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SUBMERSION OF SEMI-INVARIANT SUBMANIFOLDS OF LORENTZIAN PARA-SASAKIAN MANIFOLDS

Vibha Srivastava[1 §], P.N. Pandey[2]
[1,2]Department of Mathematics
University of Allahabad
Allahabad, 211002, INDIA

Abstract: In this paper, we discuss submersion of semi-invariant submanifolds of Lorentzian para-Sasakian manifolds and derive some results on its geometry. We also derive some curvature relations.

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1. Introduction

In 1966, O' Neill [1] initiated the study of Riemannian submersion. Semi-Riemannian submersion was introduced by O' Neill in [2]. In 1981, Kobayashi [3] studied CR-Submanifold of Sasakian manifold whereas Benjancu [4, 5] introduced the CR-submanifolds of Kähler manifold. Submersion of CR-submanifolds of nearly trans-Sasakian manifold were studied by Jamali and shahid [6] . Submersion of semi-invariant submanifolds of trans-Sasakian manifold were studied by Jamali et al [7]. In [8], Matsumoto introduced the notion of Lorentzian para-Sasakian manifold. In [9], the authors defined the same notion independently and they obtained many results about this type of manifold. In this paper we studied submersion of semi-invariant submanifolds of Lorentzian para-Sasakian manifolds.

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§Correspondence author

2. Preliminaries

Let $\overline{M}$ be an $n-$dimensional Lorentzian manifold with a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1- form η which satisfy

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2}$$

$$g(X,\xi) = \eta(X), \tag{3}$$

$$g(X,\phi Y) = g(\phi X, Y), \tag{4}$$

for any vector fields X, Y tangents to $\overline{M}$, it is called Lorentzian almost para-contact manifold [8]. Also in a Lorentzian almost para-contact structure the following relations hold :

$$\phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \text{rank } \phi = n-1.$$

A Lorentzian almost para-contact manifold $\overline{M}$ is called Lorentzian para-Sasakian ($LP-$Sasakian) manifold if [10]

$$\left(\overline{\nabla}_X \phi\right) Y = g(X,Y)\xi + \eta(Y) X + 2\eta(X)\eta(Y)\xi. \tag{5}$$

Definition 1. A $m-$dimensional Riemannian submanifold M of a Lorentzian para-Sasakian manifold $\overline{M}$ is called a semi-invariant submanifold if ξ is tangent to M and it is endowed with a pair of orthogonal differentiable distributions $\left(D, D^{\perp}\right)$, which satisfies (1) $TM = D \oplus D^{\perp} \oplus \xi$, where $\oplus$ denotes the orthogonal direct sum, (2) the distribution $D_x : x \longrightarrow D \subset T_x M$ is invariant under ϕ i.e $\phi D_x \subset D_x$ for each $x \in M$ (3) the orthogonal complementary distribution $D^{\perp} : x \longrightarrow D^{\perp} \subset T_x M$ of the distribution D on M is totally real i.e $\phi D^{\perp} \subset T_x^{\perp} M$ where $T_x M$ and $T_x^{\perp} M$ are the tangent space and the normal space of M at x respectively.

Let the dimension of $D\left(resp.\ D^{\perp}\right)$ be $2p\,$(resp. q), where $2p + q = m - 1$. If $p = 0$ $(resp.\ q = 0)$ the submanifold M becomes anti-invariant (resp. invariant)submanifold. A generic submanifold M satisfies $D^{\perp} = \dim T_x^{\perp} M$. A submanifold is called proper if it is neither invariant nor anti-invariant. It is easy to see that any hypersurface to which the vector field ξ is tangent is a typical example of semi-invariant submanifold. Where D and $D^{\perp}$ are the horizontal and vertical distribution respectively. Let $\overline{\nabla}$ (resp. ∇) be the covariant differentiation with respect to Levi-Civita connection on $\overline{M}$ (resp. M).

