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AN EFFICIENT OF DIRECT INTEGRATOR OF RUNGE-KUTTA TYPE METHOD FOR SOLVING $y''' = f(x, y, y')$ WITH APPLICATION TO THIN FILM FLOW PROBLEM

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Abstract: In this paper, we proposed a fifth-order Runge-Kutta (RK) technique for regulating coordination about third-order ordinary differential equations (ODEs) of the structure $y''' = f(x, y, y')$ indicated similarly as RKTG method is constructed. The order state about RKTG method up to order six were proved and verified. In view of those order conditions developed, four-stage fifth-order express Runge-Kutta methods of techniques were constructed. The zero Strength of the new system was indicated. The Different types for third-order ODEs need been derived utilizing the new system and also some numerical comparison were conducted when the same issue will be decreased of the first-order framework of equations which are solved using existing Runge-Kutta techniques. The numerical investigation of a third-order tribute on thin film flow for viscous liquid in applied mathematical physics. Numerical outcomes indicated that those new proposed method is more efficient in terms of accuracy and number of function evaluations of capacity assessments.

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1. Introduction

We are dealing in this work with numerical integration of third-order ODE of the form:

$$y'''(x) = f\big(x, y, y'\big), \tag{1}$$

with initial conditions

$$y(x_0) = \delta, \quad y'(x_0) = \sigma, \quad y''(x_0) = \xi, \quad x \geq x_0.$$

where $y, y', y'' \in \mathbb{R}^d, f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector-valued function. This sort of problems often found in numerous physical problems like thin film flow, gravity-driven flows and electromagnetic waves. The general solution of (1) is by decreasing it to an equal series 1^{st}-order framework which is three times extent and can be solved utilizing standard Runge-Kutta or multi-step method. A lot of researches have been solved problem (1) by converting (1) to a system of 1^{st}-order equations. Furthermore, there were several authors studied different numerical techniques which solved the problem (1) straightforwardly for instance Jator [1], Awoyemi and Idowu [2], introduced general family of hybrid methods for solving of higher-order ODEs. You and Chen [3], constructed direct integrations of RK type for special third-order ODEs. Waeleh et al. [4], proposed a new algorithm for solving higher-order IVPs of ODEs. Jator [5], constructed by hybrid multi-step method solving second order IVPs without predictors. Samat and Ismail [6], developed a block multi-step method which could straightforwardly solve general third-order equations, furthermore, Ibrahim et al. [7], found a process by using multi-step technique which could solve stiff 3^{rd}-order differential equations. Mechee et al. [8], constructed a three-stage 5^{th}-order RK type method for directly solving special 3^{rd}-order ODEs. Kasim et al. [9], proposed integration of 3^{rd}-order ODEs using improved RK technique. Subsequently, Senu et al. [10] constructed a new embedded explicit RK technique for solving special 3^{rd}-order ODEs. In this paper, the main aim is to proposed a one-step technique of order five to solve third-order ODEs easily. The derivation of order conditions are given in Section 2. In Section 3, the zero-stability of the new method is given. Four-stage fifth order is constructed in Section 4. The effectiveness of the new technique, when compared with existing method is given in Section 5. The Thin Film Flow problem discussed in Section 6.

2. Derivation of the New Method

The general type of RKTG technique with m-stage for solving the IVPs (1) can be indited as follows:

$$y_{n+1} = y_n + h\,y'_n + \frac{h^2}{2}\,y''_n + h^3 \sum_{i=1}^{m} b_i k_i, \tag{2}$$

$$y'_{n+1} = y'_n + h\,y''_n + h^2 \sum_{i=1}^{m} b'_i k_i, \tag{3}$$

$$y''_{n+1} = y''_n + h \sum_{i=1}^{m} b''_i k_i, \tag{4}$$

where

$$k_1 = f\left(x_n, y_n, y'_n\right),$$

$$k_i = f\Bigg(x_n + c_i h,\; y_n + c_i\,h\,y'_n + \frac{h^2}{2}\,c_i^2\,y''_n + h^3 \sum_{j=1}^{i-1} a_{ij} k_j,$$

$$y'_n + c_i\,h\,y''_n + h^2 \sum_{j=1}^{i-1} \bar{a}_{ij} k_j\Bigg) \tag{5}$$

for $i = 2, 3, \ldots, m$.

The new parameters $b_i, b'_i, b''_i, a_{ij}, \bar{a}_{ij}$ and c_i of the RKTG method assumed to be real and used for $i, j = 1, 2, ..., m$. The technique is explicitly if $a_{ij} = \bar{a}_{ij} = 0$ for $i \le j$ and it is implicitly if $a_{ij} \neq 0$ and $\bar{a}_{ij} \neq 0$ for $i \le j$. The new proposed technique is presented by the tableau below:

$$\begin{array}{c|c|c|c} c & A & \bar{A} & \\ \hline & b^T & b'^T & b''^T \end{array} \tag{6}$$

Expressing the new proposed method of technique parameters provided by (2)-(5), the RKTG technique is expanded utilizing Taylor's series expansion. After performing a few algebraic manipulations, this expansion is equated to the true solution that is given by Taylor's series expansion. The direct expansion of the local truncation error utilized to derive the general order conditions for the RKTG method. This conception predicated on the derivation of order conditions for the RK method proposed by [11] and [12]. The new method RKTG can be shown as follows:

$$\begin{aligned} y_{n+1} &= y_n + h\varphi(x_n, y_n, y'_n), \\ y'_{n+1} &= y'_n + h\varphi'(x_n, y_n, y'_n), \\ y''_{n+1} &= y''_n + h\varphi''(x_n, y_n, y'_n). \end{aligned} \tag{7}$$

where the increment functions are

$$\begin{aligned} \varphi(x_n, y_n, y'_n) &= y'_n + \frac{h}{2}y''_n + h^2\sum_{i=1}^{m} b_i k_i, \\ \varphi'(x_n, y_n, y'_n) &= y''_n + h\sum_{i=1}^{m} b'_i k_i, \\ \varphi''(x_n, y_n, y'_n) &= \sum_{i=1}^{m} b''_i k_i. \end{aligned} \tag{8}$$

where k_i is given in (5). If we postulate that Δ, Δ' and Δ'' are the Taylor series increment function. Thus, the local truncation errors of $y(x)$, $y'(x)$ and $y''(x)$ can be acquired by superseding the precise solution of (1) into (8) as follows:

$$\begin{aligned} t_{n+1} &= h[\varphi - \Delta], \\ t'_{n+1} &= h[\varphi' - \Delta'], \\ t''_{n+1} &= h[\varphi'' - \Delta'']. \end{aligned} \tag{9}$$

