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MORSE INDEX FOR AUTONOMOUS LINEAR HAMILTONIAN SYSTEMS

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Abstract: In this paper, we will derive a new formula for the Morse index for the linear Hamiltonian systems:

$$J\dot{x} + Ax = 0,$$

where matrix J is the standard symplectic structure matrix of order $2N$ and A is a constant, symmetric positive definite matrix of order $2N$, N is any natural number.

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1. Introduction

The Morse index for a positive definite linear Hamiltonian system was introduced by Ekeland (see [2], [3]) and has proved to be a very useful tool for analyzing Hamiltonian dynamical properties related to periodicity (see also Mawhin and Willem [4] and Weistein [6]). It was inspired by Morse celebrated index in Riemannian geometry (see Milnor [5]) and derived by using the minimax principle. Ekeland derived a formula for the Morse index of Hamiltonian system in which A is a constant matrix. His formula is based on the spectral properties of A, but is not an explicit function of the eigenvalues of A, therefore it can be somewhat unwidely when applied to the characterization of periodic solutions. In this paper, we derive a new formula (Theorem 2.1) for computing the Morse index that is expressed directly in terms of the eigenvalues of the linear Hamiltonian. It appears that this formula can be extended to include nonautonomous systems, but we shall not pursue this here - we leave this to a forthcoming paper. Our formula, which is derived using Fourier representation of the periodic solutions, is more general than Ekeland's and better suited to etsimating the number of periodic solutions.

2. Morse Index for Linear Hamiltonian Systems

First, let us introduce the Morse index for linear Hamiltonian system (see Chapter 7 in [4] for details)

$$J\dot{x} + A(t)x = 0,$$

where $A(t)$ is a continuous mapping from $\mathbf{R}$ into the space of symmetric positive definite matrices of order $2N$, i.e. $A(t)$ is continuous and $(A(t)\xi, \xi) > 0$ for all $t \in \mathbf{R}$ and $\xi \neq 0$.

The structure matrix J given by:

$$\begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},$$

is associated with a symplectic form, and satisfies $J^2 = -I$ and $(Ju, v) = -(u, Jv)$ for all $u, v \in \mathbf{R}^{2N}$. Recall that a matrix M is called *symplectic* if it preserves the 2-form associated with J:

$$(J\xi, \zeta) = (JM\xi, M\zeta), \text{ for all } (\xi, \zeta),$$

i.e. $M^T JM = J$.

Consider the following Hamiltonian system:

$$J\dot{u}(t) + H'(t, u(t)) = 0, \tag{1.1}$$

where

$$H(t, u) = \frac{1}{2}(A(t)u, u).$$

Then the corresponding Hamiltonian action is given by

$$\psi(u) = \int_0^T \frac{1}{2}[(J\dot{u}(t), u(t)) + (A(t)u(t), u(t))]dt. \tag{1.2}$$

Let $B(t) = A(t)^{-1}$. Then the Legendre transform of $H(t, u)$ (with respect to u) is

$$H^*(t, u) = \frac{1}{2}(B(t)u, u),$$

and the dual action defined on H_T^1 (defined below) for $H^*(t, u)$ is given by

$$\chi_T(u) = \int_0^T \frac{1}{2}[(J\dot{u}(t), u(t)) + (B(t)\dot{u}(t), \dot{u}(t))]dt. \tag{1.3}$$

Recall that for $1 < p < \infty$, the *Sobolev space* $W_T^{1,p}$ is the space of functions $u \in L^p(0, T; \mathbf{R}^N)$ with a weak derivative $\dot{u} \in L^p(0, T; \mathbf{R}^N)$; the norm in $W_T^{1,p}$ is defined by

$$||u||_{W_T^{1,p}} = \left(\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt\right)^{1/p}. \tag{1.4}$$

We denote the Hilbert space of $T-$periodic functions in $W_T^{1,p}$ with the inner product

$$(u, v) = \int_0^T [(u(t), v(t)) + (\dot{u}(t), \dot{v}(t))]dt \tag{1.5}$$

by H_T^1.

Note that for the case of a Hamiltonian system, it may be assumed that $H_T^1 = \{u : [0,T] \to \mathbf{R}^{2N} : u$ is absolutely continuous, $u(0) = u(T)$ and $\dot{u} \in L^2(0,T)\}$.

