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A METHOD OF SOLVING LAGRANGE'S FIRST-ORDER PARTIAL DIFFERENTIAL EQUATION WHOSE COEFFICIENTS ARE LINEAR FUNCTIONS

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Abstract: A method of solving Lagrange's first-order partial differential equation of the form

$$Pp + Qq = R,$$

where P, Q, R are linear functions of x, y, z, has been presented below.

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1. Introduction

In the book [1] (page 46-47), the partial differential equation of the form

$$(a_1x + b_1y + c_1z + d_1)p + (a_2x + b_2y + c_2z + d_2)q = a_3x + b_3y + c_3z + d_3 \quad (1)$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ and a_i, b_i, c_i, d_i $(i = 1, 2, 3)$ are all real numbers, has been discussed in brief. The present paper comprises a detailed discussion of the same including the cases of failure of the method adopted there.

2. The Method

For determining the solutions of (1.1), it is required to consider the simultaneous ordinary differential equations

$$\frac{dx}{a_1x + b_1y + c_1z + d_1} = \frac{dy}{a_2x + b_2y + c_2z + d_2} = \frac{dz}{a_3x + b_3y + c_3z + d_3}. \quad (2)$$

Suppose it is possible to find numbers λ, μ, ν, ρ such that each ratio of (2)

$$\frac{\lambda dx + \mu dy + \nu dz}{(a_1\lambda + a_2\mu + a_3\nu)x + (b_1\lambda + b_2\mu + b_3\nu)y + (c_1\lambda + c_2\mu + c_3\nu)z + (d_1\lambda + d_2\mu + d_3\nu)}$$

$$= \frac{\lambda dx + \mu dy + \nu dz}{\rho(\lambda x + \mu y + \nu z)}. \quad (3)$$

Clearly, (3) holds if:

$$\begin{aligned} (a_1 - \rho)\lambda + a_2\mu + a_3\nu &= 0, \\ b_1\lambda + (b_2 - \rho)\mu + b_3\nu &= 0, \\ c_1\lambda + c_2\mu + (c_3 - \rho)\nu &= 0, \\ d_1\lambda + d_2\mu + d_3\nu &= 0. \end{aligned} \quad (4)$$

Considering the first three equations of (4) comprising a system of linear homogeneous algebraic equations in the three variables λ, μ, ν, for a non-trivial solution $(\lambda, \mu, \nu) \neq (0, 0, 0)$ of (4), one notes that the rank of the coefficient matrix should not exceed two. So we should have

$$\begin{vmatrix} a_1 - \rho & a_2 & a_3 \\ b_1 & b_2 - \rho & b_3 \\ c_1 & c_2 & c_3 - \rho \end{vmatrix} = 0. \quad (5)$$

This leads to the following cubic in ρ:

$$\begin{aligned}\rho^3 - (a_1 + b_2 + c_3)\rho^2 + (b_2c_3 - b_3c_2 + a_2b_1 - a_1b_2 + a_3c_1 - a_1c_3)\rho \\ +(b_2c_3 - b_3c_2)a_1 + (b_3c_1 - b_1c_3)a_2 + (b_1c_2 - b_2c_1)a_3 = 0.\end{aligned} \tag{6}$$

Let the roots of the equation (2.5) be ρ_1, ρ_2, ρ_3.

The following cases arise:

Case I: ρ_1, ρ_2, ρ_3 are real and distinct,

Case II: One of ρ_1, ρ_2, ρ_3 is real, the other two are complex,

Case III: ρ_1, ρ_2, ρ_3 are real, but $\rho_1 = \rho_2 \neq \rho_3$,

Case IV: ρ_1, ρ_2, ρ_3 are real, and $\rho_1 = \rho_2 = \rho_3$.

In the following sections the four cases I-IV have been discussed citing an example in each case.

3. Case I: Roots of (6) are Real and Distinct

Let $(\lambda_i, \mu_i, \nu_i)$ be the solutions of the first three equations of the system (4) corresponding to $\rho = \rho_i$ $(i = 1, 2, 3)$.