In the terms of elementary differentials, these expressions are best given and the Taylor series might be expressed as follows:

$$\begin{aligned} \Delta &= y' + \frac{1}{2}h\,y'' + \frac{1}{6}h^2\,F_1^{(3)} + \frac{1}{24}h^3\,F_1^{(4)} + O(h^4), \\ \Delta' &= y'' + \frac{1}{2}h\,F_1^{(3)} + \frac{1}{6}h^2\,F_1^{(4)} + \frac{1}{24}h^3\,F_1^{(5)} + O(h^4), \\ \Delta'' &= F_1^{(3)} + \frac{1}{2}h\,F_1^{(4)} + \frac{1}{6}h^2\,F_1^{(5)} + O(h^3). \end{aligned} \tag{10}$$

The first few elementary differentials for the scalar case are

$$F_1^{(3)} = f,$$

$$
\begin{aligned}
F_1^{(4)} &= f_x + f_y y_x + f_{y'} y_{xx}, \\
F_1^{(5)} &= f_{xx} + y_x f_{xy} + f_{xy'} y_{xx} + y_x^2 f_{yy} + f_{yy'} y_x y_{xx} + f_y y_{xx} + f_{y'y'} y_{xx}^2 \\
&\quad + f_{y'} f.
\end{aligned} \tag{11}
$$

Substituting (11) into (8), the increment functions φ, φ' and φ'' for new method will become

$$
\begin{aligned}
&\sum_{i=1}^{m} b_i k_i = \sum_{i=1}^{m} b_i f + \sum_{i=1}^{m} b_i c_i \big(f_x + f_y y_x + f_{y'} y_{xx}\big)\, h + \frac{1}{2} \sum_{i=1}^{m} b_i c_i^2 \big(f_{xx} \\
&+ y_x f_{xy} + f_{xz} y_{xx} + y_x^2 f_{yy} + f_{yy'} y_x y_{xx} + f_y y_{xx} + f_{y'y'} y_{xx}^2 + f_{y'} f\big)\, h^2 \\
&+ O(h^3), \\
&\sum_{i=1}^{m} b'_i k_i = \sum_{i=1}^{m} b'_i f + \sum_{i=1}^{m} b'_i c_i \big(f_x + f_y y_x\big)\, h + \frac{1}{2} \sum_{i=1}^{m} b'_i c_i^2 \big(f_{xx} + y_x f_{xy} \\
&+ f_{xy'} y_{xx} + f_{y'} y_{xx} + y_x^2 f_{yy} + f_{yy'} y_x y_{xx} + f_y y_{xx} + f_{y'y'} y_{xx}^2 + f_{y'} f\big)\, h^2 \\
&+ O(h^3), \\
&\sum_{i=1}^{m} b''_i k_i = \sum_{i=1}^{m} b''_i f + \sum_{i=1}^{m} b''_i c_i \big(f_x + f_y y_x + f_{y'} y_{xx}\big)\, h + \frac{1}{2} \sum_{i=1}^{m} b''_i c_i^2 \big(f_{xx} \\
&+ y_x f_{xy} + y_x^2 f_{yy} + F_{yy'} y_x y_{xx} + f_y y_{xx} + F_{y'y'} y_{xx}^2 + f_{y'} f\big)\, h^2 + O(h^3).
\end{aligned} \tag{12}
$$

From (10) and (12), the local truncation error (9) can be expressed as follows:

$$
\begin{aligned}
t_{n+1} &= h^3 \left[\sum_{i=1}^{m} b_i k_i - \left(\frac{1}{6} F_1^{(3)} + \frac{1}{24}\, h\, F_1^{(4)} + ... \right) \right], \\
t'_{n+1} &= h^2 \left[\sum_{i=1}^{m} b'_i k_i - \left(\frac{1}{2} F_1^{(3)} + \frac{1}{6}\, h\, F_1^{(4)} + ... \right) \right], \\
t''_{n+1} &= h \left[\sum_{i=1}^{m} b''_i k_i - \left(F_1^{(3)} + \frac{1}{2}\, h\, F_1^{(4)} + \frac{1}{6} h^2 F_1^{(5)} + ... \right) \right].
\end{aligned} \tag{13}
$$

Substituting (12) into (13) by Taylors expansion employing the Maple software obtaining the truncated errors for m-stages up to order six for the proposed technique can be expressed as follows:

The order terms for y:

3^{rd}-order

$$\sum b_i = \frac{1}{6} \tag{14}$$

4^{th}-order

$$\sum b_i c_i = \frac{1}{24} \tag{15}$$

5^{th}-order

$$\sum b_i c_i^2 = \frac{1}{60}, \quad \sum b_i \bar{a}_{ij} = \frac{1}{120} \tag{16}$$

6^{th}-order

$$\sum b_i \bar{a}_{ij} c_j = \frac{1}{720}, \quad \sum b_i c_i^3 = \frac{1}{120} \tag{17}$$

$$\sum b_i \bar{a}_{ij} c_i = \frac{1}{240}, \quad \sum b_i a_{ij} = \frac{1}{720} \tag{18}$$

The order terms for $\mathbf{y'}$:

2^{nd}-order

$$\sum b'_i = \frac{1}{2} \tag{19}$$

3^{rd}-order

$$\sum b'_i c_i = \frac{1}{6} \tag{20}$$

4^{th}-order

$$\sum b'_i c_i^2 = \frac{1}{12}, \quad \sum b'_i \bar{a}_{ij} = \frac{1}{24} \tag{21}$$

