, we get $Tx = x$.

The same argument as in step 4 of the proof of Theorem 3.3 insures the uniqueness of the fixed point x. □

Now, Let us consider a complete partially ordered metric space $(X, \preceq, d)$. In the above theorem, if we take $\psi_1 = \psi_2 = \phi = \psi$ and define the modular as follows: $\forall \lambda > 0$, $\forall (x, y) \in X^2$: $\omega_\lambda(x, y) = \frac{d(x,y)}{\lambda}$, we obtain the theorem 2.5 of Azizi et al.(see [5]).

Corollary 3.4. *Let $(X, \preceq, d)$ be an ordered complete metric space. Let $T : X \to X$ be non-decreasing (with respect to $\preceq$), continuous and almost generalized $\mathcal{C}$-contractive. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point. In particular, if $\mathfrak{F}(T)$ is a totally ordered subset of X, then T has a unique fixed point.*
Where the almost generalised $\mathcal{C}-$contraction is defined for all comparable elements $x, y \in X$ by the inequality:

$$\psi(\omega_1(Tx, Ty)) \le \psi(M(x, y)) - \varphi(M_1(x, y), M_2(x, y)) + L\psi(N(x, y))$$

where M, M_1, M_2 and N are defined in the first section.

Example 3.5. Consider the space $X = [0, 1]$ ordered by " $\preceq$ " which is the reverse of the usual order between the reals ($x \preceq y \Leftrightarrow x \ge y$) and endowed with the modular metric defined for all $\lambda > 0$ as follows:

$$\omega_\lambda(x, y) = \begin{cases} \frac{x+y}{\lambda} & si \quad x \ne y \\ 0 & si \quad x = y \end{cases}$$

Consider the two self-mappings S and T defined as follows:

$$Sx = \tfrac{x}{4}, \text{ for all } x \in [0, 1] \quad \text{and} \quad Tx = \begin{cases} \frac{x}{8} & if \quad 0 \le x < 1 \\ \frac{1}{2} & if \quad x = 1 \end{cases}$$

Consider $L \ge 0$ and the four functions defined for $t, s \in [0, 1]$ as follows:

$$\psi_1(t) = t, \ \ \psi_2(t) = \tfrac{39}{40}t, \ \ \varphi(t, s) = \tfrac{3}{16}(t + s) \text{ and } \phi(t) = t^2.$$

We can see that the functions ψ_1, ψ_2, φ and ϕ satisfy all conditions (a) to (g) described in the top of the section 4. Now, we will show that the following statements hold:

(i) $X_\omega = X$ is $\omega-$complete modular metric space;

(ii) ω satisfies the Δ_2-type condition and Fatou property;

(iii) $1 \preceq S1 \preceq TS1 \preceq STS1 \preceq (TS)^2 1 \preceq S(TS)^2 1 \preceq ...$

(iv) $(X, \preceq)$ satisfies the property (P);

(v) For all comparable elements $x, y \in X$, we have:

$$\psi_1(\omega_1(Sx, Ty)) \leq \psi_2(M(x,y)) - \varphi(M_1(x,y), M_2(x,y)) + L\phi(N(x,y).$$

Proof. (i) We know that $X_\omega(0) \subset X$. Let $x \in X$. Since

$$\lim_{\lambda \to +\infty} \omega_\lambda(x, 0) = \lim_{\lambda \to +\infty} \frac{x}{\lambda} = 0,$$

then $x \in X_\omega(0)$, which prove that $X_\omega(0) = X$.
Let $\{x_n\}$ be a $\omega-$cauchy sequence in X.
From $\omega_1(x_n, 0) \leq \omega_1(x_n, x_m)$, we assert that 0 is $\omega-$limits of $\{x_n\}$. Therefore X is $\omega-$complete.

(ii) For all $\alpha, \lambda \in]0, +\infty[$, $\omega_{\frac{\lambda}{\alpha}}(x, y) = \alpha\ \omega_\lambda(x, y)$. Hence ω satisfies the Δ_2-type condition with $K_\alpha = \alpha$.
Let $\{x_n\} \subset X$ be a sequence that $\omega-$converges to some x. Then, for all $\varepsilon > 0$, there exists an integer N such that for each $n \geq N$ we have

$$x \leq x + x_n = \omega_1(x_n, x) < \varepsilon,$$

which implies that $x = 0$. From $\omega_1(0, y) = y \leq x_n + y = \omega_1(x_n, y)$ for all $y \in X$, we conclude that ω satisfies Fatou property.

