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MAIN SCALARS FOR A THREE DIMENSIONAL FINSLER SPACE WITH A GENERAL (α, β)-METRIC

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Abstract: In this paper, we find the main scalars of a three dimensional Finsler space equipped with a general (α, β)-metric $F = \alpha + \kappa\beta + \varepsilon\beta^2/\alpha$ (where κ and ε are constants which are not zero simultaneously). Consequently, we will obtain results that give the main scalars of special three dimensional (α, β)-metrics such as square metric $F = (\alpha+\beta)^2/\alpha$ and the first approximate matsumoto metric $F = \alpha + \beta + \beta^2/\alpha$. Moreover, we present a sufficient and necessary condition for Finsler spaces equipped with (α, β)-metrics to be Riemannian spaces. Some fundamental theorems on main scalars of three dimensional Finsler spaces has also been dealt.

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1. Introduction

A Finsler metric on a manifold is the set of Minkowski norms on its tangent

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bundle. These norms are not necessarily reversible and also are not necessarily induced by an inner product. When these norms are represented by an inner product on the tangent space, then desired Finsler metric is called the Riemannian metric. Therefore Finsler metrics are the generalization of Riemannian metrics. In 1917, Paul Finsler studied Finsler metrics in his doctoral thesis and defined them as follows:

Let M be an $n-$ dimensional manifold and TM be its tangent bundle. For every $x = \left(x^i\right) \in M$ and $y = \left(y^i\right) \in TM$, a Finsler structure on an n-dimensional manifold M is a function $F : TM \to [0, \infty)$ such that has the following properties [1]:

1. Regularity: $F(x, y)$ is smooth in any point of $TM - \{0\}$.
2. Positive homogeneity: $F\left(x, my\right) = mF\left(x, y\right)$, for all $m > 0$.
3. Strong convexity: The Hessian matrix $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is a positive definite at any point of $TM - \{0\}$.

In this case, the pair (M, F) is called an n-dimensional Finsler space.

Finsler metrics have many applications in various sciences like biology, engineering and physics. The most application class of Finsler metrics is class of (α, β)-metrics. In 1972, M. Matsumoto proposed the notion of (α, β)-metrics by generalization of Randers metric [1]. The (α, β)-metrics are Finsler metrics that are composed of a Riemannian metric α and an 1-form β. One of the most important properties of (α, β)-metrics is their computability. Due to this property, many worthly results has been obtained about (α, β)-metrics. Finsler spaces equipped with (α, β)- metrics are investigated with details in [1].

In three dimensional Finsler spaces, there are three scalar function J, I and H such that the sum of I and H is FC. These scalars are called the main scalars, which are of essential scalar fields for the theory of three dimensional Finsler spaces. Hence, finding them is of particular importance in geometry of three dimensional Finsler spaces. In 2009, Chaubey, Prasad and Pandy studied the main scalars in three dimensional Finsler spaces equipped with (α, β)-metrics. They also found the main scalars J, I and H of some three dimensional special (α, β)-metrics [2]. In this paper, we continue the above paper and we find main scalars of a three dimensional Finsler space equipped with general (α, β)-metric

$$F = \alpha + \kappa\beta + \frac{\varepsilon\beta^2}{\alpha}, \tag{1}$$

where κ and ε are constants which are not zero simultaneously. We first find the main scalars of J, I and H corresponding to (α, β)-metric (1) and then we determine the main scalars of some special cases of it like square metric $F=(\alpha+\beta)^2/\alpha$ and the first approximate Matsumoto metric $F=\alpha+\beta+\beta^2/\alpha$.

The ordering of this paper is as: In Section 2, we review some important geometric objects that are used throughout this paper. In Section 3, we study some fundamental theorems and results on main scalars J, I and H of three dimensional Finsler spaces equipped with (α,β)-metrics. In the Section 4, we first obtain the values of J, I and H of the general metric (1). Then according to the obtained results, we find the main scalars of square metric and the first approximate Matsumoto metric in a three dimensional Finsler space. Finally in the last section, we present a sufficient and necessary condition for Finsler spaces equipped with (α,β)-metrics to be Riemannian spaces. Moreover, we find an expression of Cartan tensor vector for each case of the (α,β)-metrics given in Section 4.

2. Preliminaries

In this section, we study briefly some important definitions and known results about Finsler spaces. We apply them in the next sections:

For any $y\neq 0$, we can define three important tensors the fundemental tensor g_{ij}, Cartan tensor C_{ijk} and the angular metric tensor h_{ij} on TM as follows [1]:

$$g_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},\quad C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k},\quad h_{ij}=F\frac{\partial^2 F}{\partial y^i\partial y^j} \tag{2}$$

Furthermore, Cartan tensor vector C_i is written as $C_i=C_{ijk}g^{ij}$, where g^{ij} is the inverse of (g_{ij}) and we also have $C^2=g^{ij}C_iC_j$ [2].