Then, from (3), we get

$$\frac{\lambda_1 dx + \mu_1 dy + \nu_1 dz}{\rho_1(\lambda_1 x + \mu_1 y + \nu_1 z)} = \frac{\lambda_2 dx + \mu_2 dy + \nu_2 dz}{\rho_2(\lambda_2 x + \mu_2 y + \nu_2 z)} = \frac{\lambda_3 dx + \mu_3 dy + \nu_3 dz}{\rho_3(\lambda_3 x + \mu_3 y + \nu_3 z)}. \tag{7}$$

If each of $(\lambda_i, \mu_i, \nu_i)$ $(i = 1, 2, 3)$ satisfies the last equation of (4), the above relations lead to the required solution of the PDE (1.1) as

$$F\Big((\lambda_1 x+\mu_1 y+\nu_1 z)^{\rho_2}(\lambda_2 x+\mu_2 y+\nu_2 z)^{-\rho_1}, (\lambda_2 x+\mu_2 y+\nu_2 z)^{\rho_3}(\lambda_3 x+\mu_3 y+\nu_3 z)^{-\rho_2}\Big) = 0,$$

where F is an arbitrary real-valued function of two real variables.

Example 1.

$$(y + 2z)p + (z + \frac{1}{2}x)q = x - y. \tag{8}$$

For this PDE, the first three equations of (4) become

$$\begin{aligned}\rho\lambda - \frac{1}{2}\mu - \nu &= 0, \\ -\lambda + \rho\mu + \nu &= 0, \\ -2\lambda - \mu + \rho\nu &= 0.\end{aligned} \tag{9}$$

In this case the fourth equation is automatically satisfied and the equation (5) becomes

$$\begin{vmatrix} \rho & -\frac{1}{2} & -1 \\ -1 & \rho & 1 \\ -2 & -1 & \rho \end{vmatrix} = 0, \tag{10}$$

leading to the equation

$$\rho^3 - \frac{3}{2}\rho = 0.$$

Roots of the above equation are 0, $\pm\frac{\sqrt{6}}{2}$.

Using $\rho = 0$ in (9) we get

$$\mu + 2\nu = 0, \quad -\lambda + \nu = 0,$$

whence we have

$$\frac{\lambda}{1} = \frac{\mu}{-2} = \frac{\nu}{1}.$$

Using $\rho = \frac{\sqrt{6}}{2}$ in (9) we get

$$\sqrt{6}\lambda - \mu - 2\nu = 0, \quad -2\lambda + \sqrt{6}\mu + 2\nu = 0.$$

It then follows that

$$\frac{\lambda}{2(\sqrt{6}-1)} = \frac{\mu}{4-2\sqrt{6}} = \frac{\nu}{6-2},$$

hence we have

$$\frac{\lambda}{5} = \frac{\mu}{\sqrt{6}-4} = \frac{\nu}{2(\sqrt{6}+1)}.$$

Similarly, using $\rho = -\frac{\sqrt{6}}{2}$ in (9) we get

$$\frac{\lambda}{-5} = \frac{\mu}{\sqrt{6}+4} = \frac{\nu}{2(\sqrt{6}-1)}.$$

So the relations (7) become

$$\begin{aligned} \frac{dx - 2dy + dz}{0} &= \frac{5dx + (\sqrt{6}-4)dy + 2(\sqrt{6}+1)dz}{\frac{\sqrt{6}}{2}(5x + (\sqrt{6}-4)y + 2(\sqrt{6}+1)z)} \\ &= \frac{-5dx + (\sqrt{6}+4)dy + 2(\sqrt{6}-1)dz}{\frac{-\sqrt{6}}{2}(-5x + (\sqrt{6}+4)y + 2(\sqrt{6}-1)z)}. \end{aligned} \tag{11}$$

From the above relations the required solution of the PDE (8) can be written as

$$F\Big(x-2y+z,\quad 6(y+z)^2-(5x-4y+2z)^2\Big)=0,$$

where F is an arbitrary real-valued function of two real variables.

4. Case II: ρ_1 is real, $\rho_2=\rho_2{}'+i\rho_2{}''$ and $\rho_3=\rho_2{}'-i\rho_2{}'',(\rho_2{}',\rho_2{}''\in\mathbb{R})$

Let (λ_1,μ_1,ν_1) be the solution of the first three equations of the system (4) corresponding to the root ρ_1 and $(\lambda_2{}'+i\lambda_2{}'',\mu_2{}'+i\mu_2{}'',\nu_2{}'+i\nu_2{}'')$, $(\lambda_2{}',\ \lambda_2{}'',\ \mu_2{}',\ \mu_2{}'',\ \nu_2{}',\ \nu_2{}''\in\mathbb{R})$, be the solution of the above mentioned first three equations of the system (4) corresponding to the root $\rho_2{}'+i\rho_2{}''$. Then the solution of the system (4) corresponding to the root $\rho_2{}'-i\rho_2{}''$ will be $(\lambda_2{}'-i\lambda_2{}'',\mu_2{}'-i\mu_2{}'',\nu_2{}'-\nu_2{}'')$.
Now, from (3), we get

