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DEFINED CONTRIBUTION PENSION PROGRAM WITH DETERMINISTIC REVENUE AND MORTALITY RISK BY APPLYING A MULTI-PERIOD MEAN-VARIANCE

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Abstract: This article examines an asset distribution with deterministic revenue and mortality risk on defined contribution (DC) pension program by applying multi-period mean-variance model. Unlike other research in this article's literature where the revenue is stochastic, this article appraise deterministic revenue that increases every period constantly. The analytical statements of the effective-investment and effective-boundary strategy are discovered by applying Lagrange multiplier method, state-variable transformation and stochastic optimal control theory. Two numerical simulations are explained at the end of this article. The first simulation is provided by the dissimilar value of contribution's percentage and the second one is explained by the dissimilar value of mortality intension.

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1. Introduction

Pension program becomes outstanding since it could support the individual ex-

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istence at their retirement term. There are two basic kinds of pension programs, i.e. defined benefit (DB) pension program and defined contribution (DC) pension program [1]. These two kinds of pension programs are corresponding to operation scheme and fund procurement [2]. The DB pension program advantages are set by the insurer beforehand [3] and the contributions are adjusted and set as well, so that the fund stay in balance. In a DB pension program, the risk is undertaken by the insurer, so the insured does not need to endure the risk on their own [2]. A DC pension program assigns the contributions each year. The retirement advantages are established by the scale of the accumulation and the return of the investment at the retirement term [2], so the risk is not endured by the insurer but it becomes undertaken by the insured.

Throughout the history, a DB pension program's popularity is more than DC pension program has. It is also well known that DB pension program is more preferable than DC pension program as DB pension program is easier to manage than DC pension program. Retirement risk of DB pension program is undertaken by the insurer. Nevertheless, in current years, DC pension program has ended up outstanding because of the capital market development and the demographic transition, particularly because of the longevity (long life) risk and the population ageing [4]. This phenomenon makes many countries turn their DB pension program into a DC pension program in a whole or in a part plan [2]. In actuality, the insurer of a DC pension program must keep an eye on the DC pension management more because the risk is endured by the insured [5]. Accordingly, DC pension program management investment research becomes outstanding subject in the literature.

The investment management of multi-period DC pension program in the course of the accumulation term and the strategy of optimal investment with dynamic method is provided in [6]. The research of optimal asset distribution of a DC pension program can be discovered in [7]. Furthermore, asset distribution of DC pension program with stochastic income and mortality risk by applying multi-period mean-variance model is provided in [2]. The mean-variance model that is discovered by Harry Markowitz is the first quantitative treatment for equalizing the advantages and the risks [8].

The asset distribution and the optimal dividend strategy of a DB pension program based on stochastic mortality by applying random walk model are inspected in [9]. The research about optimal investment and contribution strategies of DB pension program is described in [10]. Moreover, the conceptual structure and the risk of DB pension program for pension insurer are provided in [11].

This article proposes to inspect the asset distribution of DC pension program with deterministic revenue and mortality risk by applying a model multi-period mean-variance. Asset distribution of DC pension program model is comparable to [2]. This article incorporate mortality risk which is conformable to [2] and turn the stochastic revenue into the deterministic revenue at the model. Different from [2] which can be applied in the stochastic revenue, this research can be applied for DC pension program with the constantly increasing revenue each period of time. To decipher this research's model, the state variable transformation, the optimal stochastic control and the Lagrange multiplier technique are used.

2. Model Formulation of Defined Contribution (DC) Pension Program

Suppose that an insured joins at a period of time 0 to a pension program and will retire at a period of time T. Before he retires, the insured has to subscribe the currency quantity each period beforehand. After he retires, the insured affluence can be exchanged into the annuity such that the insured can get a scheduled retirement every period at pension term. The rest of fund can be withdrawn from pension program by the insured's devisor if he pass away before the pension term. Let x_0 is the fund that is paid at the beginning, y_0 as the initial revenue and the two of them are more than 0 in value. Assume that the salary revenue are deterministic and fulfills the dynamics form as follows

$$y_{k+1} = qy_k, \qquad k = 0, 1, \dots, T-1, \tag{1}$$

where q and y_k are exogenous random variable that indicates the deterministic salary revenue accretion rate which in the shape of constant variable ($q > 0$) and salary revenue at time k, respectively.