Definition 1 ([1]). Let $\alpha(x,y)=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta(x,y)=b_i(x)y^i$ is a one-form on M. A Finsler metric $F(\alpha(x,y),\beta(x,y))$ is called an (α,β)-metric if F is a positively homogeneous function of degree one in two variables α and β.

From [1], we know that a Finsler space is a Riemannian space, if and only if, Cartan tensor $C_{ijk}=0$. This condition leads to the following proposition:

Proposition 2 ([1]). *Any Finsler space of dimention $n>2$ with (α,β)-metric $F(\alpha,\beta)$ is a Riemannian space, if and only if F is written in the form*

$$F^2=u_1\alpha^2+u_2\beta^2,$$

where u_1 and u_2 are constants.

Definition 3 ([1]). The (α, β)-metric defined in the form $F = \alpha + \beta$ is called Randers metric.

In 1941, well-known physicist G. Randers introduced Randers metric from the point of view of general relativity ([3]). Randers metric is the first, most important and the simplest kind of (α, β)-metrics. In fact, (α, β)-metrics are the generalization of the Randers metric. At the first, in 1957, R.S.Ingarten recognized Randers metric as a Finsler metric and he was also the first to call it Randers metric. Later on, in 1972, M. Matsumoto proposed the notion of (α, β)-metrics by direct generalizing Randers metric [1]

Another important kind of (α, β)-metrics is the Finsler metric that L. Berwald constructed on the unit ball $\mathbb{B}^n$ with vanishing constant flag curvature in 1929 as follows [4]:

$$F\left(x,y\right) := \frac{\left(\langle x,\, y\rangle + \sqrt{|y|^2\left(1-|x|^2\right)+\langle x,\, y\rangle^{\,2}}\right)^2}{\left(1-|x|^2\right)^2\sqrt{|y|^2\left(1-|x|^2\right)+\langle x,\, y\rangle^{\,2}}}.$$

Berwald's metric can be written relative to α and β as [4]:

$$F = \frac{(\alpha+\beta)^2}{\alpha}, \tag{3}$$

where

$$\alpha = \frac{\sqrt{|y|^2\left(1-|x|^2\right)+\langle x,\, y\rangle^{\,2}}}{\left(1-|x|^2\right)^2}, \quad \beta = \frac{\langle x,\, y\rangle}{\left(1-|x|^2\right)^2}. \tag{4}$$

Definition 4 ([4]). The (α, β)-metric (3) is called a square metric.

Another interesting kind of (α, β)-metrics is the rth approximate Matsumoto metric or general approximate Matsumoto metric [5]:

$$F = \alpha \sum_{k=0}^{r} (\frac{\beta}{\alpha})^k. \tag{5}$$

If we consider $r = 0$, then (5) is reduced to Riemannian metric $F = \alpha$. If we put $r = 1$, therefore (2.4) becomes the Randers metric $F = \alpha + \beta$. If we set $r = 2$, thus (5) is converted to

$$F = \alpha + \beta + \frac{\beta^2}{\alpha}. \tag{6}$$

Definition 5 ([5]). The (α,β)-metric (6) is called the first approximate Matsumoto metric.

In the n-dimensional Finsler spaces equipped with (α,β)-metrics, the following invariants and relations are well-known [2]:

$$p=\frac{F}{\alpha}F_\alpha,\ p_{-1}=\frac{1}{\alpha}(FF_{\alpha\beta}+F_\alpha F_\beta),\ p_0=FF_{\beta\beta}+(F_\beta)^2,\ q_0=FF_{\beta\beta} \tag{7}$$

$$r_{-1}=pp_{0\beta}-3p_{-1}q_0 \tag{8}$$

$$H_{ij}=h_{ij}+\frac{r_{-1}}{3(p_{-1})^3}P_iP_j, \tag{9}$$

where $P_k=p_{-1}b_k+p_{-2}Y_k$ and $Y_i=a_{ij}y^j$.

$$2pC_{ijk}=\pi_{(ijk)}\left\{H_{ij}P_k\right\}, \tag{10}$$

where $\pi_{(ijk)}\{\,\}$ is the cyclic sum of permutations of i, j and k.

$$c_{-1}=\frac{p_{-1}C_i}{P_i}. \tag{11}$$

3. The Main Scalars of Three Dimensional (α,β)-Metrics

Now, we restrict our discussion to three dimensional Finsler spaces:

Definition 6 ([6]). In a three dimensional Finsler space, the orthogonal frame $(l_i,\quad m_i,\quad n_i)$ is called Moor's frame where $l_i=\partial L/\partial y^i$, $m_i=C^i/C$ and n_i is the unit vector field orthogonal to m_i and l_i.