$$\begin{aligned}&\frac{\lambda_1dx+\mu_1dy+\nu_1dz}{\rho_1(\lambda_1x+\mu_1y+\nu_1z)}\\ =&\frac{(\lambda_2{}'+i\lambda_2{}'')dx+(\mu_2{}'+i\mu_2{}'')dy+(\nu_2{}'+i\nu_2{}'')dz}{(\rho_2{}'+i\rho_2{}'')\Big((\lambda_2{}'+i\lambda_2{}'')x+(\mu_2{}'+i\mu_2{}'')y+(\nu_2{}'+i\nu_2{}'')z\Big)}\\ =&\frac{(\lambda_2{}'-i\lambda_2{}'')dx+(\mu_2{}'-i\mu_2{}'')dy+(\nu_2{}'-i\nu_2{}'')dz}{(\rho_2{}'-i\rho_2{}'')\Big((\lambda_2{}'-i\lambda_2{}'')x+(\mu_2{}'-i\mu_2{}'')y+(\nu_2{}'-i\nu_2{}'')z\Big)}.\end{aligned}\tag{12}$$

It is assumed that each of (λ_1,μ_1,ν_1), $(\lambda_2{}'\pm i\lambda_2{}'',\mu_2{}'\pm i\mu_2{}'',\nu_2{}'\pm i\nu_2{}'')$ satisfies the last equation of (4).

From the first equation of (12) we have a solution of the simultaneous equations (2), say u_1, as

$$\begin{aligned}u_1=(\lambda_1x+\mu_1y+\nu_1z)^{\rho_2{}'+i\rho_2{}''}\Big((\lambda_2{}'+i\lambda_2{}'')x+(\mu_2{}'+i\mu_2{}'')y+(\nu_2{}'+i\nu_2{}'')z\Big)^{-\rho_1}\\ =\text{constant}=C_1,\text{say}.\end{aligned}$$

This implies that

$$\begin{aligned}\ln u_1=&(\rho_2{}'+i\rho_2{}'')\ln(\lambda_1x+\mu_1y+\nu_1z)\\ &-\rho_1\ln\Big((\lambda_2{}'+i\lambda_2{}'')x+(\mu_2{}'+i\mu_2{}'')y+(\nu_2{}'+i\nu_2{}'')z\Big)\end{aligned}$$

$$
\begin{aligned}
&= ({\rho_2}' + i{\rho_2}'') \ln(\lambda_1 x + \mu_1 y + \nu_1 z) \\
&\qquad - \rho_1 \ln\Big(({\lambda_2}' x + {\mu_2}' y + {\nu_2}' z) + i({\lambda_2}'' x + {\mu_2}'' y + {\nu_2}'' z) \Big) \\
&= \ln C_1.
\end{aligned}
$$

Writing $\ln\Big(({\lambda_2}' x + {\mu_2}' y + {\nu_2}' z) + i({\lambda_2}'' x + {\mu_2}'' y + {\nu_2}'' z) \Big) = z_1 + i z_2$
it is noted that

$$
z_1 = \frac{1}{2} \ln\Big(({\lambda_2}' x + {\mu_2}' y + {\nu_2}' z)^2 + ({\lambda_2}'' x + {\mu_2}'' y + {\nu_2}'' z)^2 \Big), \tag{13}
$$

$$
z_2 = \arctan \frac{{\lambda_2}'' x + {\mu_2}'' y + {\nu_2}'' z}{{\lambda_2}' x + {\mu_2}' y + {\nu_2}' z}, \tag{14}
$$

and we get

$$
\begin{aligned}
\ln u_1 &= ({\rho_2}' + i{\rho_2}'') \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 (z_1 + i z_2) \\
&= {\rho_2}' \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 z_1 + i\Big({\rho_2}'' \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 z_2 \Big) \\
&= \ln C_1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
u_1 &= \exp\Big({\rho_2}' \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 z_1 \Big) \\
&\quad \Bigg(\cos\Big({\rho_2}'' \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 z_2 \Big) \\
&\quad + i \sin\Big({\rho_2}'' \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 z_2 \Big) \Bigg) \\
&= \Big((\lambda_1 x + \mu_1 y + \nu_1 z)^{{\rho_2}'} \exp(-\rho_1 z_1) \Big) \\
&\quad \Bigg(\cos\Big({\rho_2}'' \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 z_2 \Big) \\
&\quad + i \sin\Big({\rho_2}'' \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 z_2 \Big) \Bigg) \\
&= C_1.
\end{aligned}
$$