The Gauss and Weingarten formulas for M are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{6}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{7}$$

for $X, Y \in TM, N \in T^{\perp}M$, where σ(resp. A) is the second fundamental form(resp. tensor) of M in $\overline{M}$, and $\overline{\nabla}$ denote the operator of the normal connection.

$$g(\sigma(X,Y), N) = g(A_N X, Y). \tag{8}$$

The projection of TM to D and $D^{\perp}$ are denoted by h and v respectively i.e, for any $X \in TM$ we have

$$X = \sigma X + vX + \eta(X)\xi. \tag{9}$$

The normal bundle to M has the decomposition

$$T^{\perp}M = \phi D^{\perp} \oplus n_1, \tag{10}$$

where $g\left(\phi D^{\perp}, n_1\right) = 0$. For any $U \in T^{\perp}M$, we put

$$U = nU + mU, \tag{11}$$

where $nU \in \phi D^{\perp}, mU \in n_1$. From the above equation we have

$$\phi U = \phi nU + \phi mU, U \in T^{\perp}M, \phi nU \in D^{\perp}, \phi mU \in n_1. \tag{12}$$

Definition 2. Let M be a semi-invariant submanifold of a Lorentzian para-Sasakian manifolds $\overline{M}$ and M' be an Lorentzian manifold with structure (ϕ', ξ', η', g'). Assume that there is a submersion $\pi : M \to M'$ such that (i) $D^{\perp} = \ker \pi_* : TM \to TM'$ is the tangent mapping to π, (ii) $\pi_* : D_p \oplus \{\xi\} \to T_{\pi(p)}M'$ is an isometry for each $p \in M$ which satisfies $\pi_* \circ \phi = \phi' \circ \pi_*$; $\eta = \eta' \circ \pi_*$; $\pi_*(\xi_p) = \xi'_{\pi(p)}$, where $T_{\pi(p)}M'$ denotes the tangent space of M' at $\pi(p)$.

A vector X on M is said to be basic if, $X \in D_p \oplus \xi$ and X is $\pi-$related to a vector field on M' i.e there exists a vector field $X_* \in TM'$ such that $\pi_*(X_p) = X_{*\pi(p)}$ for each $p \in M$. Note that, by condition (ii) of the above definition 2, we have that the structural vector field ξ is a basic vector field.

Lemma 3. *Let X, Y be basic vector fields on M. Then*
(i) $g(X,Y) = g'(X_*, Y_*) o\pi$,
(ii) *the component* $\sigma([X,Y]) + \eta([X,Y]\xi) = [X_*, Y_*]$,
(iii) $[U, X] \in D^{\perp}$ *for any* $U \in D^{\perp}$,
(iv) $\sigma(\nabla_X Y) + \eta(\nabla_X Y)\xi$, *is a basic vector field corresponding to* $\nabla^*_{X_*} Y_*$, *where* ∇^* *denote the Levi-Civita connection on* M'.

For basic vector fields on M, we define the operator $\widetilde{\nabla}^*$ corresponding to ∇^* by setting $\widetilde{\nabla}^*_X Y = \sigma([X,Y]) + \eta([X,Y]\xi)$ for $X, Y \in (D_p \oplus \{\xi\})$. By (iv) of Lemma 3, $\widetilde{\nabla}^*_X Y$ is a basic vector field and we have

$$\pi_*\left(\widetilde{\nabla}^*_X Y\right) = \nabla^*_{X_*} Y_*. \tag{13}$$

Define the tensor field C by

$$\nabla_X Y = \widetilde{\nabla}^*_X Y + C(X,Y), \tag{14}$$

$X, Y \in (D_p \oplus \{\xi\})$, where $C(X,Y)$ is the verticle part of $\nabla_X Y$. It is known that C is skew-symmetric and satisfies

$$C(X,Y) = \frac{1}{2} v[X,Y], \tag{15}$$

where $X, Y \in (D_p \oplus \{\xi\})$.
The curvature tensor R, R^* of the connection ∇, ∇^* on M and M' respectively are related by

$$\begin{aligned} R(X,Y,Z,W) &= R^*(X_*,Y_*,Z_*,W_*) - g(C(Y,Z),C(X,W)) \\ &\quad + g(C(X,Z),C(Y,W)) + 2g(C(X,Y),C(Z,W)) \end{aligned} \tag{16}$$

for $X, Y, Z, W \in (D_p \oplus \{\xi\})$,where $\pi_* X = X_*, \pi_* Y = Y_*, \pi_* Z = Z_*$ and $\pi_* W = W_* \in \chi M'$. For Lorentzian para-Sasakian manifold $\overline{M}$ we prove

Proposition 4. *Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a Lorentzian para- Sasakian manifold $\overline{M}$ onto a Lorentzian manifold M'. Then we have*

$$\left(\widetilde{\nabla}^*_X \phi\right) Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{17}$$

$$C(X,\phi Y) = \phi n\sigma(X,Y), \tag{18}$$

$$\phi C(X,Y) = n\sigma(X,\phi Y), \tag{19}$$

$$\phi m\sigma(X,Y) = m\sigma(X,\phi Y), \tag{20}$$

for any $X, Y \in (D_p \oplus \{\xi\})$.