The components of Moor's frame satisfy in the following relations [6]:

$$(i)g^{ij}n_il_j=0,\quad g^{ij}n_im_j=0,\quad g^{ij}n_in_j=1.$$

$$(ii)n^1=\frac{l_2m_3-l_3m_2}{\sqrt{\det(g_{ij})}},\quad n^2=\frac{l_3m_1-l_1m_3}{\sqrt{\det(g_{ij})}},\quad n^3=\frac{l_1m_2-l_2m_1}{\sqrt{\det(g_{ij})}}.$$

It is known that an arbitrary tensor can be written in terms of the components of the Moor's frame in three dimensional Finsler spaces. With respect to it, we have [2,5]:

$$h_{ij}=m_im_j+n_in_j, \tag{12}$$

$$g_{ij}=l_il_j+m_im_j+n_in_j, \tag{13}$$

$$C_{ijk}=\frac{Hm_im_jm_k-J\pi_{(ijk)}\{m_im_jn_k\}+I\pi_{(ijk)}\{m_in_jn_k\}+Jn_in_jn_k}{F}. \tag{14}$$

Definition 7 ([2]). H, I and J in (14) are called main scalars of the three dimensional Finsler spaces.

Theorem 8 ([2]). *In a three dimensional Finsler space with an* (α, β)-*metric, the main scalars* J, I *and* H *are given by*

$$J = 0, \quad I = \frac{FCp_{-1}}{2pc_{-1}}, \quad H = \frac{FC(3p_{-1}(c_{-1})^2 + r_{-1}C^2)}{2p(c_{-1})^3}. \tag{15}$$

Proof. From the second component of the Moor's frame, we have $C_i = Cm_i$. Setting $C_i = Cm_i$ in (11), we get

$$P_i = \frac{Cm_i p_{-1}}{c_{-1}}. \tag{16}$$

If we put (12) and (16) in (9), we obtain

$$H_{ij} = m_i m_j \left(\frac{C^2 r_{-1}}{3p_{-1}(c_{-1})^2} + 1 \right) + n_i n_j. \tag{17}$$

Also, if we set (16) and (17) in (10), it follows that

$$C_{ijk} = \frac{\pi_{(ijk)}\{m_i m_j m_k\} C(C^2 r_{-1} + 3(c_{-1})^2 p_{-1})}{6p(c_{-1})^3} + \frac{\pi_{(ijk)}\{n_i n_j m_k\} Cp_{-1}}{2pc_{-1}}. \tag{18}$$

Since $\pi_{(ijk)}\{\}$ is the cyclic sum of permutation of i, j and k, then (18) is reduced to the following:

$$C_{ijk} = m_i m_j m_k \frac{C(C^2 r_{-1} + 3(c_{-1})^2 p_{-1})}{2p(c_{-1})^3} + \pi_{(ijk)}\{n_i n_j m_k\} \frac{Cp_{-1}}{2pc_{-1}}. \tag{19}$$

Now, if we set (19) in (14) we have

$$\begin{aligned} &F m_i m_j m_k \frac{C(C^2 r_{-1} + 3(c_{-1})^2 p_{-1})}{2p(c_{-1})^3} + F\pi_{(ijk)}\{n_i n_j m_k\} \frac{Cp_{-1}}{2pc_{-1}} \\ &= H m_i m_j m_k - J\pi_{(ijk)}\{m_i m_j n_k\} + I\pi_{(ijk)}\{m_i n_j n_k\} + J n_i n_j n_k. \end{aligned} \tag{20}$$

Comparing the two sides of the recent equation, it follows that

$$H = \frac{FC(3p_{-1}(c_{-1})^2 + r_{-1}C^2)}{2p(c_{-1})^3}, \quad I = \frac{FCp_{-1}}{2pc_{-1}}, \quad J = 0. \qquad \square$$

Theorem 9 ([2]). *Let (M, F) be a three dimensional Finsler space equipped with the (α, β)-metric $F(\alpha, \beta)$. The main scalars I and H satisfy in the following equation:*

$$I + H = FC. \tag{21}$$

Proof. From [7], we know that

$$FC_i = (H + I)m_i. \tag{22}$$

Putting $C_i = Cm_i$ in (22), we get $H + I = FC$. □

Theorem 10 ([2]). *Consider a three dimensional Finsler space with (α, β)-metric $F(\alpha, \beta)$.*

