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UNIFORM NUMBERS OF CYCLIC GRAPHS

M. Elakkiya[1], Kumar Abhishek[2 §]
[1,2]Amrita School of Engineering Coimbatore
Amrita Vishwa Vidyapeetham, INDIA

Abstract: The uniform number of a connected graph G is the least cardinality of a nonempty subset M of the vertex set of G for which the function $f_M : M^c \to \mathcal{P}(X) - \{\emptyset\}$ defined as $f_M(x) = \{D(x,y) : y \in M\}$ is a constant function, where $D(x,y)$ is the detour distance between x and y in G and $\mathcal{P}(X)$ is power set of $X = \{D(x_i,x_j) : x_i \neq x_j\}$. In this note, we determine the uniform number for the classes of graphs having at least one cycle as its induced subgraph.

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Key Words: graphs, distances, detour distance, uniform number.

1. Introduction

All the graphs unless otherwise specified are finite, undirected, connected and simple. For standard graph theory terminology and notations, we refer Buckley and Harary [1].

Let $\{x,y\}$ be an arbitrary pair of vertices in $G = (V,E)$. The detour distance $D(x,y)$ is the length of the longest path between x and y in G. As with distance, detour distance is also known to be a metric on the vertex set of any connected graph [2]. The detour eccentricity $e_D(x)$ of a vertex x is the detour distance from x to a vertex farthest from x. The detour radius of $rad_D(G)$ of a connected graph G is the minimum detour eccentricity among the vertices of G and the detour diameter $diam_D(G)$ is the maximum detour eccentricity among the vertices of

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§Correspondence author

G.

For any connected graph G, let $\mathcal{P}(X)$ be the powerset of the set $X = \{1, 2, \ldots, \mathrm{diam}_D(G)\}$ and $Y(X) = \mathcal{P}(X) - \{\emptyset\}$, and M^c be the complement of $M \subseteq V(G)$.

For a nonempty subset $M \subseteq V(G)$, Elakkiya *et al* [3] defined a function $f_M : M^c \to Y(X)$ as $f_M(x) = \{D(x, y) : y \in M\}$. The assignment $f_M(x)$ was referred as the detour pattern of the vertex x with respect to the set M in the graph G. Further, if the assignment f_M is a constant function, the set M was referred as a uniform detour pattern set of G. The authors in their introductory article observed that for any $w \in V(G)$ the set $\{w\}^c$ is trivially a uniform detour pattern set of G. A non-trivial uniform detour pattern set of a graph was referred as a uniform set of G and following facts were established in [3].

Proposition 1. Let G be a graph of order $n \geq 3$ and v be a pendant vertex of G, then the set $\{v\}$ cannot be a uniform set of G.

Proposition 2. Let S be the set of all pendant vertices of a graph G and $S' \subset S$, then S' alone cannot be a uniform set of G.

Proposition 3. If G be a graph of order $n \geq 4$ with exactly two pendant vertices, say u, v, then the set $\{u, v\}$ is a uniform set of G if and only if $G \cong P_4$.

A uniform set M of a graph G is minimal if it contains no proper uniform set. And, a uniform set M of a graph G is minimum if it is of the least cardinality. The cardinality of the minimum uniform set of G was referred as the uniform number of G, denoted as $\varsigma(G)$ and satisfies the following inequality.

Proposition 4. For any connected graph G of order n, $1 \leq \varsigma(G) \leq n-2$.

The bounds prescribed in Proposition 1 are known to be sharp. Following general results pertaining to $\varsigma(G)$ were established in [3].

Theorem 5. *A graph G of order $n \geq 3$ is Hamiltonian-connected if and only if every vertex of G is a uniform set of G.*

Theorem 6. *For P_n, $n \geq 3$,* $\varsigma(P_n) = \begin{cases} 1, & \text{if } n = 3 \\ 2, & \text{if } n = 4 \\ n-2, & \text{if } n \geq 5 \end{cases}$.

Theorem 7. *If T is a tree of order n, then $\varsigma(T) = 1$ if and only if $T \cong K_{1,n-1}$.*

Theorem 8. *If $K_{m,n}$ be a complete bipartite graph with $m, n \geq 2$, then* $\varsigma(K_{m,n}) = \begin{cases} 2, & \text{if } m = n \\ min\{m, n\}, & \text{if } m \neq n \end{cases}$.