5^{th}-order

$$\sum b'_i c_i^3 = \frac{1}{20}, \quad \sum b'_i \bar{a}_{ij} c_j = \frac{1}{120} \tag{22}$$

$$\sum b'_i \bar{a}_{ij} c_i = \frac{1}{40}, \quad \sum b'_i a_{ij} = \frac{1}{120} \tag{23}$$

6^{th}-order

$$\sum b'_i c_i^2 \bar{a}_{ij} = \frac{1}{60}, \quad \sum b'_i c_i \bar{a}_{ij} c_j = \frac{1}{180} \tag{24}$$

$$\sum b'_i c_i^4 = \frac{1}{30}, \sum b'_i c_j^2 \bar{a}_{ij} + \sum b'_i c_i \bar{a}_{ij} c_j = \frac{1}{120} \tag{25}$$

$$\frac{1}{2} \sum b'_i c_j^2 \bar{a}_{ij} = \frac{1}{720}, \frac{1}{2} \sum b'_i c_j^2 \bar{a}_{ij} + \sum b'_i a_{ij} c_j = \frac{1}{360} \tag{26}$$

$$\sum b'_i a_{ij} c_i = \frac{1}{180}, \frac{1}{2} \sum b'_i c_i^2 \bar{a}_{ij} + \sum b'_i a_{ij} c_i = \frac{1}{72} \tag{27}$$

$$\frac{1}{2}\sum b_i' c_j^2 \bar{a}_{ij} + \sum b_i' c_i \bar{a}_{ij} c_j = \frac{1}{144}, \sum b_i' a_{ij} c_j = \frac{1}{720} \tag{28}$$

$$\sum b_i' \bar{a}_{ij} \bar{a}_{jk} = \frac{1}{720}, \frac{1}{2}\sum b_i' \, \bar{a}_{ik} \, \bar{a}_{ij} = \frac{1}{240} \tag{29}$$

The order terms for y″:

1^{st}-order

$$\sum b_i'' = 1 \tag{30}$$

2^{nd}-order

$$\sum b_i'' c_i = \frac{1}{2} \tag{31}$$

3^{rd}-order

$$\sum b_i'' c_i^2 = \frac{1}{3}, \quad \sum b_i'' \bar{a}_{ij} = \frac{1}{6} \tag{32}$$

4^{th}-order

$$\sum b_i'' c_i^3 = \frac{1}{4}, \quad \sum b_i'' \bar{a}_{ij} c_j = \frac{1}{24} \tag{33}$$

$$\sum b_i'' a_{ij} = \frac{1}{24}, \quad \sum b_i'' c_i \bar{a}_{ij} = \frac{1}{8}. \tag{34}$$

5^{th}-order

$$\sum b_i'' c_i^4 = \frac{1}{5}, \quad \sum b_i'' \bar{a}_{ij} c_j^2 + \sum b_i'' \bar{a}_{ij} c_i c_j = \frac{1}{20}, \tag{35}$$

$$\sum b_i'' \bar{a}_{ij} c_j^2 = \frac{1}{60}, \quad \frac{1}{2}\sum b_i'' \bar{a}_{ij} c_j^2 + \sum b_i'' a_{ij} c_j = \frac{1}{60}, \tag{36}$$

$$\sum b_i'' a_{ij} c_i = \frac{1}{30}, \quad \frac{1}{2}\sum b_i'' c_i^2 \bar{a}_{ij} + \sum b_i'' a_{ij} c_i = \frac{1}{12}, \tag{37}$$

$$\frac{1}{2}\sum b_i'' \bar{a}_{ij} c_j^2 + \sum b_i'' \bar{a}_{ij} c_i c_j = \frac{1}{24}, \quad \sum b_i'' \bar{a}_{ij} \bar{a}_{jk} = \frac{1}{120}, \tag{38}$$

$$\frac{1}{2}\sum b_i'' \bar{a}_{ik} \bar{a}_{ij} = \frac{1}{40}, \quad \sum b_i'' a_{ij} c_j = \frac{1}{120}, \quad \sum b_i'' \bar{a}_{ij} c_i c_j = \frac{1}{30} \tag{39}$$

6^{th}-order

$$\sum b_i'' c_i^5 = \frac{1}{6}, \quad \frac{1}{2}\sum b_i'' \bar{a}_{ij} c_i^3 + \frac{1}{2}\sum b_i'' a_{ij} c_i^2 = \frac{1}{18} \tag{40}$$

$$\frac{1}{2}\sum b_i'' \bar{a}_{ij} c_j^3 + \sum b_i'' \bar{a}_{ij} c_i^2 c_j = \frac{23}{720}, \quad \frac{1}{2}\sum b_i'' \bar{a}_{ij} c_i^2 c_j = \frac{1}{72} \tag{41}$$

$$\frac{1}{6}\sum b_i'' \bar{a}_{ij} c_i^3 = \frac{1}{72}, \quad \frac{1}{2}\sum b_i'' \bar{a}_{ij} c_i c_j^2 + \sum b_i'' a_{ij} c_j^2 = \frac{7}{720} \tag{42}$$

$$\sum b''_i a_{ij} c_i c_j = \frac{1}{144}, \quad \frac{1}{6}\sum b''_i \bar{a}_{ij} c_j^3 + \frac{1}{2}\sum b''_i \bar{a}_{ij} c_i^2 c_j = \frac{11}{720} \tag{43}$$

$$\sum b''_i a_{ij} c_i c_j = \frac{1}{144}, \quad \frac{1}{6}\sum b''_i \bar{a}_{ij} c_j^3 + \frac{1}{2}\sum b''_i \bar{a}_{ij} c_i^2 c_j = \frac{11}{720} \tag{44}$$

$$\sum b''_i c_i \bar{a}_{ij} c_j^2 = \frac{1}{72}, \quad \frac{1}{2}\sum b''_i c_i^2 \bar{a}_{ij} c_j + \sum b''_i c_i a_{ij} c_j = \frac{1}{48} \tag{45}$$