(i) If the main scalar $I = 0$, then F satisfy in the following differential equation:

$$FF_{\alpha\beta} + F_\alpha F_\beta = 0 \tag{23}$$

(ii) If the main scalar $H = 0$, thus F satisfy in the following differential equation:

$$3(FF_{\alpha\beta} + F_\alpha F_\beta)^3 + 4F^4C^2(F_\alpha)^2(F_\alpha F_{\beta\beta\beta} - 3F_{\alpha\beta}F_{\beta\beta}) = 0. \tag{24}$$

Proof. (i) If $I = 0$, hence from (15), we have $FCp_{-1}/2pc_{-1} = 0$. Consequently $p_{-1} = 0$. Putting (6) in equation $p_{-1} = 0$, we get $FF_{\alpha\beta} + F_\alpha F_\beta = 0$.

(ii) Suppose the main scalar $H = 0$. Thus (21) is reduced to $I = FC$. Now if we set $I = FCp_{-1}/2pc_{-1}$ in $I = FC$, we get $c_{-1} = p_{-1}/2p$. On the other hand, if H vanishes, therefore from (16) we conclude that

$$3p_{-1}(c_{-1})^2 + r_{-1}C^2 = 0. \tag{25}$$

Now if put $c_{-1} = p_{-1}/2p$ and (8) in (25), we obtain

$$3p_{-1}\left(\frac{p_{-1}}{2p}\right)^2 + (pp_{0\beta} - 3p_{-1}q_0)\, C^2 = 0. \tag{26}$$

Setting (7) in (26), we get

$$3(FF_{\alpha\beta} + F_\alpha F_\beta)^3 + 4F^4C^2(F_\alpha)^2(F_\alpha F_{\beta\beta\beta} - 3F_{\alpha\beta}F_{\beta\beta}) = 0.$$ □

Theorem 11 ([2]). *A three dimensional Finsler space with the (α, β)-metric $F(\alpha, \beta)$ is a Riemannian space if the main scalar I vanishes.*

Proof. In the previous theorem we proved that if $I = 0$, then F satisfy in (23). The differential equation (23) has the solution $F^2 = u_1\alpha^2 + u_2\beta^2$, where u_1 and u_2 are constants. Hence according to proposition 2.2, we can conclude that F is a Riemannian metric. □

Theorem 12 ([2]). *Let (M, F) be a three dimensional Finsler space with $F(\alpha, \beta)$. The Finsler quantity c_{-1} satisfy in the following equation:*

$$2p(c_{-1})^3 = 4p_{-1}(c_{-1})^2 + r_{-1}C^2. \tag{27}$$

Proof. It is enough to put (15) in (21). □

4. The Main Scalars of Three Dimensional (α, β)-Metric $F = \alpha + \kappa\beta + \varepsilon\beta^2/\alpha$

The general (α, β)-metric $F = \alpha + \kappa\beta + \varepsilon\beta^2/\alpha$ (where κ and ε are constants which are not zero simultaneously) has been studied in [8]. In this section, we first determine the main scalars J, I and H of a three dimensional Finsler space equipped with (α, β)-metric $F = \alpha + \kappa\beta + \varepsilon\beta^2/\alpha$. According to theorem 8, the main scalar J is zero. So we just have to find the main scalars H and I: We must first calculate Finsler quantities (7), (8) from (1). Then we have:

$$p = \frac{\left(\kappa\alpha\beta + \varepsilon\beta^2 + \alpha^2\right)(\alpha^2 - \varepsilon\beta^2)}{\alpha^4}, \quad p_{-1} = \frac{-3\kappa\varepsilon\alpha\beta^2 - 4\varepsilon^2\beta^3 + \kappa\alpha^3}{\alpha^4},$$

$$p_0 = \frac{\kappa^2\alpha^2 + \kappa\varepsilon 6\alpha\beta + 6\varepsilon^2\beta^2 + 2\varepsilon\alpha^2}{\alpha^2}, \quad p_{0\beta} = \frac{2\varepsilon(3\kappa\alpha + 6\varepsilon\beta)}{\alpha^2},$$

$$q_0 = \frac{2\varepsilon\left(\kappa\alpha\beta + \varepsilon\beta^2 + \alpha^2\right)}{\alpha^2}, \quad r_{-1} = \frac{12\varepsilon^2\beta(\kappa\alpha\beta + \varepsilon\beta^2 + \alpha^2)^2}{\alpha^6} \tag{28}$$