Suppose that $c_k y_k$ is the quantity of insured's contribution at period k and c_k represents a variable with deterministic form. In consequence, pension program affluence after contribution at time k is $z_k = x_k + c_k y_k$. Represented by c_k and x_k are the deterministic form of affluence revenue percentage and pension program affluence before the contribution at period k, respectively.

In this model, c_k can rely on k. This means that pension program insured does not require to subscribe every period. Therefore, in this situation, there are two conditions, i.e. $c_k = 0$, when pension program insured does not require to subscribe at period k and $c_k > 0$, when pension program insured requires to subscribe at period k. In other situation, when $c_k < 0$, the insured's consumption over the period of k can be indicated by $c_k y_k$. Therefore, this model can be used at the de-cumulation term.

Suppose that $n+1$ asset of a pension program is invested at the market. The invested currency quantity in ith assets over period k is explained by u_k^i. Moreover, invested currency quantity on the 0th asset at time k is $z_k - \sum_{i=1}^{n} u_k^i = (x_k + c_k y_k) - \sum_{i=1}^{n} u_k^i$, for $k = 1, 2, \dots, T$, $i = 1, 2, \dots, n$. Thereupon, x_{k+1} can be formed as the following dynamics

$$x_{k+1} = (x_k + c_k y_k)e_k^0 + P_k^t u_k, \tag{2}$$

with $P_k^t = (e_k^1 - e_k^0, e_k^2 - e_k^0, \dots, e_k^n - e_k^0)$, $u_k = (u_k^1, u_k^2, \dots, u_k^n)^t$ and t indicates matrix or vector transpose, where u_k and e_k^i respectively explain the invested currency quantity in ith asset and the random returns in ith asset over period k.

By combining (1) with (2) generates

$$x_{k+1} + c_{k+1}y_{k+1} = (x_k + c_k y_k)e_k^0 + c_{k+1} q y_k + P^t u_k. \tag{3}$$

Since $z_{k+1} = x_{k+1} + c_{k+1}y_{k+1}$, so the dynamics form is fulfilled as follows

$$z_{k+1} = z_k e_k^0 + c_{k+1} q y_k + P^t u_k. \tag{4}$$

Even if the pension program insured is going retire at determined time T, he may pass away before his proper retirement time because of mortality risks. Accordingly, there is a need to determine the proper terminated term of the pension program. Suppose that the insured is alive for $t = 0$ and his mortality term is indicated by τ. Thereupon the proper end of term can be represented as follows

$$T^\tau = \begin{cases} k, & k-1 < \tau \le k \text{ and } 1 \le k \le T-1, \\ T, & \tau > T-1. \end{cases} \tag{5}$$

Let the pension program's survival probability is explained by $S(k)$ for $k > 0$. Hence, the insured's survival probability is established as follows

$$S(k) = \Pr(\tau \ge k \mid \tau > 0). \tag{6}$$

The survival probability can be presented based on [12] and [13], thus

$$S(k) = e^{-\int_0^t \beta(s)ds}, \tag{7}$$

with $\beta(s)$ is the intension of mortality. Therefore, the pmf of T^τ is

$$p_k := Pr(T^\tau = k) = \begin{cases} S(k-1) - S(k), & k = 1, \dots, T-1, \\ S(T-1), & k = T. \end{cases} \tag{8}$$

Based on equation (7) and equation (8), survival probability of the pension program insured can be acquired as follows

$$p_k = \begin{cases} e^{-\int_0^{k-1}\beta(s)ds} - e^{-\int_0^{k}\beta(s)ds}, & k = 1, \ldots, T-1, \\ e^{-\int_0^{T-1}\beta(s)ds} > 0, & k = T. \end{cases} \tag{9}$$

Along with this article, some assumptions are provided:

Assumption 1 (Yao [2]). In the market, the assets of financial are not redudant, thus $E[e_k e_k^t] > 0$, $k = 0, 1, \ldots, T-1$.