Let

$$\widetilde{H}_T^1 = \{v \in H_T^1 : \int_0^T v(t)dt = 0\}. \tag{1.6}$$

Since $\chi_T(v + w) = \chi_T(v)$ for all constant w, we only need to consider the restriction of χ_T to the subspace $\widetilde{H}_T^1$.

Definition 1.1. The index $i(A,T)$ is defined as the *Morse index* of χ_T, i.e., the supremum of the dimensions of the subspaces of H_T^1 on which χ_T is negative definite.

Define the inner product on $\widetilde{H}_T^1$ to be the symmetric bilinear form

$$((v,w)) = \int_0^T ((B(t)\dot{v}(t), \dot{w}(t))dt. \tag{1.7}$$

Then, by Wirtinger inequality, the inner product is well defined.

Let us define the linear operator K on $\widetilde{H}_T^1$, by the above defined inner product and using the Riesz representation theorem, by

$$((Kv,w)) = \int_0^T (Jv(t), \dot{w}(t))dt\,. \tag{1.8}$$

Then K is self-adjoint and the compactness of the natural embedding of H_T^1 into $C([0,T], \mathbf{R}^{2N})$ implies that K is compact.

Furthermore, we have

$$2\chi_T(v) = ((I - K)v, v))\,. \tag{1.9}$$

As a matter of fact, (1.9) can be seen from

$$\begin{aligned} 2\chi_T(v) &= \int_0^T \big[\,(J\dot{v}(t), v(t)) + (B(t)\dot{v}(t), \dot{v}(t))\,\big]dt \\ &= \int_0^T \big[-\big(Jv(t), \dot{v}(t)) + (B(t)\dot{v}(t), \dot{v}(t))\big]dt \\ &= ((v - Kv, v))\,. \end{aligned}$$

Then, by classical spectral theory, we know that $\widetilde{H}_T^1$ can be decomposed as the orthogonal sum of $\ker(I-K)$, H^+ and H^- with $I-K$ positive definite (resp. negative definite) on H^+ (resp. H^-). Since K has at most finitely many eigenvalues (with finite multiplicity) greater than one, then

$$i(A,T) = \dim H^- < \infty,$$

i.e. the index $i(A,T)$ is finite and well defined.

Now, we define

Definition 1.2. The *nullity* $\nu(A,T)$ is the dimension of $\ker(I-K)$.

In the rest of this paper, we will assume that A is a constant symmetric and positive definite matrix. Obviously, Definition 1.1 and Definition 1.2 are still valid.

3. Morse Index Formula for Autonomous Hamiltonian Systems

Consider the linear autonomous Hamiltonian system:

$$J\dot{u}(t) + Au = 0\,, \qquad (2.1)_1$$

$$u(0) = u(T)\,, \qquad (2.1)_2$$

where the matrix A is constant, symmetric and positive definite of order $2N$.

In the rest of this section, we will derive a formula for the computation of the Morse index $i(A,T)$ and the nullity $\nu(A,T)$.

For the sake of simplicity of the proof, we assume, without loss of generality, that $T = 2\pi$. Let $\lambda_1, \lambda_2, ..., \lambda_{2N}$ be the eigenvalues of matrix A.

Theorem 2.1. *Suppose that A is in block diagonal form, i.e., there exist two matrices A_1, and A_2 of order N such that*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Then

$$i(A, 2\pi) = 2\sum_{j=1}^{N} \#\{k > 0 : k^2 < \lambda_j \lambda_{j+N}\} \tag{2.2}$$

and

$$\nu(A, 2\pi) = 2\sum_{j=1}^{N} \#\{k > 0 : k^2 = \lambda_j \lambda_{j+N}\}\,. \tag{2.3}$$

Proof. Since A is block diagonal, there exists an orthogonal block diagonal matrix O, such that $OJO^T = J$ and

$$A = O\Gamma O^T,$$

where Γ =diag$\{\lambda_1, \lambda_2, ..., \lambda_{2N}\}$. Note that if A is constant, it may be assumed to be diagonal (since it can be diagonalized via a congruence transformation).