Also, we obtain c_{-1} by using (27). By neglecting the complex answers of c_{-1}, we have

$$c_{-1} = \frac{2\alpha T(B^{1/3} + 2\alpha T) + B^{2/3}}{-3B^{1/3}(\beta^2\varepsilon - \alpha^2)\left(\kappa\alpha\beta + \varepsilon\beta^2 + \alpha^2\right)\alpha}, \tag{29}$$

where

$$\begin{aligned}
T =& -3\kappa\varepsilon\alpha\beta^2 - 4\varepsilon^2\beta^3 + \kappa\alpha^3\\
B =&\alpha\left(-9C\varepsilon\left(\beta^2\varepsilon-\alpha^2\right)\left(\kappa\alpha\beta+\varepsilon\beta^2+\alpha^2\right)^2\sqrt{E\beta}+A+8\alpha^{11}\kappa^3\right.\\
&\left.+\beta^9\varepsilon^6\left(81C^2\beta^4\varepsilon^2-512\alpha^2\right)+D\right),\\
E =&81C^2\beta\alpha^{12}\varepsilon^2+\left(324\varepsilon^2C^2\beta^2\kappa+16\kappa^3\right)\alpha^{11}\\
&+\left(486\varepsilon^2C^2\beta^3\kappa^2+162\varepsilon^3C^2\beta^3\right)\alpha^{10}\\
&+\left(324\varepsilon^2C^2\beta^4\kappa^3+324\varepsilon^3C^2\beta^4\kappa-144\beta^2\varepsilon\kappa^3\right)\alpha^9\\
&+\left(81\varepsilon^2C^2\beta^5\kappa^4-81\varepsilon^4C^2\beta^5-192\beta^3\varepsilon^2\kappa^2\right)\alpha^8\\
&+\left(-324\varepsilon^3C^2\beta^6\kappa^3-648\varepsilon^4C^2\beta^6\kappa+432\beta^4\varepsilon^2\kappa^3\right)\alpha^7\\
&+\left(-162\varepsilon^3C^2\beta^7\kappa^4-972\varepsilon^4C^2\beta^7\kappa^2-324\varepsilon^5C^2\beta^7+1152\beta^5\varepsilon^3\kappa^2\right)\alpha^6\\
&+\left(-324\varepsilon^4C^2\beta^8\kappa^3-648\varepsilon^5C^2\beta^8\kappa-432\varepsilon^3\kappa^3\beta^6+768\varepsilon^4\kappa\beta^6\right)\alpha^5\\
&+\left(81\varepsilon^4C^2\beta^9\kappa^4-81\varepsilon^6C^2\beta^9-1728\beta^7\varepsilon^4\kappa^2\right)\alpha^4\\
&+\left(324\varepsilon^5C^2\beta^{10}\kappa^3+324\varepsilon^6C^2\beta^{10}\kappa-2304\beta^8\varepsilon^5\kappa\right)\alpha^3\\
&+\left(486\varepsilon^6C^2\beta^{11}\kappa^2+162\varepsilon^7C^2\beta^{11}-1024\beta^9\varepsilon^6\right)\alpha^2\\
&+324C^2\beta^{12}\kappa\alpha\varepsilon^7+81C^2\beta^{13}\varepsilon^8,\\
A =&81C^2\beta\alpha^{12}\varepsilon^2+162C^2\alpha^{10}\beta^3\varepsilon^3-72\varepsilon\beta^2\alpha^9\kappa^3\\
&+\left(-81C^2\beta^5\varepsilon^4-96\beta^3\varepsilon^2\kappa^2\right)\alpha^8\\
&+216\kappa^3\alpha^7\beta^4\varepsilon^2+\left(-324C^2\beta^7\varepsilon^5+576\kappa^2\beta^5\varepsilon^3\right)\alpha^6\\
&+\left(-216\beta^6\varepsilon^3\kappa^3+384\beta^6\varepsilon^4\kappa\right)\alpha^5+\left(-81C^2\beta^9\varepsilon^6-864\varepsilon^4\beta^7\kappa^2\right)\alpha^4\\
&-1152\varepsilon^5\kappa\beta^8\alpha^3+162C^2\beta^{11}\alpha^2\varepsilon^7,\\
D =&81\varepsilon^2C^2\beta^2\kappa\alpha\left(\beta^2\varepsilon-\alpha^2\right)^2\left(\alpha\beta\kappa+2\beta^2\varepsilon+2\alpha^2\right)\\
&\left(\alpha^2\beta^2\kappa^2+2\alpha\beta^3\varepsilon\kappa+2\beta^4\varepsilon^2+2\alpha^3\beta\kappa+4\alpha^2\beta^2\varepsilon+2\alpha^4\right).
\end{aligned}$$