$$\frac{1}{2}\sum b''_i \bar{a}_{ij} c_j^3 + \sum b''_i c_i\, a_{ij} c_j = \frac{1}{90}, \quad \frac{1}{2}\sum b''_i c_i^2 a_{ij} = \frac{1}{72} \tag{46}$$

$$\frac{1}{2}\sum b''_i a_{ij} c_j^2 + \sum b''_i c_i a_{ij} c_j = \frac{1}{120}, \quad \sum b''_i \bar{a}_{ij} \bar{a}_{jk} c_k = \frac{1}{720} \tag{47}$$

$$\frac{1}{2}\sum b''_i c_i^2 \bar{a}_{ij} c_j + \frac{1}{2}\sum b''_i \bar{a}_{ij} c_j^3 = \frac{13}{720}, \quad \sum b''_i a_{ij} \bar{a}_{ik} = \frac{1}{72} \tag{48}$$

$$\frac{1}{2}\sum b''_i c_i^3 \bar{a}_{ij} + \sum b''_i c_i^2 a_{ij} = \frac{5}{75}, \quad \frac{1}{2}\sum b''_i a_{ij} c_j^2 = \frac{1}{720} \tag{49}$$

$$\frac{1}{2}\sum b''_i a_{ij} c_j^2 + \frac{1}{2}\sum b''_i c_i \bar{a}_{ij} c_j^2 = \frac{1}{120} \tag{50}$$

$$\frac{1}{2}\sum b''_i c_i^2 \bar{a}_{ij} c_j + \frac{1}{2}\sum b''_i \bar{a}_{ij} c_j^3 + \sum b''_i c_i a_{ij} c_j = \frac{1}{40} \tag{51}$$

$$\sum b''_i \bar{a}_{ij} a_{jk} + \sum b''_i a_{ij} \bar{a}_{jk} = \frac{1}{360} \tag{52}$$

$$\sum b''_i a_{ij} c_j^2 + \sum b''_i c_i a_{ij} c_j = \frac{7}{720} \tag{53}$$

$$\frac{1}{2}\sum b''_i c_i \bar{a}_{ij} c_j^2 + \frac{1}{2}\sum b''_i c_i^2 \bar{a}_{ij} c_j + \sum b''_i a_{ij} c_j^2 + \sum b''_i c_i a_{ij} c_j = \frac{11}{360} \tag{54}$$

$$\frac{1}{2}\sum b''_i c_i \bar{a}_{ij} \bar{a}_{ik} = \frac{1}{48}, \sum b''_i \bar{a}_{ij} \bar{a}_{ik} c_j = \frac{1}{72} \tag{55}$$

$$\sum b''_i \bar{a}_{ij} \bar{a}_{ik} c_k = \frac{1}{72} \tag{56}$$

$$\sum b''_i \bar{a}_{ij} \bar{a}_{jk} c_i + \sum b''_i \bar{a}_{ij} \bar{a}_{jk} c_j + \sum b''_i \bar{a}_{ij} \bar{a}_{ik} c_j = \frac{1}{40} \tag{57}$$

All indexes are run from one to m. To obtain the higher-order RKTG technique we assume the following equation to eliminate and reduce the problem as stated below:

$$\begin{aligned} \sum \bar{a}_{ij} &= \frac{c_i^2}{2}, \\ b'_i &= b''_i (1 - c_i), \\ b_i &= b''_i \frac{(1 - c_i)^2}{2}. \quad i = 1, \ldots, m. \end{aligned} \tag{58}$$

3. Zero-Stability of the New Model

Here, we will discuss the zero-stability of new technique of convergence. It is stable at zero significance to prove the convergence of multi-step techniques and stability (see [13], [14]). Hairer et al. [15], also discussed on the zero-stability to obtained the upper boundedness of the multi-steps methods. Now, the first characteristic polynomial for the RKTG method (2)-(5) is based on the following equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2y''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ hy'_n \\ h^2y''_n \end{bmatrix},$$

where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix coefficient of $y_{n+1}, h\,y'_{n+1}$ and $h^2y''_{n+1}$

and $A = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is matrix coefficient of y_n, $h\,y'_n$ and $h^2y''_n$, respectively.

Then, the first characteristic polynomial of new method is

$$\rho(\zeta) = det[I\zeta - A] = \begin{vmatrix} \zeta - 1 & -1 & -\frac{1}{2} \\ 0 & \zeta - 1 & -1 \\ 0 & 0 & \zeta - 1 \end{vmatrix}.$$

thus,

$$\rho(\zeta) = (\zeta - 1)^3.$$

Therefore, this technique is stable at zero interval whereby the roots, $\zeta 1, 2, 3$ are all equivalence to unity.

4. Construction of the RKTG Methods

By the order conditions stated in Section 2 above we proceed to construct explicit RKTG methods. The local truncated error for the p order RKTG technique is defined as follows:

$$\| t_g^{(p+1)} \|_2 = \left(\sum_{i=1}^{n_p+1} \left(t_i^{(p+1)}\right)^2 + \sum_{i=1}^{n'_p+1} \left(t_i'^{(p+1)}\right)^2 + \sum_{i=1}^{n''_p+1} \left(t_i''^{(p+1)}\right)^2 \right)^{\frac{1}{2}} \tag{59}$$

where $t^{(p+1)}, t'^{(p+1)}$ and $t''^{(p+1)}$ are the local truncation error terms for y, y' and y'' respectively, ${t_g}^{(p+1)}$ is the global local truncation error.