Theorem 9. *If G be a strongly Hamiltonian-laceable graph of order $n \geq 4$, then $\varsigma(G) = 2$.*

Apart from the above mentioned results, uniform number of various classes of graphs has been determined in [3]. However, the problem of finding the uniform number of cycles remains open. In the quest to find the uniform number of cycles, we initiate the study of uniform number for the classes of graphs having at least one cycle as its induced subgraph.

2. New results on uniform numbers

We begin our discussion with triangular book graph. A triangular book [4], is a graph consisting of n copies of K_3 sharing a common edge and is isomorphic to the complete tripartite graph $K_{1,1,n}$.

Theorem 10. *If $G \cong K_{1,1,n}$ where $n \geq 2$, then $\varsigma(G) = 2$.*

Proof. Let $V_1 = \{u\}, V_2 = \{v\}$ and $V_3 = \{w_i : 1 \leq i \leq n\}$ be the color classes of $K_{1,1,n}$. Since $D(u, w_i) = 3 = D(v, w_i)$ for all $w_i \in V_3$, the set $M = \{u, v\}$ is a uniform set of G. We claim that M is minimum too.

Suppose not then, there exists a uniform set M' of G such that $|M'| = 1$. Since $|M'| = 1$, $M' = \{u\}$ or $M' = \{v\}$ or $M' = \{w_k\}$ for some $w_k \in V_3$.
Owing to the geometric symmetry of G about the edge uv it is enough to consider $M' = \{u\}$. Let $M' = \{u\}$. Since $D(v, u) = 2$ and $D(w_i, u) = 3$ for all $w_i \in V_3$, $f_{M'}(v) \neq f_{M'}(w_i)$. A contradiction to the assumption that M' is a uniform set of G. Hence, $M' \neq \{u\}$ and $M' \neq \{v\}$.
So, let $M' = \{w_k\}$ for some arbitrarily chosen $w_k \in V_3$. In view of the fact that $D(v, w_k) = 3$ and $D(w_i, w_j) = 4$ for all $1 \leq i \neq j \leq n$, therefore $f_{M'}(v) \neq f_{M'}(w_i)$ for all $1 \leq k \neq i \leq n$. Yet another contradiction to the assumption that M' is a uniform set of G, implying that $M' \neq \{w_k\}$ for any $w_k \in V_3$. Hence the claim. □

The problem of determining the uniform number of a general complete tripartite $K_{a,b,c}$ remains open.

The next three classes of cyclic graphs are obtainable from wheel W_n. A switch graph of a wheel W_n is a graph obtained by subdividing the spokes W_n.

Theorem 11. *If G be the switch graph of W_n, where $n \geq 3$, then $\varsigma(G) = n + 1$.*

Proof. Let u be the central vertex, $\{v_i : 1 \leq i \leq n\}$ be the vertices of the rim and $\{w_i : 1 \leq i \leq n\}$ be the vertices obtained by subdividing the spokes of

W_n so that the resultant is the switch graph G. For $1 \leq i, j \leq n$, the detour distance matrix D of G is given by

$$\mathrm{D} = \begin{bmatrix} & u & v_i & v_j & w_i & w_j \\ u & 0 & n+1 & n+1 & n+2 & n+2 \\ v_i & n+1 & 0 & n+2 & n+2 & n+3 \\ v_j & n+1 & n+2 & 0 & n+3 & n+2 \\ w_i & n+2 & n+2 & n+3 & 0 & n+4 \\ w_j & n+2 & n+3 & n+2 & n+4 & 0 \end{bmatrix}$$

The set $M = \{u\} \bigcup \{v_i : 1 \leq i \leq n\}$ is a uniform set of G, since $f_M(w_i) = \{n+2, n+3\}$ for all $w_i \in V(G)$.

To see that M is a minimum uniform set of G, let M' be a uniform set of G such that $|M'| < |M|$. Since $|M| = n+1$ and $|M'| < |M|$, we have the following exhaustive cases: either $M' \subseteq M^c$ or $M' \subset M$ or $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$.

Case 1: If $M' \subseteq M^c$, then either $M' = M^c$ or $M' \subset M^c$. Since $f_{M'}(v_i) = \{n+2, n+3\}$ for all $v_i \notin M^c$ and $f_{M'}(u) = \{n+2\}$ for $u \notin M^c$, $f_{M'}(v_i) \neq f_{M'}(u)$. A contradiction to our assumption that M' is a uniform set of G. Hence, $M' \neq M^c$. So, let $M' \subset M^c$ then there exists $w_k \in M^c$ such that $M' = M^c - \{w_k\}$. Since $f_{M'}(w_k) = \{n+4\}$ and $f_{M'}(u) = \{n+2\}$, $f_{M'}(w_k) \neq f_{M'}(u)$. Contradicting our assumption that M' is a uniform set of G. Hence, $M' \nsubseteq M^c$.