Putting (28) and (29) in (15), we obtain the main scalar I as follows:

$$I=\frac{3CB^{1/3}T(\kappa\alpha\beta+\varepsilon\beta^2+\alpha^2)}{8\alpha^2T^2+2B^{1/3}\left(2T+B^{1/3}\right)}, \tag{30}$$

Also, using (21) and (30), we have

$$H=\frac{C(\kappa\alpha\beta+\varepsilon\beta^2+\alpha^2)\left(8\alpha^2T^2+B^{1/3}\left(\alpha T+2B^{1/3}\right)\right)}{\alpha\left(4\alpha^2T^2+B^{1/3}\left(2\alpha T+B^{1/3}\right)\right)}. \tag{31}$$

Then we have the following:

Theorem 13. *The main scalars of J, H and I of a three dimentional Finsler space equipped with the general (α, β)-metric $F = \alpha + \kappa\beta + \varepsilon\beta^2/\alpha$ (where κ and ε are constants which are not zero simultaneously) are given by* 0, (31) *and* (30) *respectively.*

Now based on the above calculations, we can obtain main scalars of many special three dimensional (α, β)-metrics such as Randers metric, first approximate Matsumoto metric and square metric:

Case 1. Taking $\varepsilon = 0$ and $\kappa = 1$ in (1), we obtain Randers metric $F = \alpha + \beta$. In this case, (28) and (29) are converted to the following:

$$p = \frac{\alpha + \beta}{\alpha}, \quad p_{-1} = \frac{1}{\alpha}, \quad p_0 = 1, \quad p_{0\beta} = q_0 = r_{-1} = 0,$$

$$c_{-1} = \frac{2}{\alpha + \beta}, \tag{32}$$

where $T = \alpha^3, \quad E = 16\alpha^{11}, \quad B = 8\alpha^{12}$ and $A = D = 0$.

Thus

$$I = \frac{C(\alpha + \beta)}{4}, \tag{33}$$

and

$$H = \frac{3C(\alpha + \beta)}{4}. \tag{34}$$

Corollary 14. *The main scalars of J, H and I of a three dimentional Randers space ($F = \alpha + \beta$) are given by* 0, (34) *and* (33) *respectively.*

Case 2. Considering $\varepsilon = 1$ and $\kappa = 1$ in (1), we get the first approximate Matsumoto metric $F = \alpha + \beta + \beta^2/\alpha$. In this case, (28) and (29) are reduced to the following:

$$p = \frac{\left(\alpha\beta + \beta^2 + \alpha^2\right)(\alpha^2 - \beta^2)}{\alpha^4}, \quad p_{-1} = \frac{\alpha^3 - 3\alpha\beta^2 - 4\beta^3}{\alpha^4},$$

$$p_0 = \frac{3(\alpha^2 + 2\alpha\beta + 2\beta^2)}{\alpha^2},$$

$$p_{0\beta} = \frac{6(\alpha + 2\beta)}{\alpha^2}, \quad q_0 = \frac{2\left(\alpha\beta + \beta^2 + \alpha^2\right)}{\alpha^2}, \quad r_{-1} = \frac{12\beta(\alpha\beta + \beta^2 + \alpha^2)^2}{\alpha^6},$$

$$c_{-1}=\frac{2\alpha T(B^{1/3}+2\alpha T)+B^{2/3}}{-3B^{1/3}\left(\beta^2-\alpha^2\right)\left(\alpha\beta+\beta^2+\alpha^2\right)\alpha}, \tag{35}$$

where

$$\begin{aligned}
T=&-3\alpha\beta^2-4\beta^3+\alpha^3\\
B=&\alpha\Big(-9C\left(\beta^2-\alpha^2\right)\left(\alpha\beta+\beta^2+\alpha^2\right)^2\sqrt{E\beta}\\
&+A+8\alpha^{11}+\beta^9\left(81C^2\beta^4-512\alpha^2\right)+D\Big),\\
E=&81C^2\beta(\alpha-\beta)^2(\alpha+\beta)^2\left(\alpha\beta+\beta^2+\alpha^2\right)^4\\
&+16\alpha^2\left(-3\alpha\beta^2-4\beta^3+\alpha^3\right)^3,
\end{aligned}$$

$$\begin{aligned}
A=&81C^2\alpha^2\beta\left(\alpha^2\beta^2-\beta^4+\alpha^4\right)\left(\alpha^4\beta^2-\alpha^2\beta^4-2\beta^6+\alpha^6\right)\\
&-24\alpha^3\beta^2\left(3\alpha^6+4\alpha^5\beta-9\alpha^4\beta^2-24\alpha^3\beta^3-7\alpha^2\beta^4+36\alpha\beta^5+48\beta^6\right),\\
D=&81C^2\alpha\beta^2(\alpha-\beta)^2(\alpha+\beta)^2\left(\alpha\beta+2\beta^2+2\alpha^2\right)\\
&\left(2\alpha^4+2\alpha^3\beta+5\alpha^2\beta^2+2\alpha\beta^3+2\beta^4\right).
\end{aligned}$$