So one solution of the simultaneous equations (12), say u, can be taken as

$$
u = \Big((\lambda_1 x + \mu_1 y + \nu_1 z)^{{\rho_2}'} \exp(-\rho_1 z_1) \Big) \cos\Big({\rho_2}'' \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 z_2 \Big)
$$

$$= \text{constant}. \quad (15)$$

From the last equation of (12) we have another solution of the simultaneous equations (12), say v_1, as

$$\begin{aligned} v_1 =& \Big((\lambda_2{}'x + \mu_2{}'y + \nu_2{}'z) + i(\lambda_2{}''x + \mu_2{}''y + \nu_2{}''z)\Big)^{\rho_2{}' - i\rho_2{}''} \\ & \Big((\lambda_2{}'x + \mu_2{}'y + \nu_2{}'z) - i(\lambda_2{}''x + \mu_2{}''y + \nu_2{}''z)\Big)^{-\rho_2{}' - i\rho_2{}''} = \text{constant} \\ =& C_2(\text{say}), \end{aligned}$$

or,

$$\ln v_1 = (\rho_2{}' - i\rho_2{}'')(z_1 + iz_2) - (\rho_2{}' + i\rho_2{}'')(z_1 - iz_2) = \ln C_2,$$

where z_1, z_2 are given by (13) and (14) respectively.
This shows that

$$\ln v_1 = 2i(\rho_2{}'z_2 - \rho_2{}''z_1) = \ln C_2.$$

Hence another solution of the simultaneous equations (12), say v, can be taken as

$$v = \rho_2{}'z_2 - \rho_2{}''z_1 = \text{constant}. \quad (16)$$

Therefore, in this case, the required solution of the PDE (1.1) is given by

$$F\left(\Big((\lambda_1 x + \mu_1 y + \nu_1 z)^{\rho_2{}'} \exp(-\rho_1 z_1)\Big) \cos\Big(\rho_2{}'' \ln(\lambda_1 x + \mu_1 y + \nu_1 z) - \rho_1 z_2\Big), \rho_2{}'z_2 - \rho_2{}''z_1\right) = 0, \quad (17)$$

where z_1, z_2 are given by (13) and (14) respectively, and as in Case I and Example 1, F is an arbitrary real-valued function of two real variables.

Example 2.

$$yp + zq = x. \quad (18)$$

The simultaneous ordinary differential equations corresponding to this PDE are

$$\frac{dx}{y} = \frac{dy}{z} = \frac{dz}{x}. \quad (19)$$

It is noted that the fourth equation of (4) is automatically satisfied.

In this case, the equation (5) becomes

$$\begin{vmatrix} -\rho & 0 & 1 \\ 1 & -\rho & 0 \\ 0 & 1 & -\rho \end{vmatrix} = 0 \tag{20}$$

or,

$$\rho^3 - 1 = 0.$$

Roots of this equation are 1, ω, ω^2; where $\omega = \frac{-1\pm i\sqrt{3}}{2}$.

$$\text{For } \rho = 1, \quad \frac{\lambda}{1} = \frac{\mu}{1} = \frac{\nu}{1}.$$

$$\text{For } \rho = \omega, \quad \frac{\lambda}{\omega} = \frac{\mu}{1} = \frac{\nu}{\omega^2}.$$

$$\text{For } \rho = \omega^2, \quad \frac{\lambda}{\omega^2} = \frac{\mu}{1} = \frac{\nu}{\omega}.$$

So each ratio of (19) is equal to

$$\begin{aligned} \frac{dx+dy+dz}{y+z+x} &= \frac{\omega dx + dy + \omega^2 dz}{\omega y + z + \omega^2 x} = \frac{\omega dx + dy + \omega^2 dz}{\omega(\omega x + y + \omega^2 z)} \\ &= \frac{\omega^2 dx + dy + \omega dz}{\omega^2 y + z + \omega x} = \frac{\omega^2 dx + dy + \omega dz}{\omega^2(\omega^2 x + y + \omega z)}. \end{aligned}$$

The required solutions of the PDE (18) are then obtained as

$$F\Big((x+y+z)^{\omega}(\omega x + y + \omega^2 z)^{-1}, \ \ (x+y+z)^{\omega^2}(\omega^2 x + y + \omega z)^{-1}\Big) = 0,$$

where F is an arbitrary real-valued function of two real variables.