Assumption 2 (Yao [2]). The financial assets returns and salary revenue $\Upsilon_k = (P_k^t, q_k)^t$ do not depend on the dissimilar time of terms.

Assumption 3 (Yao [2]). The time of mortality τ does not depend on Υ_k, with $k = 0, 1, \ldots, T-1$.

Assumption 4 (Yao [2]). $E[P_k] \neq \vec{0}$, with $\vec{0}$ is the zero vector.

3. Model of Multi-Period Mean-Variance

The mean-variance model of DC pension program is applied to discover the optimal investment strategy. In this model, the terminal affluence variance is minimized and the functional constraint is explained by d as the expectation of terminal affluence. Therefore, the model of mean-variance is provided as follows

$$\begin{cases} \min\limits_{u \in \Theta_0} & \{Var[x_{T^\tau}] := [x_{T^\tau}^2] - d^2\}, \\ \text{s.t.} & E[x_{T^\tau}] = d, \quad (1) - (2). \end{cases} \tag{10}$$

The effective investment strategy of (10) indicated by $u^* = \{u_k^*; k = 0, 1, \ldots, T-1\}$. The effective point is the ordered pairs of variance axis and d axis. Hence, point $(Var[x_{T^\tau}], d)$ can be constructed at the space of mean-variance. The entire effective points construct the effective boundary.

4. Solution Scheme

This research establishes $p_0 = 0$. Corresponding to the total probability law and Assumption 3, we have

$$\begin{cases} E[x_{T^\tau}] = \sum\limits_{s=0}^{T} E\left[x_{T^\tau} \mid T^\tau = s\right] \Pr(T^\tau = s) = E\left[\sum\limits_{s=0}^{T} p_s x_s\right], \\ E[x_{T^\tau}^2] = \sum\limits_{s=0}^{T} E\left[x_{T^\tau}^2 \mid T^\tau = s\right] \Pr(T^\tau = s) = E\left[\sum\limits_{s=0}^{T} p_s x_s^2\right]. \end{cases} \tag{11}$$

Accordingly, (10) is equal in value with

$$\begin{cases} \min\limits_{u\in\Theta_0} & \left\{ E\left[\sum_{s=0}^{T} p_s x_s^2\right] - d^2 \right\}, \\ \text{s.t.} & E\left[\sum_{s=0}^{T} p_s x_s\right] = d, \qquad (1)-(2). \end{cases} \tag{12}$$

The Lagrange multiplier method can be applied to eliminate functional constraint d from (12). Set 2μ as Lagrange multiplier, so optimization form (12) becomes

$$\begin{cases} \min\limits_{u\in\Theta_0} & \left\{ E\left[\sum_{s=0}^{T} p_s x_s^2\right] - d^2 + 2\mu \left(E\left[\sum_{s=0}^{T} p_s x_s\right] - d \right) \right\}, \\ \text{s.t.} & \text{equation } (1)-(2). \end{cases} \tag{13}$$

Moreover, $(-d^2 - 2\mu d)$ and the following result

$$\begin{aligned} E\left[\sum_{s=0}^{T} p_s x_s^2\right] - d^2 + 2\mu \left(E\left[\sum_{s=0}^{T} p_s x_s\right] - d \right) \\ = E\left[\sum_{s=0}^{T} p_s x_s^2 + 2\mu p_s x_s\right] - d^2 - 2\mu d \end{aligned} \tag{14}$$

are established and (13) is equivalent with the optimization form. Consequently, both of them have identical optimal solution, i.e.

$$\min_{u\in\Theta_0} E\left[\sum_{s=0}^{T} \left(p_s x_s^2 + 2\mu p_s x_s\right)\right], \text{ s.t. equation (1)-(2).} \tag{15}$$