Case 2: If $M' \subset M$, there exist at least one element $x \in M$ such that $M' = M - \{x\}$. If $x = u$, then $M' = M - \{u\}$. Since $f_{M'}(u) = \{n+1\}$ and $f_{M'}(w_j) = \{n+2, n+3\}$ for all $w_j \in V(G)$, $f_{M'}(u) \neq f_{M'}(w_j)$. A contradiction to our assumption that M' is a uniform set of G. It follows that $u \notin M'$ is not true.

So, let $x = v_k$ for some $k \leq n$, then $M' = M - \{v_k\}$. Since $f_{M'}(v_k) = \{n+1, n+2\}$ and $f_{M'}(w_j) = \{n+2, n+3\}$ for all $w_j \in V(G)$, $f_{M'}(v_k) \neq f_{M'}(w_j)$. Again a contradiction to our assumption that M' is a uniform set of G. Since v_k was arbitrarily chosen, $v_k \notin M'$ for all $k \leq n$. Hence our assumption $M' \subset M$ is false.

Case 3: Finally, let $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$. Since $M' \bigcap M^c \neq \emptyset$, $M' \neq M$ and there exists at least one $x \in M$ such that $x \notin M'$. Similarly, there exists at least one $w_s \in M^c$ such that $w_s \notin M'$. If $x = u$, then $f_{M'}(u) = \{n+1, n+2\}$ and $f_{M'}(w_s) = \{n+2, n+3, n+4\}$ and therefore $f_{M'}(u) \neq f_{M'}(w_s)$. A contradiction to our assumption that M' is a uniform set of G. Hence $x \neq u$. So, let $x = v_k$ for some $k \leq n$, then $f_{M'}(v_k) = \{n+1, n+2, n+3\}$ and $f_{M'}(w_s) = \{n+2, n+3, n+4\}$ and therefore $f_{M'}(v_k) \neq f_{M'}(w_s)$. Again a contradiction to our assumption that M' is a uniform set of G. Hence $x \neq v_k$.

Since for all possibilities of $x \in M$ such that $x \notin M'$, we arrive at a contradiction to our assumption that M' is a uniform set of G, M is seen to be a minimum uniform set of G. □

Helms H_n [4] are the graphs obtained from a wheel W_n by attaching one pendent edge to each vertex on the rim of it's cycle.

Theorem 12. *If H_n be a helm graph with $n \geq 3$, then $\varsigma(H_n) = n+1$.*

Proof. Let u be the central vertex and $\{v_i : 1 \leq i \leq n\}$ be the vertices on the rim of H_n, and $\{w_i : 1 \leq i \leq n\}$ be the vertices that are made adjacent to v_i such that $v_i w_i$ is a pendant edge H_n. For $1 \leq i, j \leq n$, the detour distance matrix D of H_n is given by

$$\mathrm{D} = \begin{bmatrix} & u & v_i & v_j & w_i & w_j \\ u & 0 & n & n & n+1 & n+1 \\ v_i & n & 0 & n & 1 & n+1 \\ v_j & n & n & 0 & n+1 & 1 \\ w_i & n+1 & 1 & n+1 & 0 & n+2 \\ w_j & n+1 & n+1 & 1 & n+2 & 0 \end{bmatrix}$$

The set $M = \{u\} \bigcup \{v_i : 1 \leq i \leq n\}$ is a uniform set of H_n, since $f_M(w_i) = \{1, n+1\}$ for all $w_i \in V(H_n)$. We claim that M is a minimum too. Suppose not, then there exists a uniform set M' of H_n such that $|M'| < |M|$. Since $|M| = n+1$ and $|M'| < |M|$, we have the following exhaustive cases: either $M' \subseteq M^c$ or $M' \subset M$ or $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$. In view of Proposition 1, Proposition 2 and Proposition 3, $M' \nsubseteq M^c$. Hence either $M' \subset M$ or $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$.