5. Case III: $\rho_1 = \rho_2 \neq \rho_3$

In this case, the method described above for the cases Case I and II does not work. This is exhibited by citing the following example.

Example 3.

$$(y-z)p + (z+x)q = x + \frac{3}{4}y. \tag{21}$$

The simultaneous ordinary differential equations corresponding to this PDE are

$$\frac{dx}{(y-z)} = \frac{dy}{(z+x)} = \frac{dz}{x+\frac{3}{4}y}. \tag{22}$$

For this PDE the equation (5) becomes

$$\begin{vmatrix} -\rho & 1 & 1 \\ 1 & -\rho & \frac{3}{4} \\ -1 & 1 & -\rho \end{vmatrix} = 0, \tag{23}$$

which leads to

$$(\rho-1)(2\rho+1)^2 = 0.$$

Roots of this equation are 1, $-\frac{1}{2}$, $-\frac{1}{2}$.

$$\text{For } \rho = 1, \quad \frac{\lambda}{1} = \frac{\mu}{1} = \frac{\nu}{0}.$$

$$\text{For } \rho = -\frac{1}{2}, \quad \frac{\lambda}{2} = \frac{\mu}{5} = \frac{\nu}{-6}.$$

So each ratio of (22) is equal to

$$\frac{dx+dy}{y+x} = \frac{2dx+5dy-6dz}{-\frac{1}{2}(2x+5y-6z)}.$$

This relation gives us the single solution of the equation (2), in the present case, as

$$u = (x+y)(2x+5y-6z)^2 = \text{constant}.$$

Being unable to find another solution of the simultaneous equations (2), in the present case, the method described above fails to derive solutions of the PDE (21).

6. Case IV: $\rho_1 = \rho_2 = \rho_3$

In this case, no solution of the simultaneous equations (2) can be derived and hence the method presented fails to find any solution of the PDE under consideration. The following example exhibits the difficulty.

Example 4.

$$(x-2y+z)p + (2x+y+z)q = 2x+2y+z. \tag{24}$$

The simultaneous ordinary differential equations corresponding to this PDE are

$$\frac{dx}{(x-2y+z)} = \frac{dy}{(2x+y+z)} = \frac{dz}{2x+2y+z}. \tag{25}$$

For this PDE the equation (5) becomes

$$\begin{vmatrix} 1-\rho & 2 & 2 \\ -2 & 1-\rho & 2 \\ 1 & 1 & 1-\rho \end{vmatrix} = 0, \tag{26}$$

which leads to the equation

$$(1-\rho)^3 = 0.$$

Roots of this equation are 1, 1, 1.

$$\text{For } \rho = 1, \quad \frac{\lambda}{1} = \frac{\mu}{-1} = \frac{\nu}{1}.$$

So, in this case, the equation (3) gives us the only ratio

$$\frac{dx - dy + dz}{x - y + z}. \tag{27}$$

Hence the method described above fails to give us the required solutions of the PDE (24).

7. Alternative Approach to Solve the Problems in Case III and Case IV

To find the solutions of the PDEs (21) and (24) the following approch has been found to be suitable in case of the examples cited.

7.1. Solution of the PDE (21)

The simultaneous ordinary differential equations corresponding to the PDE (21) are

$$\frac{dx}{(y-z)} = \frac{dy}{(z+x)} = \frac{dz}{x+\frac{3}{4}y}. \tag{28}$$

Suppose it is possible to find numbers α_i, β_i, γ_i $(i = 1, 2, 3)$ and ρ such that each ratio of (28) is equal to

$$\frac{(\alpha_1 x + \beta_1 y + \gamma_1 z)dx + (\alpha_2 x + \beta_2 y + \gamma_2 z)dy + (\alpha_3 x + \beta_3 y + \gamma_3 z)dz}{(\alpha_1 x + \beta_1 y + \gamma_1 z)(y-z) + (\alpha_2 x + \beta_2 y + \gamma_2 z)(z+x) + (\alpha_3 x + \beta_3 y + \gamma_3 z)(x+\frac{3}{4}y)}$$