Furthermore, we convert the equation (15) to state variable transformation as follows

$$\begin{cases} \min\limits_{u\in\Theta_0} & \left[\sum_{s=0}^{T} \left(p_s(z_s - c_s y_s)^2 + 2\mu p_s(z_s - c_s y_s)\right)\right], \\ \text{s.t.} & \text{equation (1) and (4).} \end{cases} \tag{16}$$

The function of optimal value for (16) with z_k and y_k as initial states of $f_k(z_k, y_k)$, i.e.

$$\begin{cases} f_k(z_k, y_k) = & \min\limits_{u\in\Theta_0} \left[\sum_{s=0}^{T} (p_s(z_s - c_s y_s)^2 + 2\mu p_s(z_s - c_s y_s)) \mid (z_k, y_k)\right], \\ \text{s.t.} & \text{equation (1) and (4).} \end{cases} \tag{17}$$

By applying the principle of dynamic programming, we have the Bellman form of (16) as follows

$$\begin{cases} f_k(z_k, y_k) &= p_k z_k^2 + p_k c_k^2 y_k^2 - 2p_k z_k c_k y_k + 2p_k \mu z_k - 2p_k c_k \mu y_k \\ & \quad + \min_{u \in \Theta_0} E[f_{k+1}(z_k e_k^0 + c_{k+1} q y_k + P_k^t u_k, q y_k)], \\ f_T(z_T, y_T) &= p_T z_T^2 - 2p_T c_T z_T y_T + p_T c_T^2 y_T^2 + 2\mu p_T z_T \\ & \quad -2\mu p_T c_T y_T. \end{cases} \tag{18}$$

Let $t = 0$, then the optimal value of (13) is $f_0(z_0, y_0)$. Afterwards, we have $f_0(z_0, y_0) - d^2 - 2\mu d$ as the optimal value of (16).

The explicit statement for $f_k(z_k, y_k)$ can be acquired by constructing w_k, h_k, α_k, γ_k and g_k as the series of computational formula that complies the relations of recurrence and the boundary conditions. The series can be formed in the following result

$$w_k = p_k + w_{k+1} A_k, \qquad w_T = p_T, \tag{19a}$$

$$h_k = p_k + h_{k+1} J_k, \qquad h_T = p_T, \tag{19b}$$

$$\alpha_k = \alpha_{k+1} - \frac{h_{k+1}^2}{w_{k+1}} D_k, \qquad \alpha_T = 0, \tag{19c}$$

$$\lambda_k = \lambda_{k+1} C_k - p_k c_k + w_{k+1} c_{k+1} C_k, \qquad \lambda_T = -c_T p_T, \tag{19d}$$

$$\begin{cases} \gamma_k = & \gamma_{k+1} q^2 + p_k c_k^2 + (w_{k+1} c_{k+1} + 2\lambda_{k+1}) c_{k+1} B_k - \dfrac{\lambda_{k+1}^2}{w_{k+1}} (q^2 - B_k), \\ \gamma_T = & c_T^2 p_T, \end{cases} \tag{19e}$$

$$\begin{cases} g_k = & g_{k+1} q - p_k c_k + h_{k+1} \times \left(c_{k+1} M_k - \dfrac{\lambda_{k+1}}{w_{k+1}} (q - M_k) \right), \\ g_T = & -c_T p_T, \end{cases} \tag{19f}$$

with

$$\begin{cases} A_k = & E[(e_k^0)^2] - E[e_k^0 P_k^t] E^{-1}[P_k P_k^t] E[e_k P_k], \\ B_k = & q^2 - qE[P_k^t] E^{-1}[P_k P_k^t] q E[P_k], \\ C_k = & qE[e_k^0] - E[e_k^0 P_k^t] E^{-1}[P_k P_k^t] q E[P_k], \\ D_k = & E[P_k^t] E^{-1}[P_k P_k^t] E[P_k], \\ J_k = & E[e_k^0] - E[e_k^0 P_k^t] E^{-1}[P_k P_k^t] E[P_k], \\ M_k = & q - qE[P_k^t] E^{-1}[P_k P_k^t] E[P_k]. \end{cases} \tag{20}$$

The following Lemma is used to establish the previous series of computational formula.