Now let $v = O^T u$ and $B = A^{-1}$. Then

$$J\dot{v}(t) + \Gamma v(t) = 0,$$

and

$$\chi_{2\pi}(u) = \int_0^{2\pi} \frac{1}{2}\left[(J\dot{u}(t), u(t)) + (B\dot{u}(t), \dot{u}(t))\right] dt \tag{2.4}$$

is equivalent to

$$\chi_{2\pi}(v) = \int_0^{2\pi} \frac{1}{2}\left[(J\dot{v}(t), v(t)) + (\Gamma^{-1}\dot{v}(t), \dot{v}(t))\right] dt. \tag{2.5}$$

Furthermore,

$$||u||_{L^2_{2\pi}} = ||v||_{L^2_{2\pi}}, \qquad ||u||_{H^1_{2\pi}} = ||v||_{H^1_{2\pi}}. \tag{2.6}$$

Therefore, without loss of generality, we may assume that A is diagonalized, i.e., A =diag$\{\lambda_1, \lambda_2, ..., \lambda_{2N}\}$.

Let $u = \begin{pmatrix} u^I \\ u^{II} \end{pmatrix}$ with $u^I, u^{II} \in \mathbf{R}^N$ and let

$$u^{(j)}(t) = \sum_{k=-\infty}^{\infty} u_k^{(j)} e^{ikt}, \quad u_k^{(j)} \in \mathbf{R}$$

be the Fourier expansion for the j-th component $u^{(j)}(t)$ of the solution $u(t)$. Then, by Parseval equality, we have

$$\int_0^{2\pi} (J\dot{u}(t), u(t))\,dt = \int_0^{2\pi} \left(\begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \begin{pmatrix} \dot{u}^I \\ \dot{u}^{II} \end{pmatrix}, \begin{pmatrix} u^I \\ u^{II} \end{pmatrix} \right) dt$$

$$= \int_0^{2\pi} \left(\begin{pmatrix} \dot{u}^{II} \\ -\dot{u}^I \end{pmatrix}, \begin{pmatrix} u^I \\ u^{II} \end{pmatrix} \right) dt = \int_0^{2\pi} \left((\dot{u}^{II}, u^I) - (\dot{u}^I, u^{II})\right) dt$$

$$= \sum_k \left[-ik\left((u_k^{II}, u_k^I) - (u_k^I, u_k^{II})\right)\right]$$

$$= \sum_k \sum_{j=1}^N \left[-ik\left((u_k^{(j+N)}, u_k^{(j)}) - (u_k^{(j)}, u_k^{(j+N)})\right)\right].$$

Similarly, we have

$$\int_0^{2\pi} \left(A^{-1}\dot{u}(t), \dot{u}(t)\right) dt$$

$$= \sum_{j=1}^{2N} \int_0^{2\pi} \lambda_j^{-1} \left(\dot{u}^{(j)}(t), \dot{u}^{(j)}(t)\right) dt = \sum_k \sum_{j=1}^{2N} \lambda_j^{-1} k^2 (u_k^{(j)}, u_k^{(j)})$$

$$= \sum_k \sum_{j=1}^N \left[\lambda_j^{-1} k^2 (u_k^{(j)}, u_k^{(j)}) + \lambda_{j+N}^{-1} k^2 (u_k^{(j+N)}, u_k^{(j+N)})\right].$$

Hence, by (1.3) and the above two equalities, we get

$$\int_0^{2\pi} \left(A^{-1}\dot{u}(t), \dot{u}(t)\right) dt$$

$$= \sum_{j=1}^{2N} \int_0^{2\pi} \lambda_j^{-1} \left(\dot{u}^{(j)}(t), \dot{u}^{(j)}(t)\right) dt = \sum_k \sum_{j=1}^{2N} \lambda_j^{-1} k^2 (u_k^{(j)}, u_k^{(j)})$$

$$= \sum_k \sum_{j=1}^N \left[\lambda_j^{-1} k^2 (u_k^{(j)}, u_k^{(j)}) + \lambda_{j+N}^{-1} k^2 (u_k^{(j+N)}, u_k^{(j+N)})\right].$$

Recall that the index $i(A, 2\pi)$ is the Morse index of $\chi_{2\pi}$, which is the supremum of the dimension of the subspaces of $H^1_{2\pi}$ on which $\chi_{2\pi}$ is negative definite. Now, for each pair of indices (j,k) with $1 \leq j \leq N, k \in \mathbf{Z}$, consider the term:

$$\Big[- ik\Big((u_k^{(j+N)}, u_k^{(j)}) - (u_k^{(j)}, u_k^{(j+N)}) \Big) \Big]$$

$$+ \lambda_j^{-1} k^2 (u_k^{(j)}, u_k^{(j)}) + \lambda_{j+N}^{-1} k^2 (u_k^{(j+N)}, u_k^{(j+N)}).$$