4.1. A Four-stage Fifth-order RKTG Method

In this part, we discuss mainly on the deriving the four-stage RKTG technique of 5^{th}-order and the algebraic conditions $\big((14)$–(16), (19)–(23), (30)–$(39)\big)$ will be use because of the high number consisting of some non-linear equations, therefore we use the simplifying assumption (58) to reduce the system of equations to 19 equations with 16 unknowns and left with 3 degree of freedom. Solving the system simultaneously and the family of solution in term of a_{42}, a_{43} and c_3 are given as follows:

$$\bar{a}_{21} = \frac{(5c_3-3)^2}{50(2c_3-1)^2},\ \bar{a}_{31} = -\frac{c_3(-35\,c_3^2+19\,c_3-3+20\,c_3^3)}{10(c_3-3)},$$

$$\bar{a}_{32} = \frac{(-10\,c_3+3+10\,c_3^2)\,c_3\,(2\,c_3-1)}{(10\,c_3-3)},$$

$$\bar{a}_{41} = -\frac{10\,c_3^3-17\,c_3^2+10\,c_3-2}{2\,c_3\,(3-12\,c_3+10\,c_3^2)\,(5\,c_3-3)},$$

$$\bar{a}_{42} = \frac{(-85\,c_3^2+45\,c_3+50\,c_3^3-7)\,(2\,c_3-1)\,(-2+5\,c_3)}{2\,(-10\,c_3+3+10\,c_3^2)\,(5\,c_3-3)\,(3-12\,c_3+10\,c_3^2)},$$

$$\bar{a}_{43} = -\frac{(-2+5\,c_3)\,(-1+\,c_3)^2}{2\,(-10\,c_3+3+10\,c_3^2)\,(3-12\,c_3+10\,c_3^2)},\ a_{21}=0,$$

$$a_{31} = -\frac{(-10\,c_3+3+10\,c_3^2)\,c_3\,(450\,a_{43}\,c_3^2-75\,a_{43}\,c_3+255\,a_{42}\,c_3}{(5\,c_3-3)\,(-2+5\,c_3)}$$

$$\frac{+250\,a_{42}\,c_3^3+2-45\,a_{42}-15\,c_3+35\,c_3^2-25\,c_3^3}{(5\,c_3-3)\,(-2+5\,c_3)}$$

$$\frac{-850\,a_{43}\,c_3^3+500\,a_{43}\,c_3^4)}{(5\,c_3-3)\,(-2+5\,c_3)},$$

$$a_{32} = -\frac{(-10\,c_3+3+10\,c_3^2)\,c_3\,(-11\,c_3+2+19\,c_3^2-10\,c_3^3+102\,c_3\,a_{42}}{(5\,c_3-3)\,(-2+5\,c_3)}$$

$$\frac{-18\,a_{42}-180\,a_{42}\,c_3^2+100\,a_{42}\,c_3^3-30\,a_{43}\,c_3+180\,a_{43}\,c_3^2-340\,a_{43}\,c_3^3}{(5\,c_3-3)\,(-2+5\,c_3)}$$

$$\frac{+200\,a_{43}\,c_3^4)}{(5\,c_3-3)\,(-2+5\,c_3)},$$

$$a_{41}=-\frac{(100\,a_{43}\,c_3^2-25\,c_3^2+100\,c_3^2\;-120\,a_{43}\,c_3 120\,a_{43}\,c_3-120\,c_3\,a_{42}}{10\,(3-12\,c_3+10\,c_3^2)}$$

$$+\frac{+30\,c_3+30\,a_{42}+30\,a_{43}-8)}{10\,(3-12\,c_3+10\,c_3^2)},\ a_{42}=a_{42},\ a_{43}=a_{43},$$

$$b_1'=\frac{10\,c_3^2-8\,c_3+1}{12\,c_3\,(5\,c_3-3)},\ b_2'=\frac{25\,(-12\,c_3^2+6\,c_3-1+8\,c_3^2)}{12\,(5\,c_3-3)\,(-10\,c_3+3+10\,c_3^2)},$$

$$b_3'=\frac{25\,(10\,c_4^2-12\,c_4+3}{48\,(-4+5\,c_4)\,(11\,c_4-4)},\ b_4'=0,\ c_1=0,\ c_2=\frac{5\,c_3-3}{5\,(2\,c_3-1)},$$

$$c_3=c_3,\ c_4=1,\ b_1=\frac{10\,c_3^2-8\,c_3+1}{24\,c_3\,(5\,c_3-3)},$$

$$b_2=\frac{5\,(2\,c_3-1)\,(10\,c_3^2-9\,c_3+2)}{24\,(5\,c_3-3)\,(-10\,c_3+3+10\,c_3^2},$$

$$b_3=-\frac{-1+\;c_3}{24\,c_3\,(5\,c_3-3)},b_4=0,\ b_1''=\frac{10\,c_3^2-8\,c_3+1}{12\,c_3\,(5\,c_3-3)},$$

$$b_2''=\frac{16\,c_3^4-32\,c_3^3+24\,c_3^2-8\,c_3+1)}{(-2+5\,c_3)\,(5\,c_3-3)\,(-10\,c_3+3+10\,c_3^2)},$$

$$b_3''=-\frac{1}{12\,c_3\,(-1+c_3)\,(-10\,c_3+3+10\,c_3^2)},$$

$$b_4''=\frac{3-12\,c_3+10\,c_3^2}{12\,(-2+5\,c_3)\,(-1+\;c_3)\,(-4+5\,c_4)}.$$

Letting $c_3=\frac{2}{3}$, the local truncation error in two free parameters given by

$$\|\ t_g^{(6)}\ \|_2=\frac{1}{21600}\big(-61200\,a_{42}-401400\,a_{43}+596000\,a_{43}^2+18746 \\ +202500\,a_{42}^2+1875000\,a_{42}\,a_{43}\big)^{\frac{1}{2}}. \qquad (60)$$

By using minimize command in Maple we obtain
$a_{42}=-0.0176230497844326$, $a_{43}=0.0364465787986446$ and the minimum local truncation error is 0.005065254267. For the optimized value in fractional form

then we choose $a_{42} = -\frac{1}{50}$, and $a_{43} = \frac{1}{25}$. Finally, all the coefficients of four-stage fifth-order RKTG method denoted by RKTG5 can be written as follows (see Table 1):

5. Numerical Experiments

In this subsection, some of the problems involving $y''' = f(x, y, y')$ are tested upon. The numerical results are compared with the results obtained when the same set of problems is reduced to a system of first-order equations and is solved using the existing RK of the same order.