Case 1: If $M' \subset M$, then there exist at least one element $x \in M$ such that $M' = M - \{x\}$. If $x = u$, then $M' = M - \{u\}$. Since $f_{M'}(u) = \{n\}$ and $f_{M'}(w_i) = \{1, n+1\}$ for all $w_i \in V(H_n)$, $f_{M'}(u) \neq f_{M'}(w_i)$. A contradiction to our assumption that M' is a uniform set of H_n, hence $x \neq u$. So, let $x = v_k$ for some k, then $M' = M - \{v_k\}$. Since $f_{M'}(v_k) = \{n\}$ and $f_{M'}(w_i) = \{1, n+1\}$ for all $w_i \in V(H_n)$, $f_{M'}(v_k) \neq f_{M'}(w_j)$. Again a contradiction to our assumption that M' is a uniform set of H_n, hence $x \neq v_k$ for any k. Since for all possibilities of $M' \subset M$, we have a contradiction to our assumption that M' is a uniform set of H_n, it follows that our assumption $M' \subset M$ is false.

Case 2: Finally, let $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$. Since $M' \bigcap M^c \neq \emptyset$, $M' \neq M$. Therefore there exists at least one $y \in M$ such that $y \notin M'$. Similarly, there exists a $w_r \in M^c$ such that $w_r \notin M'$. If $y = u$, then $f_{M'}(u) = \{n, n+1\}$ and $f_{M'}(w_r) = \{1, n+1, n+2\}$. Clearly, $f_{M'}(u) \neq f_{M'}(w_r)$. A contradicting our assumption that M' is a uniform set of H_n, hence $y \neq u$. Let $y = v_k$ for some $v_k \in M$. Then $f_{M'}(w_r) = \{1, n+1, n+2\}$ and $f_{M'}(v_k) = \{1, n, n+1\}$. Again, $f_{M'}(w_r) \neq f_{M'}(v_k)$. Yet another contradiction to our assumption that M' is a uniform set of H_n, hence $y \neq v_k$. But $v_k \in M$ was arbitrary chosen, therefore $y \neq v_k$ for any $v_k \in M$. Since for all possibilities of $y \in M$ we have a contradiction to the assumption that M' is a uniform set of H_n. Hence, M is seen to be a minimum uniform set of H_n. □

Flower graph [4] F_n is a graph obtained from helm H_n by joining each pendent vertex to the central vertex of the helm by an edge.

Theorem 13. *If F_n be a flower graph, where $n \geq 3$, then $\varsigma(F_n) = n + 1$.*

Proof. Let u be the central vertex, $\{v_j : 1 \leq j \leq n\}$ be the vertices of the rim and $\{w_i : 1 \leq i \leq n\}$ be the pendent vertices of H_n which are joined to the central vertex u by an edge to form a flower graph F_n. For $1 \leq i, j \leq n$, the detour distance matrix D of F_n is given by

$$\mathrm{D} = \begin{bmatrix} & u & v_i & v_j & w_i & w_j \\ u & 0 & n+1 & n+1 & n+2 & n+2 \\ v_i & n+1 & 0 & n+2 & n+2 & n+3 \\ v_j & n+1 & n+2 & 0 & n+3 & n+2 \\ w_i & n+2 & n+2 & n+3 & 0 & n+4 \\ w_j & n+2 & n+3 & n+2 & n+4 & 0 \end{bmatrix}$$

The set $M = \{u\} \bigcup \{v_i : 1 \leq i \leq n\}$ is a uniform set of F_n, since $f_M(w_j) = \{n+2, n+3\}$ for all $w_j \in V(F_n)$. We claim that M is a minimum uniform set of F_n. Suppose not, then there exist a uniform set M' of F_n such that $|M'| < |M| = n+1$. Since $|M| = n+1$ and $|M'| < |M|$, we have the following exhaustive cases: either $M' \subseteq M^c$ or $M' \subset M$ or $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$.

Case 1: If $M' \subseteq M^c$, then either $M' = M^c$ or $M' \subset M^c$. To begin with, let $M' = M^c$. Since $f_{M'}(v_i) = \{n+2, n+3\}$ and $f_{M'}(u) = \{n+2\}$, $f_{M'}(u) \neq f_{M'}(v_i)$. A contradiction to our assumption that M' is an uniform set of F_n. Therefore $M' \neq M^c$. Let $M' \subset M^c$, then there exists at least one $w_k \in M$ such that $M' = M^c - \{w_k\}$. Since $f_{M'}(w_k) = \{n+4\}$ and $f_{M'}(u) = \{n+2\}$, $f_{M'}(w_k) \neq f_{M'}(u)$. Yet another contradiction to our assumption that M' is a uniform set of F_n, hence $M' \nsubseteq M^c$.