Proof. For any $X, Y \in (D \oplus \{\xi\})$ and by using Gauss formula (6), decomposition equation (11) and (14), we obtain

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) = \nabla_X Y + n\sigma(X,Y) + m\sigma(X,Y)$$

$$= \widetilde{\nabla}^*_X Y + C(X,Y) + n\sigma(X,Y) + m\sigma(X,Y). \tag{21}$$

And

$$\phi\widetilde{\nabla}_X Y = \phi\widetilde{\nabla}^*_X Y + \phi C(X,Y) + \phi n\sigma(X,Y) + \phi m\sigma(X,Y). \tag{22}$$

Putting $Y = \phi Y$ in equation (21), we get

$$\widetilde{\nabla}_X \phi Y = \widetilde{\nabla}^*_X \phi Y + C(X,\phi Y) + n\sigma(X,\phi Y) + m\sigma(X,\phi Y). \tag{23}$$

Using the definition of Lorentzian para-Sasakian manifold, we find

$$\left(\widetilde{\nabla}^*_X \phi\right) Y = \widetilde{\nabla}_X \phi Y - \phi\widetilde{\nabla}_X Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{24}$$

Substituting (22) and (23) in (24), we get

$$\begin{aligned} & \widetilde{\nabla}^*_X \phi Y + C(X,\phi Y) + n\sigma(X,\phi Y) + m\sigma(X,\phi Y) - \phi\widetilde{\nabla}^*_X Y \\ & -\phi C(X,Y) - \phi n\sigma(X,Y) - \phi m\sigma(X,Y) \\ = \; & g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \end{aligned} \tag{25}$$

Comparing the components of $(D \oplus \{\xi\})$, $D^\perp, \phi D^\perp$ and n_1 respectively on both sides in the above equation, we get the required results. □

Proposition 5. *Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a Lorentzian para- Sasakian manifold $\overline{M}$ onto a Lorentzian manifold M'. Then M' is also a Lorentzian para-Sasakian manifold.*

Proof. From equation (17) of proposition (4), we have

$$\left(\widetilde{\nabla}^*_X \phi\right) Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{26}$$

Applying π_* to the above equaction and using Lemma 3, equation (13) and definition of submersion, we derive

$$\left(\widetilde{\nabla}^*_{X_*} \phi'\right) Y_* = g'(X_*,Y_*)\xi' + \eta'(Y_*)X' + 2\eta'(X_*)\eta'(Y_*)\xi'. \tag{27}$$

Hence M' is a Lorentzian para-Sasakian manifold. □

Proposition 6. *Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a Lorentzian para- Sasakian manifold $\overline{M}$ onto a Lorentzian manifold M'. Then*

(i) $n\sigma(X,\phi Y) + n\sigma(\phi X,Y) = 0,$

(ii) $n\sigma(\phi X,\phi Y) = -n\sigma(Y,X),$

(iii) $m\sigma(\phi X,\phi Y) = m\sigma(X,Y),$

(iv) $C(\phi X,\phi Y) = -C(X,Y),$

for any $X, Y \in (D \oplus \{\xi\})$.

Proof. (i) Interchaging X and Y in equation (19) gives

$$\phi C(Y,X) = n\sigma(Y,\phi X) = n\sigma(\phi X,Y), \tag{28}$$

Then

$$n\sigma(X,\phi Y)+n\sigma(\phi X,Y) = \phi C(X,Y)+\phi C(Y,X) = \phi C(X,Y)-\phi C(X,Y) = 0.$$

(ii) Putting $X = \phi X$ in (19), we get

$$n\sigma(\phi X,\phi Y) = \phi C(\phi X,Y) = -\phi C(Y,\phi X). \tag{29}$$

Using (18) in (29), we deduce

$$\begin{aligned} n\sigma(\phi X,\phi Y) &= -\phi C(Y,\phi X) = -\phi(\phi n\sigma(Y,X)) = -\phi^2 n\sigma(Y,X) \\ &= -n\sigma(Y,X) - \eta(\sigma(X,Y))\xi = -n\sigma(Y,X). \end{aligned}$$