Therefore

$$I=\frac{3CB^{1/3}T(\alpha\beta+\beta^2+\alpha^2)}{8\alpha^2T^2+2B^{1/3}\left(2T+B^{1/3}\right)}, \tag{36}$$

and

$$H=\frac{C(\alpha\beta+\beta^2+\alpha^2)\left(8\alpha^2T^2+B^{1/3}\left(\alpha T+2B^{1/3}\right)\right)}{\alpha\left(4\alpha^2T^2+B^{1/3}\left(2\alpha T+B^{1/3}\right)\right)}. \tag{37}$$

Corollary 15. *The main scalars of J, H and I of a three dimentional Finsler space with the first approximate Matsumoto metric $F=\alpha+\beta+\beta^2/\alpha$ are given by* 0, (37) *and* (36) *respectively.*

Case 3. Assuming $\varepsilon=1$ and $\kappa=2$ in (1), we obtain the square metric $F=(\alpha+\beta)^2/\alpha$. In this case, the Finsler quantities (28) and (29) are converted to the following:

$$p=\frac{(\beta-\alpha)(\alpha+\beta)^3}{\alpha^4},\quad p_{-1}=\frac{2\alpha^3-6\alpha\beta^2-4\beta^3}{\alpha^4},\quad p_0=\frac{6(\alpha+\beta)^2}{\alpha^2},$$

$$p_{0\beta} = \frac{12(\alpha+\beta)}{\alpha^2}, \quad q_{_0} = \frac{2(\alpha+\beta)^2}{\alpha^2}, \quad r_{-1} = \frac{12\beta(\alpha+\beta)^2}{\alpha^6},$$

$$c_{-1} = \frac{2\alpha T(B^{1/3} + 2\alpha T) + B^{2/3}}{-3B^{1/3}(\beta-\alpha)(\alpha+\beta)^3\alpha}, \tag{38}$$

where

$$\begin{aligned}
T =& -6\alpha\beta^2 - 4\beta^3 + 2\alpha^3 \\
B =& \alpha\Big(-9C\left(\beta^2-\alpha^2\right)(\alpha+\beta)^4\sqrt{E\beta} + A + 64\alpha^{11} \\
& +\beta^9(81C^2\beta^4 - 512\alpha^2) + D\Big), \\
E =& 81C^2\beta(\alpha-\beta)^2(\alpha+\beta)^{10} + 128\alpha^2(\alpha-2\beta)^3(\alpha+\beta)^6, \\
A =& 81C^2\alpha^2\beta\left(\alpha^2\beta^2 - \beta^4 + \alpha^4\right)\left(\alpha^4\beta^2 - \alpha^2\beta^4 - 2\beta^6 + \alpha^6\right) \\
& -192\alpha^3\beta^2\Big(3(\alpha^6 + 4\beta^6) + 2\alpha\beta(\alpha^4 + 9\beta^4) \\
& -\alpha^2\beta^2(9\alpha^2 - 12\alpha\beta + 5\beta^2)\Big), \\
D =& 648C^2\alpha\beta^2(\alpha-\beta)^2(\alpha+\beta)^2\left(\alpha\beta + \beta^2 + \alpha^2\right) \\
& \left(\alpha^4 + 2\alpha\beta(\alpha+\beta)^2 + \beta^4\right).
\end{aligned}$$

Hence

$$I = \frac{3CB^{1/3}T(\alpha+\beta)^2}{8\alpha^2T^2 + 2B^{1/3}\left(2T + B^{1/3}\right)}, \tag{39}$$

and

$$H = \frac{C(\alpha+\beta)^2\left(8\alpha^2T^2 + B^{1/3}\left(\alpha T + 2B^{1/3}\right)\right)}{\alpha\left(4\alpha^2T^2 + B^{1/3}\left(2\alpha T + B^{1/3}\right)\right)}. \tag{40}$$

Corollary 16. *The main scalars of J, H and I of a three dimentional square space ($F = (\alpha+\beta)^2/\alpha$) are given by* 0, (40) *and* (39) *respectively.*