Case 2: If $M' \subset M$, then there exists at least one element $x \in M$ such that $M' = M - \{x\}$. If $x = u$, then $M' = M - \{u\}$. Since $f_{M'}(u) = \{n+1\}$ and

$f_{M'}(w_i) = \{n+2, n+3\}$, $f_{M'}(u) \neq f_{M'}(w_j)$. A contradiction to our assumption that M' is a uniform set of F_n, hence $x \neq u$. So, let $x = v_k$ for some k, then $M' = M - \{v_k\}$. Since $f_{M'}(v_k) = \{n+1, n+2\}$ and $f_{M'}(w_i) = \{n+2, n+3\}$, $f_{M'}(v_k) \neq f_{M'}(w_i)$. Again a contradiction to our assumption that M' is a uniform set of F_n. Hence $x \neq v_k$ for any k. Since for all possibilities of $M' \subset M$, we have a contradiction to our assumption that M' is a uniform set of F_n, it follows that our assumption that $M' \subset M$ is false.

Case 3: Finally, let $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$. Since $M' \bigcap M^c \neq \emptyset$, $M' \neq M$. Therefore there exists at least one $y \in M$ such that $y \notin M'$. Similarly, there exists a $w_r \in M^c$ such that $w_r \notin M'$. If $y = u$, then $f_{M'}(u) = \{n+1, n+2\}$ and $f_{M'}(w_r) = \{n+2, n+3, n+4\}$. Clearly, $f_{M'}(u) \neq f_{M'}(w_r)$. A contradiction to our assumption that M' is a uniform set of F_n, hence $y \neq u$. Let $y = v_k$ for some $v_k \in M$. Then $f_{M'}(w_r) = \{n+2, n+3, n+4\}$ and $f_{M'}(v_k) = \{n+1, n+2, n+3\}$. Again, $f_{M'}(w_r) \neq f_{M'}(v_k)$. Yet another contradiction to assumption that is M' is a uniform set of F_n, hence $y \neq v_k$. But $v_k \in M$ was arbitrary chosen, therefore $y \neq v_k$ for any $v_k \in M$. Since for all possibilities of $y \in M$ we have a contradiction to the assumption that M' is a uniform set of F_n. Hence, M is a minimum uniform set of F_n. ☐

In 1963, Ore [5] defined a graph G to be Hamilton-connected if it has a spanning path for all pairs of vertices x and y in G. The $(n, m)-$lollipop graph [6] is the graph obtained by joining a complete graph K_n to a path graph P_m with a bridge. In view of the fact that a complete graph is a Hamilton connected graph, we extend the notion of $(n, m)-$lollipop graph by replacing K_n by a Hamilton connected graph of order n.

Theorem 14. *If G be a $(n, m)-$lollipop graph obtained by joining a hamiltonian connected graph H of order $n \geq 3$ to a path graph P_m with a bridge, then $\varsigma(G) = m + 1$.*

Proof. Let G be a $(n, m)-$lollipop graph as in the statement with $V(H) = \{v_i : 1 \leq i \leq n\}$ and $V(P_m) = \{w_j : 1 \leq j \leq m\}$. Let $v_n w_1$ be the bridge in G. For any two vertices $x, y \in V(G)$ we have

$$\begin{aligned} D(v_l, v_m) &= n-1, \\ D(v_i, w_j) &= \begin{cases} j, & i = n, \\ n-1+j, & i \neq n, \end{cases} \\ D(w_r, w_s) &= |r-s|,\ 1 \leq r, s \leq m. \end{aligned}$$

The set $M = \{v_n\} \bigcup \{w_j : 1 \leq j \leq m\}$ is a uniform set of G, since for all i such that $1 \leq i \leq n-1$, we have

$$\begin{aligned} f_M(v_i) &= \{D(v_i, v_n)\} \bigcup \{D(v_i, w_j) : 1 \leq j \leq m\} \\ &= \{n-1\} \bigcup \{n-1+j : 1 \leq j \leq m\} \\ &= \{n-1, n, n+1, n+2, \ldots, n+m-2, n+m-1\}. \end{aligned}$$

We claim that M is minimum too. Suppose not, then there exists a uniform set M' of G such that $|M'| < |M|$. Since $|M| = n+1$ and $|M'| < |M|$, we have the following exhaustive cases: either $M' \subseteq M^c$ or $M' \subset M$ or $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$.