(iii) Putting $X = \phi X$ in (20) and using again the same equation, we find

$$m\sigma(\phi X,\phi Y) = \phi m\sigma(\phi X,Y) = \phi m\sigma(Y,\phi X) = \phi^2 m\sigma(Y,X) = m\sigma(X,Y).$$

(iv) Putting $X = \phi X$ in (18) and then using (19) yields

$$\begin{aligned} C(\phi X,\phi Y) &= \phi n\sigma(\phi X,Y) = \phi n\sigma(Y,\phi X) = \phi^2 C(Y,X) \\ &= C(Y,X) + \eta(C(Y,X))\xi = -C(X,Y). \quad \square \end{aligned}$$

3. Curvature Relation

Proposition 7. *Let $\pi : M \to M'$ be a submersion of semi -invariant submanifold of a Lorentzian para-Sasakian manifold $\overline{M}$ onto a Lorentzian manifold M'. Then the $\phi-$sectional curvature of $\overline{M}$ and M' are related by*

$$\overline{B}(X,Y) = B'(X_*,Y_*) - 2g(n\sigma(X,X), n\sigma(Y,Y)).$$

where $X, Y \in (D \oplus \{\xi\})$.

Proof. We know

$$\overline{B}(X,Y) = \overline{R}(X,\phi X,\phi Y,Y).$$

Putting $Y = \phi X, Z = \phi Y, W = Y$ in Gauss equation

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(\sigma(X,W),\sigma(Y,Z))$$

$$+g\left(\sigma\left(X,Z\right),\sigma\left(Y,W\right)\right),$$

we get

$$\begin{aligned}\overline{R}\left(X,\phi X,\phi Y,Y\right) &= R\left(X,\phi X,\phi Y,Y\right)-g\left(\sigma\left(X,Y\right),\sigma\left(\phi X,\phi Y\right)\right)\\ &\quad +g\left(\sigma\left(X,\phi Y\right),\sigma\left(\phi X,Y\right)\right).\end{aligned}$$

Subsitituting $\sigma = n\sigma + m\sigma$, in above equation, we get

$$\begin{aligned}&\overline{R}\left(X,\phi X,\phi Y,Y\right)=R\left(X,\phi X,\phi Y,Y\right)-g(n\sigma\left(X,Y\right)\\ &\quad + m\sigma\left(X,Y\right),n\sigma\left(\phi X,\phi Y\right)+m\sigma\left(\phi X,\phi Y\right))\\ &\quad + g(n\sigma\left(X,\phi Y\right)+m\sigma\left(X,\phi Y\right),n\sigma\left(\phi X,Y\right)+m\sigma\left(\phi X,Y\right)),\\ &=R\left(X,\phi X,\phi Y,Y\right)-g\left(n\sigma\left(X,Y\right),n\sigma\left(\phi X,\phi Y\right)\right)\\ &\quad - g\left(n\sigma\left(X,Y\right),m\sigma\left(\phi X,\phi Y\right)\right)-g\left(m\sigma\left(X,Y\right),n\sigma\left(\phi X,\phi Y\right)\right)\\ &\quad - g\left(m\sigma\left(X,Y\right),m\sigma\left(\phi X,\phi Y\right)\right)+g\left(n\sigma\left(X,\phi Y\right),n\sigma\left(\phi X,Y\right)\right)\\ &\quad + g\left(n\sigma\left(X,\phi Y\right),m\sigma\left(\phi X,Y\right)\right)+g\left(m\sigma\left(X,\phi Y\right),n\sigma\left(\phi X,\phi Y\right)\right)\\ &\quad + g\left(m\sigma\left(X,\phi Y\right),m\sigma\left(\phi X,Y\right)\right),\end{aligned}$$

$$\begin{aligned}&\overline{R}\left(X,\phi X,\phi Y,Y\right)=R\left(X,\phi X,\phi Y,Y\right)-g\left(n\sigma\left(X,Y\right),n\sigma\left(\phi X,\phi Y\right)\right)\\ &\quad - g\left(m\sigma\left(X,Y\right),m\sigma\left(\phi X,\phi Y\right)\right)+g\left(n\sigma\left(X,\phi Y\right),n\sigma\left(\phi X,Y\right)\right)\\ &\quad + g\left(\phi m\sigma\left(X,Y\right),\phi m\sigma\left(X,Y\right)\right),\end{aligned}$$

$$\overline{R}\left(X,\phi X,\phi Y,Y\right)=R\left(X,\phi X,\phi Y,Y\right)+\left\|n\sigma\left(X,Y\right)\right\|^{2}-\left\|n\sigma\left(X,\phi Y\right)\right\|^{2}. \quad (30)$$