(iii) It is trivial.

(iv) Since 0, the greatest element in $(X, \preceq)$, is the $\omega-$limit of all $\omega-$convergent sequences in X, then the property (P) holds.

(v) Let x and y be two comparable elements in X. We will show that

$$\psi_1(\omega_1(Sx, Ty)) \leq \psi_2(M(x,y)) - \varphi(M_1(x,y), M_2(x,y)) + L\phi(N(x,y)) \qquad (4)$$

To see this, we consider the following cases:

- Let $x, y \in]0, 1[$ and $x = y$, then
  $M(x,x) = M_1(x,x) = M_2(x,x) = \frac{5x}{4}$, $N(x,x) = \frac{9x}{8}$ and

$$\begin{aligned}\psi_1(\omega_1(Sx, Tx)) &= \omega_1(\frac{x}{4}, \frac{x}{8}) = \frac{3x}{8} \\ &\leq \frac{39}{40} \times \frac{5x}{4} - \frac{3}{16} \times \frac{5x}{2} + L(\frac{9x}{8})^2. \\ &\leq \psi_2(M(x,x)) - \varphi(M_1(x,x), M_2(x,x)) + L\phi(N(x,x)).\end{aligned}$$

- Let $0 < x < y < 1$.
  - If $x \le \frac{y}{8} < y$, then $M(x,y) = M_2(x,y) = \frac{9y}{8}$, $M_1(x,y) = x + y$ and $N(x,y) = \frac{5x}{4}$ and
$$\begin{aligned}&\psi_1(\omega_1(Sx,Ty)) + \varphi(M_1(x,y), M_2(x,y))\\&= \frac{x}{4} + \frac{y}{8} + \frac{3}{16}(\frac{9y}{8} + x + y)\\&\le \frac{351y}{320} = \psi_2(M(x,y)) \le \psi_2(M(x,y)) + L\phi(N(x,y)).\end{aligned}$$
    Hence the inequality (11) holds.
  - If $\frac{y}{8} < x < y$, then $M(x,y) = M_1(x,y) = M_2(x,y) = x + y$ and
$$\begin{aligned}&\psi_1(\omega_1(Sx,Ty)) = \frac{x}{4} + \frac{y}{8} \le \frac{3}{5}(x+y)\\&\le \psi_2(M(x,y)) - \varphi(M_1(x,y), M_2(x,y)) + L\phi(N(x,y).\end{aligned}$$
- Let $0 < y < x < 1$.
  - If $y < \frac{x}{4} < x$, then $M(x,y) = M_1(x,y) = \frac{5x}{4}$, $M_2(x,y) = x + y$, $N(x,y) = y + \frac{x}{4}$ and
$$\begin{aligned}&\psi_1(\omega_1(Sx,Ty)) + \varphi(M_1(x,y), M_2(x,y))\\&= \frac{x}{4} + \frac{y}{8} + \frac{3}{16}(\frac{5x}{4} + x + y) \le \frac{39x}{32} = \psi_2(M(x,y)).\end{aligned}$$
    Hence the inequality (11) holds.
  - If $\frac{x}{4} \le y < x$, then $M(x,y) = M_1(x,y) = M_2(x,y) = x + y$. Again (11) is valid as in the above case.
- Let $x = y = 0$. In this case (11) is obviously verified.
- Let $x = y = 1$. Then $M(x,y) = M_1(1,1) = M_2(1,1) = \frac{3}{2}$, $N(1,1) = \frac{5}{4}$ and
$$\begin{aligned}&\psi_1(\omega_1(S1,T1)) = \frac{3}{4} \le \frac{9}{10} + \frac{25L}{16}\\&\le \psi_2(M(1,1)) - \varphi(M_1(1,1), M_2(1,1)) + L\phi(N(1,1)).\end{aligned}$$
- Let $y = 1$ and $0 \le x < 1$.