For simplicity of notation, we introduce

$$p = u_k^{(j)}, \quad q = v_k^{(j+N)}, \qquad p, q \in \mathbf{C}.$$

Then the above term becomes

$$F(p,q) = \lambda_j^{-1} k^2 (p,p) + \lambda_{j+N}^{-1} k^2 (q,q) - ik\left[(q,p) - (p,q)\right]. \tag{2.7}$$

To guarantee that $\chi_{2\pi}$ is negative definite on some subspace of $H^1_{2\pi}$, we need $F(p,q) < 0$ for some $p, q \in \mathbf{C}$.

Consider the one-dimensional subspace $p = rq, r = a + bi \in \mathbf{C}$. Then

$$F(p,q) = \lambda_j^{-1} k^2 (p,p) + \lambda_{j+N}^{-1} k^2 |r|^2 (p,p) - ik\left(\bar{r}(p,p) - r(q,q)\right)$$

$$= \Big(\lambda_j^{-1} k^2 (a^2 + b^2) + \lambda_{j+N}^{-1} k^2 + 2kb \Big) |q|^2.$$

Let $a = 0, b \neq 0$. Then

$$F(biq, q) = (\lambda_j^{-1} k^2 b^2 + 2kb + \lambda_{j+N}^{-1} k^2) |q|^2. \tag{2.8}$$

In order to make $F(biq,q) < 0$ for some $b \neq 0$, the discriminant $\Delta = 4k^2 - 4\lambda_j^{-1}\lambda_{j+n}^{-1} k^4$ must be positive, which requires that

$$k^2 < \lambda_j \lambda_{j+N}, \qquad (k \neq 0). \tag{2.9}$$

Since, for each such a pair (j,k), there corresponds a 1-D subspace of $\widetilde{H}^1_{2\pi}$ with basis

$$e_j e^{ikt} = (0, ..., 0, \underbrace{e^{ikt}}_{j-th\ place}, 0, ..., 0)^T,$$

we see that

$$i(A, 2\pi) = 2\sum_{j=1}^{N} \#\{k > 0 : k^2 < \lambda_j \lambda_{j+N}\}.$$

Now, we prove the second part of Theorem 2.1.

Let

$$u^{(j)}(t) = \sum_{k=-\infty}^{\infty} u_k^{(j)} e^{ikt}, \quad u_k^{(j)} \in \mathbf{R}$$

and

$$v^{(j)}(t) = \sum_{k=-\infty}^{\infty} v_k^{(j)} e^{ikt}, \quad v_k^{(j)} \in \mathbf{R}$$

be the Fourier representations for the j-th component of the solutions u and v respectively, and let

$$w^{(j)}(t) = \sum_{k=-\infty}^{\infty} w_k^{(j)} e^{ikt}, \quad w_k^{(j)} \in \mathbf{R}$$

be the Fourier representation of the j-th component of $w = Ku$.

Recall that $\nu(A, 2\pi)$ is defined to be the dimension of $\ker(I - K)$, where K is the linear operator defined on $\widetilde{H}^1_{2\pi}$ by (1.7) and the inner product is defined by (1.8).

Then, again by Parseval equality, we have

$$((Ku, v)) = ((w, v)) = \int_0^{2\pi} \big(A^{-1}\dot{w}(t), \dot{v}(t)\big)\, dt$$

$$= \sum_{j=1}^{2N} \int_0^{2\pi} k^2 \lambda_j^{-1} \big(w(t), v(t)\big) dt = \sum_k \sum_{j=1}^{2N} k^2 \lambda_j^{-1} \left(w_k^{(j)}, v_k^{(j)}\right)$$

$$= \sum_k \sum_{j=1}^{N} k^2 \left[\lambda_j^{-1} \left(w_k^{(j)}, v_k^{(j)}\right) + \lambda_{j+N}^{-1} \left(w_k^{(j+N)}, v_k^{(j+N)}\right)\right]$$

$$= \sum_k \sum_{j=1}^{N} \frac{k^2}{\lambda_j \lambda_{j+N}} \left[\lambda_{j+N} \left(w_k^{(j)}, v_k^{(j)}\right) + \lambda_j \left(w_k^{(j+N)}, v_k^{(j+N)}\right)\right].$$