Lemma 1 (Yao[2]). *Let l_T is provided for this Lemma and $\{l_k\}$ fulfills $l_k = l_{k+1}t_k + s_k$ form as the recursion formula, for $k = 0, 1, \ldots, T-1$, so that*

$$l_k = l_T \prod_{i=k}^{T-1} t_i + \sum_{i=k}^{T-1} s_i \prod_{j=k}^{i-1} t_j. \tag{21}$$

Towards the research's convenience, $\sum_{i=k}^{k-1}(\cdot) = 0$ and $\prod_{i=k}^{k-1}(\cdot) = 1$ are interpreted [2]. For $k = 0, 1, \ldots, T$, the series w_k, h_k, α_k, γ_k and g_k can be established as follows

$$\begin{cases} w_k = & \sum_{i=k}^{T} p_i \prod_{j=k}^{i-1} A_j, \\ h_k = & \sum_{i=k}^{T} p_i \prod_{j=k}^{i-1} J_j, \end{cases} \tag{22a}$$

$$\begin{cases} \lambda_k = & \sum_{i=k}^{T} w_{i+1} c_{i+1} \prod_{j=k}^{i-1} C_j - \sum_{i=k}^{T-1} c_i p_i \prod_{j=k}^{i-1} C_j, \\ \alpha_k = & -\sum_{i=k}^{T-1} \frac{h_{i+1}^2}{w_{i+1}} D_i. \end{cases} \tag{22b}$$

Let

$$\begin{cases} \eta_k = & c_k^2 p_k + (w_{k+1}c_{k+1} + 2\lambda_{k+1})c_{k+1}B_k - \frac{\lambda_{k+1}^2}{w_{k+1}}(q^2 - B_k), \\ \xi_k = & -c_k p_k + h_{k+1}\left(c_{k+1}M_k - \frac{\lambda_{k+1}}{w_{k+1}}(q - M_k)\right), \end{cases} \tag{23}$$

so (19e) and (19f) can be established as follows

$$\begin{cases} \gamma_k = & \gamma_{k+1}q^2 + \eta_k, \qquad \gamma_k = c_T^2 p_T, \\ g_k = & g_{k+1}q + \xi_k, \qquad g_T = -c_T p_T. \end{cases} \tag{24}$$

By applying Lemma 1 and based on (24), the explicit statements of γ_k and g_k are

$$\begin{cases} \gamma_k = & \gamma_T \prod_{i=k}^{T-1} q^2 + \sum_{i=k}^{T-1} \eta_i \prod_{j=k}^{i-1} q^2 = c_T^2 p_T (q^2)^T + (q^2)^{T-1} \sum_{i=k}^{T-1} \eta_i, \\ g_k = & g_T \prod_{i=k}^{T-1} q + \sum_{i=k}^{T-1} \xi_k \prod_{j=k}^{i-1} q = -c_T p_T q^T + q^{T-1} \sum_{i=k}^{T-1} \xi_k. \end{cases} \tag{25}$$

Based on Assumption 1, we have

$$A_k = E[(e_k^0)^2] - [e_k^0 P_k^t] E^{-1}[P_k P_k^t] E[e_k^0 P_k] > 0.$$

Proposition 2 (Yao[2]). *The value of w_k is greater than 0 for $k = 0, 1, \ldots, T$.*

Theorem 3 (Yao[2]). *To simplify, define $z_k = z$ and $y_k = y$. The solution of* (18) *is the function of optimal value for* (16), *i.e.*

$$f_k(z, y) = w_k z^2 + 2\lambda_k zy + \gamma_k y^2 + 2h_k \mu z + 2g_k \mu y + \alpha_k \mu^2. \tag{26}$$

Proposition 2 and Theorem 3 can be used to show that (26) fulfilled for k. Assume that (26) fulfilled for $k+1$, i.e.