Case 1: Let $M' \subseteq M^c$, then either $M' = M^c$ or $M' \subset M^c$. It's easy to verify in either of the cases, $f_{M'}(v_n) \neq f_{M'}(w_1)$. A contradiction to the assumption that M' is a uniform set of G. Hence, $M' \nsubseteq M^c$.

Case 2: Let $M' \subset M$, then there exists at least one element $x \in M$ such that $M' = M - \{x\}$. If possible let $x = v_n$, then

$$\begin{aligned} f_{M'}(v_n) &= \{D(v_n, w_j) : 1 \leq j \leq m\} = \{j : 1 \leq j \leq m\}, \text{and} \\ f_{M'}(v_1) &= \{D(v_1, w_j) : 1 \leq j \leq m\} = \{n-1+j : 1 \leq j \leq m\}. \end{aligned}$$

A contradiction to the assumption that M' is a uniform set of G. Hence $x \neq v_n$.

Let $x = w_k$ for some $k, 1 \leq k \leq m$ and we have

$$\begin{aligned} f_{M'}(w_k) &= \{D(w_k, w_j) : 1 \leq j \neq k \leq m\} \bigcup \{D(w_k, v_n)\} \\ &= \begin{cases} \{1, 2, 3, \ldots, m-k\} \bigcup \{k\}, & \text{if } k \leq \lceil \frac{m}{2} \rceil \\ \{1, 2, 3, \ldots, k-1\} \bigcup \{k\}, & \text{if } k > \lceil \frac{m}{2} \rceil \end{cases}, \text{and} \\ f_{M'}(v_1) &= \{D(v_1, w_j) : 1 \leq j \neq k \leq m\} \bigcup \{D(v_1, v_n)\} \\ &= \{n-1+j : 1 \leq j \neq k \leq m\} \bigcup \{n-1\}. \end{aligned}$$

Since $n \geq 3$ and $j \geq, 1$ it follows that $1 \notin f_{M'}(v_1)$ and therefore $f_{M'}(w_k) \neq f_{M'}(v_1)$. Yet another contradiction to the assumption that M' is a uniform set of G. Hence, $x \neq w_k$. Since for all possible choices of $x \in M$ such that $x \notin M'$ we arrive at a contradiction to the assumption that M' is a uniform set of G, $M' \subset M$ is false.

Case 3: Finally, let $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$, then M' is not a subset of M. Then there exists at least one element in $y \in M^c$ such that $y \in M'$ and $z \in M$ such that $z \notin M'$. We now have the following possibilities:

When $y = v_i$ for some i say $i = 1$ and $z = v_n$. Since $|M'| \leq m$, therefore there exists a w_k such that $w_k \notin M'$.

Hence, $M' = \{v_1\} \bigcup \{w_j : 1 \leq j \neq k \leq m\}$, and we have

$$\begin{aligned} f_{M'}(v_n) &= \{D(v_n, v_1)\} \bigcup \{D(v_n, w_j) : 1 \leq j \neq k \leq m\} \\ &= \{n-1\} \bigcup \{j : 1 \leq j \neq k \leq m\}, \text{and} \\ f_{M'}(v_2) &= \{D(v_2, v_1)\} \bigcup \{D(v_2, w_j) : 1 \leq j \neq k \leq m\} \\ &= \{n-1\} \bigcup \{n-1+j : 1 \leq j \neq k \leq m\}. \end{aligned}$$

Since $j \geq 1$ and $n \geq 3$, it follows that $f_{M'}(v_n) \neq f_{M'}(v_2)$. A contradiction, hence M' is not of the form $\{v_1\} \bigcup \{w_j : 1 \leq j \neq k \leq m\}$. It is easy to verify that $M' \neq \{v_i : 1 \leq i \leq n\}$, as $f_{M'}(w_1) \neq f_{M'}(w_m)$.
Hence the only possibility remaining is that $M' = \{v_1, v_n\} \bigcup \{w_j : 1 \leq j \neq r \neq s \leq m\}$. Without loss of generality let $r > s$. Since w_r and w_s lies on the path graph P_m and $v_n w_1$ is a bridge in G, for any vertex $x \in V(G)$ we have $D(w_r, x) = D(w_r, w_s) + D(w_s, x)$ and $D(w_r, w_s) > 0$ which implies $f_{M'}(w_r) \neq f_{M'}(w_s)$. Which is again a contradiction to the assumption that M' is a uniform set of G. Hence the claim. □

3. Conclusion

In this note we have exhibited the existence of graphs for which the uniform number is almost at the middle of the bounds prescribed by Proposition 4.

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