On the other hand, let $u = \begin{pmatrix} u^I \\ u^{II} \end{pmatrix}$ with $u^I, u^{II} \in \mathbf{R}^N$ and $v = \begin{pmatrix} v^I \\ v^{II} \end{pmatrix}$ with $v^I, v^{II} \in \mathbf{R}^N$. Then we have

$$\int_0^{2\pi} \big(Ju(t), \dot{v}(t)\big)dt = \int_0^{2\pi} \left(\begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \begin{pmatrix} u^I \\ u^{II} \end{pmatrix}, \begin{pmatrix} \dot{v}^I \\ \dot{v}^{II} \end{pmatrix} \right) dt$$

$$= \int_0^{2\pi} \left(\begin{pmatrix} u^{II} \\ -u^I \end{pmatrix}, \begin{pmatrix} \dot{v}^I \\ \dot{v}^{II} \end{pmatrix} \right) dt = \int_0^{2\pi} \left((u^{II}, \dot{v}^I) - (u^I, \dot{v}^{II}) \right) dt$$

$$= \sum_k \left[k\Big((u_k^{II}, v_k^I) - (u_k^I, v_k^{II}) \Big) \right]$$

$$= \sum_k \sum_{j=1}^N \left[k \left((u_k^{(j+N)}, v_k^{(j)}) - (u_k^{(j)}, v_k^{(j+N)}) \right) \right].$$

According to (1.7) and (1.8), by comparing the above two equalities, we have

$$k\lambda_j^{-1} w_k^{(j)} = u_k^{(j+N)}, \tag{2.10}$$

and

$$k\lambda_{j+N}^{-1} w_k^{(j+N)} = u_k^{(j)}, \tag{2.11}$$

for all $1 \leq j \leq N$.

Now $u \in \ker(I - K)$ if and only if $(I - K)u = 0$, or $u = Ku = w$. So, by (2.10) and (2.11), we have

$$u_k^{(j)} = w_k^{(j)} = k^{-1}\lambda_j u_k^{(j+N)} = k^{-2}\lambda_j\lambda_{j+N} u_k^{(j)}. \tag{2.12}$$

Since

$$e_j e^{ikx} = (0, ..., 0, \underbrace{e^{ikx}}_{j-th\ place}, 0, ..., 0)^T$$

is a basis of one such 1-D subspace if and only if

$$\frac{\lambda_j \lambda_{j+N}}{k^2} = 1,$$

or

$$k^2 = \lambda_j \lambda_{j+N}, \tag{2.13}$$

the dimension of $\ker(I-K)$ is

$$2\sum_{j=1}^{N}\#\{k>0 : k^2=\lambda_j\lambda_{j+N}\}. \tag{2.14}$$

This completes the proof of Theorem 2.1. $\square$

Remark 2.1. Ekeland [2] derived an indirect formula of a similar type in terms of the eigenvalues of JA, in contrast to our formula (2.2). For the special case, where $A = cI$ with c a positive constant, he did derive a direct formula. His approach started with the following concept.

Definition 2.1. A point $\tau > 0$ is called conjugate to 0 for the linear Hamiltonian system with multiplicity m if the periodic boundary value problem

$$J\dot{u}(t) + Au = 0\,,$$
$$u(0) = u(\tau)$$

has m linearly indepedent solutions.

Then by the maximum-minimum principal, he proved the following theorem.

Theorem. (Ekeland [2]) *The index $i(A,T)$ is equal to the sum of the multiplicities of the conjugate points to 0 for (2.1) contained in* $(0,T)$.

Suppose that the set of all eigenvalues of JA is $\sigma(JA)$.

Now consider the energy integral

$$(Au(t),u(t)) = c.$$

Then all the solutions of the linear Hamiltonian equation

$$J\dot{u}(t) + Au(t) = 0\,,$$

are bounded on $\mathbf{R}$. So, all the eigenvalues of JA must be pure imaginary. Hence the spectrum $\sigma(JA)$ can be written as

$$\sigma(JA) = \{i\alpha_k : \alpha_k > 0, \alpha_{N+k} = -\alpha_k, k = 1,2,...,N\}.$$

Applying the above theorem, he obtained

Proposition. (Ekeland [2]) *The Morse index of* (2.1) *with constant matrix* A *is*

$$i(A,T) = 2\sum_{j=1}^{N} \#\{k > 0 : k < \alpha_j\}.$$

Remark 2.2. From the proof of the theorem and proposition above, it can be seen that one cannot follow this line of reasoning to give an explicit formula for $i(A,T)$ in terms of the eigenvalues of A.