Putting $Y=\phi X, Z=\phi Y, W=Y$ in equation (16) it follows

$$\begin{aligned}R\left(X,\phi X,\phi Y,Y\right) &= R^{*}\left(X_{*},\phi^{'}X_{*},\phi^{'}Y_{*},Y_{*}\right)\\ &\quad -g\left(C\left(\phi X,\phi Y\right),C\left(X,Y\right)\right)\\ &\quad +g\left(C\left(X,\phi Y\right),C\left(\phi X,Y\right)\right)\\ &\quad +2g\left(C\left(X,\phi X\right),C\left(\phi Y,Y\right)\right).\end{aligned}$$

$$\begin{aligned}R\left(X,\phi X,\phi Y,Y\right) &= R^{*}\left(X_{*},\phi^{'}X_{*},\phi^{'}Y_{*},Y_{*}\right)\\ &\quad -g\left(C\left(\phi X,\phi Y\right),C\left(X,Y\right)\right)\\ &\quad -g\left(C\left(X,\phi Y\right),C\left(Y,\phi X\right)\right)\end{aligned}$$

$$-2g\left(C\left(X,\phi X\right),C\left(Y,\phi Y\right)\right). \tag{31}$$

Applying ϕ on both side of equation (19) ,we obtain

$$C\left(X,Y\right)=\phi n\sigma\left(X,\phi Y\right). \tag{32}$$

Using equation (32) in equation (31) give

$$\begin{aligned} R\left(X,\phi X,\phi Y,Y\right) &= R^{*}\left(X_{*},\phi^{'}X_{*},\phi^{'}Y_{*},Y_{*}\right) \\ &\quad -\left\|n\sigma\left(X,Y\right)\right\|^{2}+\left\|n\sigma\left(X,\phi Y\right)\right\|^{2} \\ &\quad -2g\left(n\sigma\left(X,X\right),n\sigma\left(Y,Y\right)\right). \end{aligned}$$

Putting the value of $R\left(X,\phi X,\phi Y,Y\right)$ in (30) , we obtain

$$\overline{R}\left(X,\phi X,\phi Y,Y\right)=R^{*}\left(X_{*},\phi^{'}X_{*},\phi^{'}Y_{*},Y_{*}\right)-2g\left(n\sigma\left(X,X\right),n\sigma\left(Y,Y\right)\right)$$

or

$$\overline{B}\left(X,Y\right)=B^{'}\left(X_{*},Y_{*}\right)-2g\left(n\sigma\left(X,X\right),n\sigma\left(Y,Y\right)\right). \tag{33}$$

□

Proposition 8. *Let $\pi : M \rightarrow M'$ be a submersion of semi-invariant submanifold of a Lorentzian para- Sasakian manifold $\overline{M}$ onto a Lorentzian manifold M'. Then the $\phi-$sectional curvature of $\overline{M}$ and M' are related by*

$$\overline{H}\left(X\right)=H'\left(X_{*}\right)-2\left\|n\sigma\left(X,X\right)\right\|^{2},$$

where $X,Y\in(D\oplus\{\xi\})$.

Proof. Putting $X=Y$ in equation (33) we obtain

$$\begin{aligned} \overline{B}\left(X,X\right) &= \overline{H}\left(X\right)=H'\left(X_{*}\right)-2g\left(n\sigma\left(X,X\right),n\sigma\left(X,X\right)\right) \\ &= H'\left(X_{*}\right)-2\left\|n\sigma\left(X,X\right)\right\|^{2} \end{aligned}$$

Thus we get

$$\overline{H}\left(X\right)=H'\left(X_{*}\right)-2\left\|n\sigma\left(X,X\right)\right\|^{2}.$$

□

Application: Lorentzian para-Sasakian manifolds are used in the theory of Relativity and Newtons law of gravitational field.

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