– If $y = 1$ and $x = \frac{1}{2}$, then
$M(\frac{1}{2}, 1) = M_2(\frac{1}{2}, 1) = M_1(\frac{1}{2}, 1) = \frac{3}{2}$, $N(\frac{1}{2}, 1) = \frac{5}{8}$ and

$$\begin{aligned} &\psi_1(\omega_1(S\frac{1}{2}, T1)) = \frac{5}{8} \leq \frac{9}{10} + \frac{25L}{64} \\ &\leq \psi_2(M(\frac{1}{2}, 1)) - \varphi(M_1(\frac{1}{2}, 1), M_2(\frac{1}{2}, 1)) + L\phi(N(\frac{1}{2}, 1)). \end{aligned}$$

– If $y = 1$ and $x < \frac{1}{2}$,then
$M(x, 1) = M_2(x, 1) = \frac{3}{2}$, $M_1(x, 1) = 1 + x$, $N(x, 1) = \frac{5x}{4}$ and

$$\begin{aligned} &\psi_1(\omega_1(Sx, T1)) + \varphi(M_1(x, 1), M_2(x, 1)) \\ &= \frac{x}{4} + \frac{1}{2} = \frac{7x}{16} + \frac{15}{32} \leq \frac{7}{16} \times \frac{1}{2} + \frac{15}{32} \\ &\leq \psi_2(M(x, 1)) \leq \psi_2(M(x, 1)) + L\phi(N(x, 1)). \end{aligned}$$

Therefore, the inequality (11) holds.

– If $y = 1$ and $x > \frac{1}{2}$, then
$M(x, 1) = M_1(x, 1) = M_2(x, 1) = 1 + x$ and

$$\begin{aligned} &\psi_1(\omega_1(Sx, T1)) = \frac{x}{4} + \frac{1}{2} \leq \frac{3}{5}(x + 1) \\ &\leq \psi_2(M(x, 1)) - \varphi(M_1(x, 1), M_2(x, 1)) + L\phi(N(x, 1)). \end{aligned}$$

• Let $y = 0$ and $0 < x \leq 1$, then
$M(x, 0) = M_1(x, 0) = \frac{5x}{4}$, $M_2(x, 0) = x$, $N(x, 0) = \frac{x}{4}$ and

$$\psi_1(\omega_1(Sx, T0)) = \omega_1(\frac{x}{4}, 0) = \frac{x}{4}$$

and

$$\psi_2(M(x, 0)) - \varphi(M_1(x, 0), M_2(x, 0)) + L\phi(N(x, 0)) = \frac{51x}{64} + \frac{Lx^2}{16}$$

Therefore

$$\psi_1(\omega_1(Sx, T0)) \leq \psi_2(M(x, 0)) - \varphi(M_1(x, 0), M_2(x, 0)) + L\phi(N(x, 0)).$$

• Let $x = 1$ and $0 \leq y < 1$.

– If $y \leq \frac{1}{4}$, then $M(1,y) = M_1(1,y) = \frac{5}{4}$, $M_2(1,y) = 1+y$ and

$$\begin{aligned}&\psi_1(\omega_1(S1,Ty)) + \varphi(M_1(1,y), M_2(1,y)\\&= \frac{1}{4} + \frac{y}{8} + \frac{27}{64} + \frac{3y}{16} = \psi_2(M(1,y))\\&\leq \psi_2(M(1,y)) + L\phi(N(1,y)).\end{aligned}$$

Hence (11) holds.

– If $y > \frac{1}{4}$, then $M(1,y) = M_1(1,y) = M_2(1,y) = y+1$ and $N(1,y) = y + \frac{1}{4}$.Hence

$$\begin{aligned}&\psi_1(\omega_1(S1,Ty)) = \omega_1(\frac{1}{4}, \frac{y}{8}) = \frac{1}{4} + \frac{y}{8}\\&\leq \frac{3}{5}(1+y) + L(y + \frac{1}{4})^2\\&\leq \psi_2(M(1,y)) - \varphi(M_1(1,y), M_2(1,y)) + L\phi(N(1,y)).\end{aligned}$$

- Let $x = 0$ and $0 < y < 1$,then $M(x,y) = M_2(0,y) = \frac{9y}{8}$, $M_1(0,y) = y$, $N(0,y) = 0$, $\psi_1(\omega_1(S0,Ty)) = \frac{y}{8}$ and

$$\psi_2(M(0,y)) - \varphi(M_1(0,y), M_2(0,y)) + L\phi(N(0,y)) = \frac{447y}{640}.$$

  It follows that

$$\psi_1(\omega_1(S0,Ty)) \leq \psi_2(M(0,y)) - \varphi(M_1(0,y), M_2(0,y)) + L\phi(N(0,y)).$$

Consequently, the statement (v) holds. □

Then, according to Theorem 5.1, S and T admit a unique common fixed point in X, which is 0.

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