- RKTG5: the four-stage fifth-order RKTG method derived in this paper.
- RK5B: the six-stage fifth-order RK method given by Butcher [13].
- RKF5: the six-stage fifth-order RK method given by Lambert [14].
- DOPRI5: the seven-stage fifth-order RK method derived by Dormand [11].
- RK4: the fourth-order classical RK method as given in Butcher [13].

Problem 1: (Homogeneous Linear Problem)

$$y'''(x) = -25\, y'(x),$$
$$y(0) = 0, y'(0) = 0, y''(0) = 1,$$

The exact solution is given by $y(x) = \frac{1}{25} - \frac{1}{25}\cos(5x)$.

Problem 2: (Inhomogeneous Linear Problem)

$$y'''(x) = y'(x) + \cos^2(x) - 1,$$
$$y(0) = 0, y'(0) = 0, y''(0) = 1,$$

The exact solution is given by

$$y(x) = -\frac{1}{5}e^{-x} + \frac{4}{5}e^{x} - \frac{1}{20}\sin(2x) + \frac{1}{2}x - 1\,.$$

Problem 3: (Homogeneous Non-linear Problem)

$$y'''(x) = \frac{3\,y'(x)}{4(\,y(x))^4},$$
$$y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = -\frac{1}{4},$$

The exact solution is given by $y(x) = \sqrt{x+1}$.

Problem 4: (Inhomogeneous Non-linear Problem)

$$y'''(x) = y^2(x) + \cos^2(x) - y'(x) - 1,$$
$$y(0) = 0, y'(0) = 1, y''(0) = 0,$$

The exact solution is given by $y(x) = \sin(x)$.

Problem 5: (Non-linear System)

$$y_1'''(x) = \frac{1}{2}\, e^{4x}\, y_3(x)\, y_2'(x),$$
$$y_2'''(x) = \frac{8}{3}\, e^{2x}\, y_1(x)\, y_3'(x),$$
$$y_3'''(x) = 27\, y_2(x)\, y_1'(x)$$

$$y_1(0) = 1, y_1'(0) = -1, y_1''(0) = 1,$$
$$y_2(0) = 1, y_2'(0) = -2, y_2''(0) = 4,$$
$$y_3(0) = 1, y_3'(0) = -3, y_3''(0) = 9,$$

The exact solution is given by

$$y_1(x) = e^{-x},$$
$$y_2(x) = e^{-2x},$$
$$y_3(x) = e^{-3x}.$$

Problem 6: (Inhomogeneous Non-linear Problem)

$$y'''(x) = 6\,y'(x)\,y^2(x),$$
$$y(0) = 1, y'(0) = -1, y''(0) = 2,$$

The exact solution is given by $y(x) = \frac{1}{1+x}$.

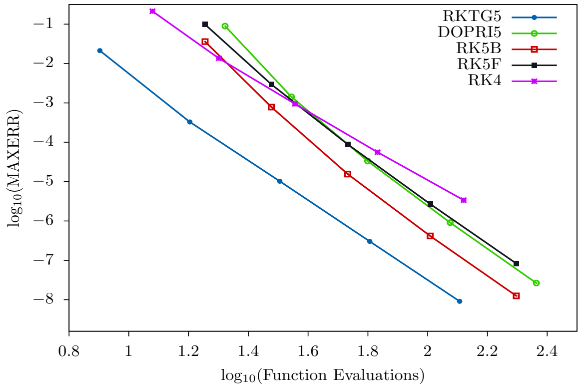

Figure 1: Comparison for RKTG5, RK5B, RK5F, DOPRI5 and RK4 Problem 1 with X_{end}=1

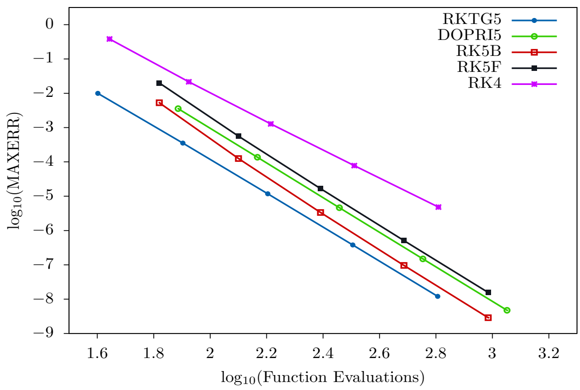

Figure 2: Comparison for RKTG5, RK5B, RK5F, DOPRI5 and RK4 Problem 2 with X_{end}=1

6. An Application to a Problem in Thin Film Flow

Here, we will use the suggested method to a famous problem in engineering and physics based on the thin film flow of a liquid. Many researchers in the literature explain more on this problem. Momoniat and Mahomed [16], constructed symmetry reduction and numerical solution of a third-order ODE from thin film flow. Tuck and Schwartz [17], discussed the movement of a thin film of viscous fluid over a solid surface and taken into account Tension, gravity, as well as viscosity. The problem was evaluated and solved using 3^{rd}-order ODE

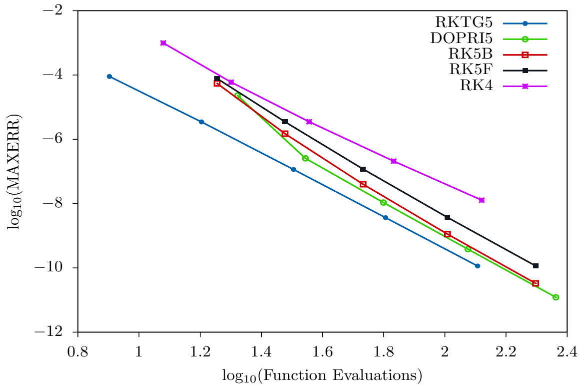

Figure 3: Comparison for RKTG5, RK5B, RK5F, DOPRI5 and RK4 Problem 3 with X_{end}=1