$$f_{k+1}(z, y) = w_{k+1} z^2 + \gamma_{k+1} y^2 + 2\lambda_{k+1} zy + 2h_{k+1} \mu z + 2g_{k+1} \mu y + \alpha_{k+1} \mu^2.$$

For k with (18), the result can be acquired as follows

$$\begin{aligned} f_k(z, y) =& p_k z^2 + p_k c_k^2 y^2 - 2p_k c_k zy + 2p_k \mu z - 2p_k c_k \mu y + w_{k+1} z^2 E(e_k^0)^2 \\ & + w_{k+1} c_{k+1}^2 y^2 q^2 + 2w_{k+1} c_{k+1} yzqE[e_k^0] + \gamma_{k+1} y^2 q^2 + 2\lambda_{k+1} yz \\ & qE[e_k^0] + 2\lambda_{k+1} c_{k+1} y^2 q^2 + 2h_{k+1} \mu z E[e_k^0] + 2h_{k+1} \mu c_{k+1} yq \\ & + 2g_{k+1} \mu q y + \alpha_{k+1} \mu^2 + \min_{u_k} \{ w_{k+1} u_k^t \times E[P_k P_k^t] u_k + 2(w_{k+1} z \\ & E[e_k^0 P_k^t] + (w_{k+1} c_{k+1} + \lambda_{k+1}) yqE[P_k^t] \\ & + h_{k+1} \mu E[P_k^t] u_k) \}. \end{aligned} \tag{27}$$

In this article $E[P_k P_k^t]$ is a positive definite matrix, based on Assumption 1. Accordingly, the optimal strategy can be acquired with $\dfrac{\partial H}{\partial u} = 0$ as a requirement, i.e.

$$\begin{aligned} u_k^* =& - E^{-1}[P_k P_k^t] \Bigg(zE[e_k^0 P_k] + y \left(c_{k+1} + \frac{\lambda_{k+1}}{w_{k+1}} \right) \times qE[P_k] \\ & + \mu \frac{h_{k+1}}{w_{k+1}} E[P_k] \Bigg). \end{aligned} \tag{28}$$

By (20), then combined form of (27) and (28) can be turned into

$$\begin{aligned} f_k(z, y) =& (p_k + w_{k+1} A_k) z^2 + 2[-p_k c_k + (w_{k+1} c_{k+1} + \lambda_{k+1}) C_k] zy \\ & + 2(p_k + h_{k+1} J_k) \mu z + \left(\alpha_{k+1} - \frac{h_{k+1}^2}{w_{k+1}} D_k \right) \mu^2 + \Bigg[p_k c_k^2 \end{aligned}$$

$$+\gamma_{k+1}q^2+(w_{k+1}c_{k+1}^2+2\lambda_{k+1}c_{k+1})B_k-\frac{\lambda_{k+1}^2}{w_{k+1}}(q^2-B_k)\Bigg]y^2$$
$$+2\Bigg[-p_kc_k+g_{k+1}q+h_{k+1}\Bigg(c_{k+1}M_k-\frac{\lambda_{k+1}}{w_{k+1}}$$
$$(q-M_k)\Bigg)\Bigg]\mu y. \tag{29}$$

Moreover, by substituting (19a)-(19f) into (29) yields

$$f_k(z,y)=w_kz^2-2\lambda_kzy+\gamma_ky^2+2h_k\mu z+2g_k\mu y+\alpha_k\mu^2. \tag{30}$$

Equation (30) shows (26) is fulfilled for k.

5. Effective Investment Strategy and Effective Boundary

The effective investment strategy and the effective boundary of the model are provided in this section. The foregoing analysis explains the optimal value of (13), i.e.

$$H(z_0,y_0,\mu)=f_0(z_0,y_0)-d^2-2\mu d. \tag{31}$$

By using Theorem 3 and equation (31), we have

$$H(z_0,y_0)=\alpha_0\mu^2+2\mu(h_0z_0+g_0y_0-d)+w_0z_0^2+2\lambda_0z_0y_0$$
$$+\gamma_0y_0^2-d^2. \tag{32}$$

The optimal value of (10) that equal in value with (12) is acquired by maximizing (31) on μ, such that

$$Var^*[x_{T^\tau}]=\max_\mu f_0(z_0,y_0)-d^2-2\mu d=\max_\mu H(z_0,y_0,\mu). \tag{33}$$

The next proposition is provided to indicate the solution existence of (33).