Similarly, applying the Ekeland Theorem for the case $A = cI$, $c > 0$, the index is given by

$$i(A,T) = 2nE[c]\,,$$

where $E[c]$ is the integer part of the real number c defined as follows:

$$E[c] = k <=> k < c \leq k+1.$$

As a matter of fact, the results in this case are equivalent to our formula in Theorem 2.1. Let us take the above proposition for $N = 1$ as an example.

Let

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

where $\alpha > 0$, $\gamma > 0$ and $\alpha\gamma - \beta^2 > 0$. Let λ_1, λ_2 be the eigenvalues of A. Then we have $\lambda_1\lambda_2 = \alpha\gamma - \beta^2$. On the other hand, $JA = \begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix}$. Let μ_1, μ_2 be the eigenvalues of JA. Then $\mu_1 = -\mu_2$, and $\alpha_1^2 := \mu_1^2 = \mu_2^2 = \alpha\gamma - \beta^2$. Therefore, we have $\alpha_1^2 = \lambda_1\lambda_2$.

From the proof of Theorem 2.1, we can directly obtain

Corollary 2.1. *Suppose that there exists a nondegenerate matrix* M *of order* $2N$ *such that*

$$M^{-1}JM = J\,,$$

and

$$M^{-1}AM$$

is block diagonalizable. Then the conclusion of Theorem 2.2 is still true.

Now consider the second order differential equation:

$$x'' + \widetilde{A}x = 0\,, \tag{2.15$_1$}$$

$$\dot{x}(0) = \dot{x}(2\pi), \quad x(0) = x(2\pi), \tag{2.15$_2$}$$

where $\widetilde{A}$ is a symmetric, positive definite matrix of order N.

The corresponding action is defined on $H^1_{2\pi}$ by

$$\chi_{2\pi}(u) = \frac{1}{2}\int_0^{2\pi} \left[|\dot{u}(t)|^2 - \left(\widetilde{A}u(t), u(t)\right)\right] dt.$$

Let $j(\widetilde{A}, 2\pi)$ be the Morse index of $\chi_{2\pi}$. Define the linear operator K on $H^1_{2\pi}$ by

$$((Ku, v)) = \int_0^{2\pi} \left(u(t) + \widetilde{A}u(t), v(t)\right) dt,$$

and the nullity $\nu_N(\widetilde{A}, 2\pi)$ by the dimension of $\ker(I - K)$. Then we have the following corollary.

Corollary 2.2. *Let $\widetilde{A}$ be constant with eigenvalues $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_N$. Then,*

$$j(\widetilde{A}, 2\pi) = 2\sum_{k=1}^{N} \#\{j \in \mathbf{N}^* : j^2 < \alpha_k\}\,, \tag{2.16}$$

and

$$\nu_N(\widetilde{A}, 2\pi) = 2\sum_{k=1}^{N} \#\{j \in \mathbf{N}^* : j^2 = \alpha_k\}, \tag{2.17}$$

where $\mathbf{N}^* = \mathbf{Z}\backslash\{0\}$.

Proof. Let $\dot{x} = y$. Then equations $(2.15)_1$ and $(2.15)_2$ can be rewritten as

$$\begin{aligned} \dot{y} + \widetilde{A}x &= 0\,, \\ \dot{x} - y &= 0\,, \end{aligned}$$

$$x(0) = x(2\pi), \qquad y(0) = y(2\pi).$$

Let $u = \begin{pmatrix} x \\ y \end{pmatrix}$. Then the above equation can be written in the form of $(2.1)_1 - (2.1)_2$, where

$$A = \begin{pmatrix} \widetilde{A} & 0_N \\ 0_N & I_N \end{pmatrix}.$$

For the case of a constant symmetric and positive definite matrix $\widetilde{A}$, the matrix A is also constant symmetric and positive definite, and the eigenvalues of A are

$$\alpha_1, \alpha_2, ..., \alpha_N, \underbrace{1, 1, ..., 1}_{N}.$$

No matter what the values of $\alpha_i, i = 1, 2, ..., N$, are, all the eigenvalues 1 are consecutive so that we can apply Theorem 2.1 to get the conclusion of Corollary 2.2. □

Remark 2.3. Corollary 2.2 is Proposition 9.1 in Mawhin and Willem [1989] for a positive definite matrix A.

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