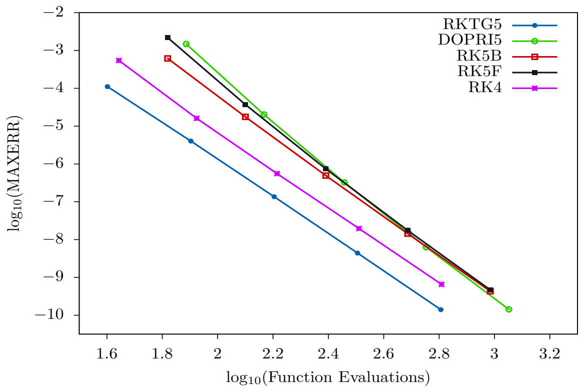

Figure 4: Comparison for RKTG5, RK5B, RK5F, DOPRI5 and RK4 Problem 4 with X_{end}=1

as follows:

$$\frac{d^3y}{dx^3} = f(y) \tag{61}$$

Many forms of the function were studied by [17]. for the drainage dry surface it has the form of $f(y)$ can be stated as:

$$\frac{d^3y}{dx^3} = -1 + \frac{1}{y^2}. \tag{62}$$

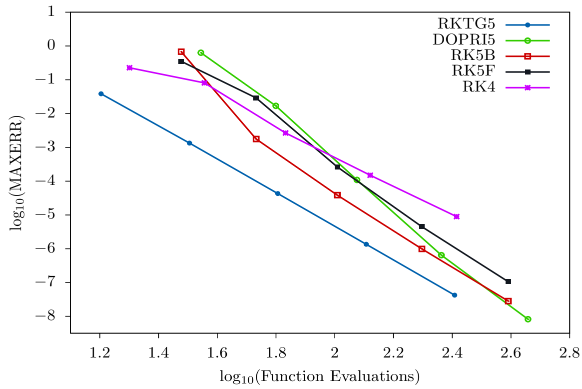

Figure 5: Comparison for RKTG5, RK5B, RK5F, DOPRI5 and RK4 Problem 5 with X_{end}=1

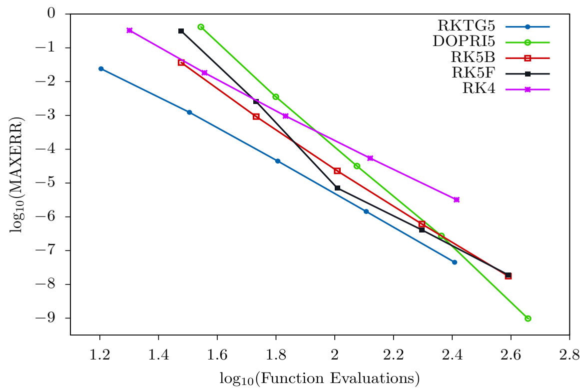

Figure 6: Comparison for RKTG5, RK5B, RK5F, DOPRI5 and RK4 Problem 6 with X_{end}=1

When the surface is pre-wetted by a thin film with thickness $\omega > 0$ (where $\omega > 0$ is very small), the function f is given by

$$f(y) = -1 + \frac{1+\omega+\omega^2}{y^2} - \frac{\omega+\omega^2}{y^3}. \tag{63}$$

Problems concerning the flow of thin films of viscous fluid with a free surface in which surface tension effects play a role typically lead to 3^{rd}-order ODEs governing the shape of the free surface of the fluid, $y = y(x)$. As indicated by [17], one such equation is

$$y''' = y^{-k},\; x \geq x_0 \tag{64}$$

with initial conditions

$$y(x_0) = \delta,\ y'(x_0) = \sigma,\ y''(x_0) = \xi, \tag{65}$$

where δ, σ, and ξ are constants, is of specific significance since it portrays the dynamic balance amongst surface and gooey strengths in a thin fluid layer in the disregard of gravity. For compare and contrast, we utilized Runge-Kutta strategies which are 5^{th}-order (RK5B and DOPRI5) strategies, individually. To utilize Runge-Kutta techniques we write (1) as a system of three 1^{st}-order equations. Following [18], we can write (64) as the following system:

$$\frac{dy_1}{dx} = y_2(x),\ \frac{dy_2}{dx} = y_3(x),\ \frac{dy_3}{dx} = y_1^{-k}(x), \tag{66}$$

where

$$y_1(0) = 1,\ y_2(0) = 1,\ y_3(x) = 1, \tag{67}$$

we have taken $x_0 = 0$ and $\delta = \sigma = \xi$=1. Unfortunately, for general k, (64) cannot be solved analytically. However, we can use these reductions to determine an the efficient way to solve (1) numerically. Here, we are focusing on the cases $k = 2$ and k= 3.

The results are displayed in Tables 2 to 3 for the case k= 2 and Tables 4 and 5 for the case k= 3.