Proposition 4 (Yao [2]). $\alpha_k<0$, $k=0,1,\ldots,T$.

Since $\alpha_0<0$, then optimal solution from (33) exists. Based on $\frac{\partial H}{\partial \mu}$ as a requirement, the optimal solution can be acquired as the following result

$$\mu^*=-\frac{h_0z_0+g_0y_0-d}{\alpha_0}. \tag{34}$$

Substitute equation (34) into equation (28) and let $z_k=z$, $y_k=y$, the effective investment strategy can be obtained as follows

$$u_k^*=-E^{-1}[P_kP_k^t]\Bigg[(x_k+c_ky_k)E[e_k^0P_k]+y_k\left(c_{k+1}+\frac{\lambda_{k+1}}{w_{k+1}}\right)qE[P_k]$$

$$- \frac{(h_0(x_0 + c_0y_0) + g_0y_0 - d)h_{k+1}}{\alpha_0 w_{k+1}} E[P_k]\Bigg]. \tag{35}$$

Furthermore, substitute (34) into (33). Regard that $z_0 = x_0 + c_0y_0$, then the optimal value for (10) is

$$\begin{aligned} Var^*[x_{T^\tau}] = &- \frac{1+\alpha_0}{\alpha_0}\left(d - \frac{h_0(x_0+c_0y_0)+g_0y_0}{1+\alpha_0}\right)^2 + w_0(x_0+c_0y_0)^2 \\ &+ 2\lambda_0(x_0+c_0y_0)y_0 + \gamma_0 y_0^2 - \frac{1}{1+\alpha_0}(h_0(x_0+c_0y_0) \\ &+ g_0y_0)^2. \end{aligned} \tag{36}$$

6. Numerical Simulation

Suppose that the pension program insured joins a DC pension program at 0 and arrange to retire at $T = 20$ beforehand. The fund that paid in advance is $x_0 = 12$ and his initial revenue established by $y_0 = 3$. The pension program insured is required to subscribe 20% from his salary revenue every period and the mortality intension which defined by $\beta(s)$ is always equal to 0.1. This means that $\beta(s)$ is independent on s, $s \in [0, T]$.

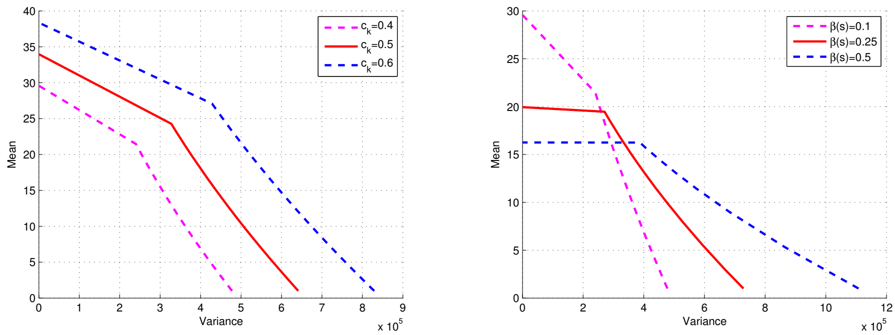
We arrange the parameters which are independent on k, with $k = 0, 1, \ldots, T$. These parameters are settled as:

$$E[P_kP_k^t] = \begin{pmatrix} 0.2365 & 0.0719 & 0.1184 \\ 0.0719 & 0.3449 & 0.1378 \\ 0.1184 & 0.1378 & 0.3262 \end{pmatrix}, \qquad \begin{aligned} E[e_k^0] &= 1.0430, \\ E[(e_k^0)^2] &= 1.2468, \\ q &= 1.0284. \end{aligned}$$

$$\begin{aligned} E[P_k] &= \begin{pmatrix} -0.0255 & 0.0015 & 0.0004 \end{pmatrix}^t, \\ E[e_k^0P_k] &= \begin{pmatrix} -0.0827 & -0.0924 & -0.0446 \end{pmatrix}^t, \end{aligned}$$