5. Cartan Tensor Vector

An expression of Cartan tensor vector $C_i = C_{ijk}g^{jk}$ is given as [2]:

$$C_i = c_{-1}b_i + c_{-2}Y_i. \tag{41}$$

Since $\beta = b_i y^i$, $Y_i = a_{ij} y^j$ and $\alpha^2 = a_{ij} y^i y^j$ thus $C_i y^i = 0$ yields

$$c_{-1}\beta + c_{-2}\alpha^2 = 0. \tag{42}$$

Hence, we can find the coefficients $c_i (i = 1, 2)$ in the expression (41) from the relations (27) and (42). Based on (41) and (42), we also give the following necessary and sufficient condition for a Finsler space to be a Riemannian space:

Theorem 17. *An n-dimensional Finsler space with the (α, β)-metric $F(\alpha, \beta)$ is a Riemannian space if c_{-1} vanishes.*

Proof. If $c_{-1} = 0$, then from (42) it follows that c_{-2} is also zero. Hence Cartan tensor vector vanishes. According to Deicke's theorem, a Finsler space with $C_i = 0$ is a Riemannian space. □

Now, we want to obtain the coefficients $c_i (i = 1, 2)$ in the expression (41) for the general (α, β)-metric (1). We found c_{-1} in (29). Then setting (29) in (42), we have

$$c_{-2} = \frac{\left(2\alpha T(B^{1/3} + 2\alpha T) + B^{2/3}\right)\beta}{-3B^{1/3}\left(-\beta^2\varepsilon + \alpha^2\right)\left(\kappa\alpha\beta + \varepsilon\beta^2 + \alpha^2\right)\alpha^3}, \tag{43}$$

Therefore we have:

Corollary 18. *Cartan tensor vector C_i in a three dimensional Finsler space equipped with the general (α, β)-metric $F = \alpha + \kappa\beta + \varepsilon\beta^2/\alpha$ (where κ and ε are constants which are not zero simultaneously) can be expressed as $C_i = c_{-1} b_i + c_{-2} Y_i$, where $b_i = a_{ij} \mathrm{b}^j$, $Y_i = a_{ij} y^j$ and the forms of coefficients c_{-1} and c_{-2} are given by* (29) *and* (43) *respectively.*

Consequently:

Case 1. If $\varepsilon = 0$ and $\kappa = 1$, , thus (43) is reduced to the following form:

$$c_{-2} = \frac{-2\beta}{(\alpha + \beta)\alpha^2} \tag{44}$$

Corollary 19. *In a three dimensional Randers space, Cartan tensor vector C_i can be expressed as follows:*

$$C_i = \frac{2}{\alpha + \beta} b_i + \frac{-2\beta}{(\alpha + \beta)\alpha^2} Y_i,$$

where $b_i = a_{ij} \mathrm{b}^j$, $Y_i = a_{ij} y^j$.

Case 2. If $\varepsilon = 1$ and $\kappa = 1$, therefore (43) is converted to the following form:

$$c_{-2} = \frac{\left(2\alpha T(B^{1/3} + 2\alpha T) + B^{2/3}\right)\beta}{-3B^{1/3}\left(\beta - \alpha\right)\left(\alpha + \beta\right)^3\alpha^3}. \tag{45}$$

Corollary 20. *In a three dimensional Finsler space equipped with the first approximate Matsumoto metric $F = \alpha + \beta + \beta^2/\alpha$, Cartan tensor vector C_i can be expressed as $C_i = c_{-1}b_i + c_{-2}Y_i$, where $b_i = a_{ij}\mathrm{b}^j$, $Y_i = a_{ij}y^j$ and the forms of coefficients c_{-1} and c_{-2} are given by* (35) *and* (45) *respectively.*

Case 3. If $\varepsilon = 1$ and $\kappa = 2$, (42) is converted to the following form:

$$c_{-2} = \frac{\left(2\alpha T(B^{1/3} + 2\alpha T) + B^{2/3}\right)\beta}{-3B^{1/3}\left(-\beta^2 + \alpha^2\right)\left(2\alpha\beta + \varepsilon\beta^2 + \alpha^2\right)\alpha^3}. \tag{46}$$

Corollary 21. *In a three dimensional square space, Cartan tensor vector C_i can be expressed as $C_i = c_{-1}b_i + c_{-2}Y_i$, where $b_i = a_{ij}\mathrm{b}^j$, $Y_i = a_{ij}y^j$ and the forms of coefficients c_{-1} and c_{-2} are given by* (38) *and* (46) *respectively .*

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