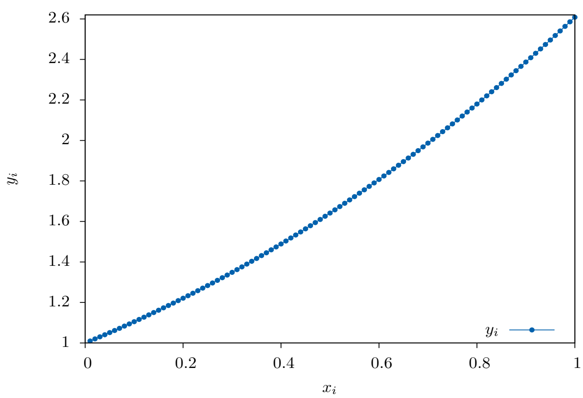

Figure 7: Plot of the solution y_i for problem (64) for $k = 2, h = 0.01$

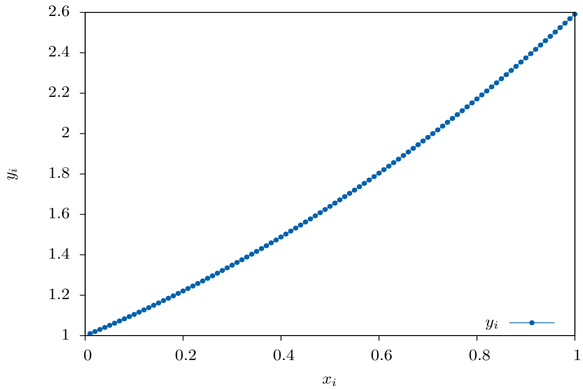

Figure 8: Plot of the solution y_i for problem (64) for $k = 3, h = 0.01$

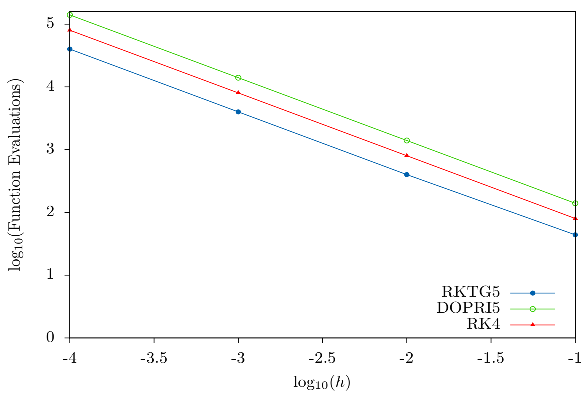

Figure 9: Plot of graph for function evaluations against step-size h for $k = 3, h = 1/10^i, i = 1...4$.

7. Discussion and Conclusion

In this review, we have inferred the order conditions for a RK techniques which can be utilized to unravel 3^{rd}-order ODEs specifically. A fourth-stage 5^{th}-order RKTG5 has been introduced and the comparison are made with existing RK methods and we used in numerical comparison the criteria based on computing the maximum error in the solution $\big(max(\mid y(t_n) - y_n \mid)\big)$ which is equal to the maximum between absolute errors of the actual solutions and computed solutions. The numerical outcomes are plotted in Figures 1, 2, ... , 6. Those Figures show the proficiency bends where the common logarithm of the maximum global error throughout the integration versus computational cost measured by the number of function evaluations. In Figures 7 and 8, we plot the numerical arrangement, y_i for $k = 2$ and $k = 3$, individually, with $h = 0.01$. Figure 9 demonstrates that the new RKTG5 technique requires less capacity assessments than the RK4 and DOPRI5 strategies. This is on account of when issue (64) is unraveled utilizing RK4 and DOPRI5 technique, it should be decreased to a system of 1^{st}-order equations which is three times the dimension. From numerical outcomes, we saw that the new RKTG5 strategy is more proficient compared with existing RK strategies and its has demonstrated that the new technique is more precise and able when solving 3^{rd}-order ODEs of the form $y''' = f(x, y, y')$ straightforwardly.

Table 1: The RKTG5 Method:

0	0				0							
$\frac{1}{5}$	0	0			$\frac{1}{50}$	0						
$\frac{2}{3}$	$-\frac{49}{4860}$	$\frac{301}{4860}$	0		$-\frac{1}{27}$	$\frac{7}{27}$	0					
1	$\frac{7}{50}$	$-\frac{1}{50}$	$\frac{1}{25}$	0	$\frac{3}{10}$	$-\frac{2}{35}$	$\frac{9}{35}$	0				
	$\frac{1}{48}$	$\frac{5}{42}$	$\frac{3}{112}$	0	$\frac{1}{24}$	$\frac{25}{84}$	$\frac{9}{56}$	0	$\frac{1}{24}$	$\frac{125}{336}$	$\frac{27}{56}$	$\frac{5}{48}$

Table 2: Comparison of RK5B, DOPRI5 and RKTG5 methods when solving Problem (64) with $h = 0.1$ and $k = 2$

x	Exact Solution	RK5B	DOPRI5	RKTG5
0.0	1.000000000	1.0000000000	1.0000000000	1.0000000000
0.2	1.221211030	1.2212103654	1.2212100041	1.2212100039
0.4	1.488834893	1.4888507091	1.4888347796	1.4888347797
0.6	1.807361404	1.8074895468	1.8073613979	1.8073613988
0.8	2.179819234	2.1803393022	2.1798192349	2.1798192371
1.0	2.608275822	2.6097383114	2.6082748696	2.6082748735

Table 3: Comparison of RK5B, DOPRI5 and RKTG5 methods when solving Problem (64) with $h = 0.01$ and $k = 2$

x	Exact Solution	RK5B	DOPRI5	RKTG5
0.0	1.000000000	1.0000000000	1.0000000000	1.0000000000
0.2	1.221211030	1.2212103651	1.2212100045	1.2212100045
0.4	1.488834893	1.4888507105	1.4888347799	1.4888347799
0.6	1.807361404	1.8074895516	1.8073613977	1.8073613977
0.8	2.179819234	2.1803393119	2.1798192339	2.1798192339
1.0	2.608275822	2.6097383271	2.6082748676	2.6082728676

Table 4: Comparison of RK5B, DOPRI5 and RKTG5 methods when solving Problem (64) with $h = 0.1$ and $k = 3$

x	RK5B	DOPRI5	RKTG5
0.0	1.0000000000	1.0000000000	1.0000000000
0.2	1.2211557749	1.2211551421	1.2211551394
0.4	1.4881300287	1.4881052848	1.4881052807
0.6	1.8044424216	1.8042625503	1.8042625459
0.8	2.1721917263	2.1715228023	2.1715227987
1.0	2.5927035854	2.5909582657	2.5909582638

Table 5: Comparison of RK5B, DOPRI5 and RKTG5 methods when solving Problem (64) with $h = 0.01$ and $k = 3$

x	RK5B	DOPRI5	RKTG5
0.0	1.0000000000	1.0000000000	1.0000000000
0.2	1.2211557725	1.2211551424	1.2211551424
0.4	1.4881300313	1.4881052842	1.4881052842
0.6	1.8044424292	1.8042625481	1.8042625481
0.8	2.1721917529	2.1715227981	2.1715227981
1.0	2.5927036287	2.5909582591	2.5909582591

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