Thereupon, the effective investment strategy and effective boundary respectively can be acquired as described form

$$\begin{aligned} u_k^* = &\begin{pmatrix} 0.3174 & 0.2324 & -0.0766 \end{pmatrix}^t (x_k + c_ky_k) + y_k\left(c_k + \frac{\lambda_{k+1}}{w_{k+1}}\right) \\ &\begin{pmatrix} 0.1380 & -0.0153 & -0.0449 \end{pmatrix}^t + \frac{(h_0(12+3c_0)+3g_0-d)h_{k+1}}{\alpha_0 w_{k+1}} \\ &\begin{pmatrix} 0.1342 & -0.0149 & -0.0436 \end{pmatrix}^t, \end{aligned}$$



(a) (b)

Figure 1: (a) Simulation I with $c_k = 0.4, 0.5, 0.6$ and $\beta(s) = 0.1$; (b) Simulation II with $\beta(s) = 0.1, 0.25, 0.5$ and $c_k = 0.4$.

and

$$Var^*[x_{T^\tau}] = -\frac{1+\alpha_0}{\alpha_0}\left(d - \frac{h_0(12+3c_0)+3g_0}{1+\alpha_0}\right)^2 + w_0(12+3c_0)^2 + 6\lambda_0(12+3c_0) + 9\gamma_0 - \frac{1}{1+\alpha_0}(h_0(12+3c_0)+3g_0)^2.$$

This problem is described into two simulation. Simulation I is provided in $c_k = 0.4, 0.5, 0.6$ with $\beta(s) = 0.1$ in value and Simulation II is explained with $\beta(s) = 0.1, 0.25, 0.5$ and $c_k = 0.4$ constantly.

From the figure 1(a), we see that when $c_k = 0.4$, the values of effective point at time $k = 0$ and $k = 20$ respectively are $(4.7789 \times 10^5, 1.0000)$ and $(0.0000, 29.5755)$. When $c_k = 0.5$, the values of effective point at time $k = 0$ and $k = 20$ respectively are $(6.4153 \times 10^5, 1.0000)$ and $(0.0000, 33.9693)$. For $c_k = 0.6$, the values of effective point at time $k = 0$ and $k = 20$ are $(8.2928 \times 10^5, 1.0000)$ and $(0.0000, 38.3632)$, respectively. Therefore, the higher the value of c_k the higher the value of effective point can get, so the value of c_k and effective point are directly proportional.

Figure 1(b) illustrates when $\beta(s) = 0.1$ the values of effective point at time $k = 0$ and $k = 20$ are $(4.7789 \times 10^5, 1.0000)$ and $(0.0000, 29.5755)$, respectively. Second, when $\beta(s) = 0.25$, the values of effective point at time $k = 0$ and $k = 20$ respectively provided by $(7.2838 \times 10^5, 1.0000)$ and $(0.0000, 19.9421)$. Third, when $\beta(s) = 0.5$, the values of effective point at time $k = 0$ and $k = 20$ are $(1.1081 \times 10^6, 1.0000)$ and $(0.0000, 16.2522)$ respectively. Based on the numerical result, if the intension of mortality is smaller, then the contribution decreases as well. However, the distance between mean of contribution becomes higher. Accordingly, the value of mortality intension and contribution are directly proportional, while the value of mortality intensity and variance are inversely proportional.

7. Conclusion

The deterministic revenue and mortality risk are capable to control the asset distribution risk of defined contribution (DC) pension program investment management issue. The effective investment strategy and effective boundary can be acquired by applying dynamic program and Lagrange multiplier method. In this article, two numerical simulations and their interpretations for each simulation are provided. The management of multi-period mean-variance of defined benefit (DB) pension program with deterministic revenue and mortality risk can be examined for future research.

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