$$= \frac{(\alpha_1 x + \beta_1 y + \gamma_1 z)dx + (\alpha_2 x + \beta_2 y + \gamma_2 z)dy + (\alpha_3 x + \beta_3 y + \gamma_3 z)dz}{(\alpha_2 + \alpha_3)x^2 + (\beta_1 + \frac{3}{4}\beta_3)y^2 + (-\gamma_1 + \gamma_2)z^2 + (\alpha_1 + \beta_2 + \frac{3}{4}\alpha_3 + \beta_3)xy}$$
$$+ (-\beta_1 + \gamma_1 + \beta_2 + \frac{3}{4}\gamma_3)yz + (-\alpha_1 + \alpha_2 + \gamma_2 + \gamma_3)zx$$
$$= \frac{dD}{\rho d}, \tag{29}$$

where

$$D = (\alpha_2 + \alpha_3)x^2 + (\beta_1 + \frac{3}{4}\beta_3)y^2 + (-\gamma_1 + \gamma_2)z^2$$
$$+ (\alpha_1 + \beta_2 + \frac{3}{4}\alpha_3 + \beta_3)xy + (-\beta_1 + \gamma_1 + \beta_2 + \frac{3}{4}\gamma_3)yz + (-\alpha_1 + \alpha_2 + \gamma_2 + \gamma_3)zx,$$

and dD denotes the total derivative of D.

We see that (29) holds if

$$\begin{aligned}
\rho\alpha_1 &= 2\alpha_2 + 2\alpha_3,\\
\rho\beta_1 &= \alpha_1 + \beta_2 + \frac{3}{4}\alpha_3 + \beta_3,\\
\rho\gamma_1 &= -\alpha_1 + \alpha_2 + \gamma_2 + \gamma_3,\\
\rho\alpha_2 &= \alpha_1 + \beta_2 + \frac{3}{4}\alpha_3 + \beta_3,\\
\rho\beta_2 &= 2\beta_1 + \frac{3}{2}\beta_3,\\
\rho\gamma_2 &= -\beta_1 + \gamma_1 + \beta_2 + \frac{3}{4}\gamma_3,\\
\rho\alpha_3 &= -\alpha_1 + \alpha_2 + \gamma_2 + \gamma_3,\\
\rho\beta_3 &= -\beta_1 + \gamma_1 + \beta_2 + \frac{3}{4}\gamma_3,\\
\rho\gamma_3 &= -2\gamma_1 + 2\gamma_2.
\end{aligned}$$

The above equations give us a linear homogeneous system of equations

$$\begin{aligned}
-\rho\alpha_1 + 2\beta_1 + 2\gamma_1 &= 0,\\
\alpha_1 - \rho\beta_1 + \beta_2 + \frac{3}{4}\gamma_1 + \gamma_2 &= 0,\\
2\beta_1 - \rho\beta_2 + \frac{3}{2}\gamma_2 &= 0,\\
-\alpha_1 + \beta_1 - \rho\gamma_1 + \gamma_2 + \gamma_3 &= 0,\\
-\beta_1 + \beta_2 + \gamma_1 - \rho\gamma_2 + \frac{3}{4}\gamma_3 &= 0,\\
-2\gamma_1 + 2\gamma_2 - \rho\gamma_3 &= 0,
\end{aligned} \tag{30}$$

where $\alpha_2 = \beta_1$, $\alpha_3 = \gamma_1$, $\beta_3 = \gamma_2$. The linear homogeneous system (30) will have a non-trivial solution for $(\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3)$ if the determinant of the coefficient matrix of the system (30) is zero, i.e.

$$\begin{vmatrix} -\rho & 2 & 0 & 2 & 0 & 0 \\ 1 & -\rho & 1 & \frac{3}{4} & 1 & 0 \\ 0 & 2 & -\rho & 0 & \frac{3}{2} & 0 \\ -1 & 1 & 0 & -\rho & 1 & 1 \\ 0 & -1 & 1 & 1 & -\rho & \frac{3}{4} \\ 0 & 0 & 0 & -2 & -2 & -\rho \end{vmatrix} = 0.$$

This leads to the equation

$$\rho^6 - \frac{15}{4}\rho^4 - \frac{7}{4}\rho^3 + \frac{9}{4}\rho^2 + \frac{3}{4}\rho - \frac{1}{2} = 0.$$

Roots of this equation are found to be 2, -1, -1, -1, $\frac{1}{2}$, $\frac{1}{2}$.

Using these values of ρ in (30) solutions for $(\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3)$ are found to be as follows:

For $\rho = 2$, $\alpha_1 = \beta_1 = \beta_2 = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = 0$.

For $\rho = -1$, $\alpha_1 = -4$, $\beta_1 = -10$, $\beta_2 = -25$,$\gamma_1 = 12$, $\gamma_2 = 30$, $\gamma_3 = -36$.

For $\rho = \frac{1}{2}$, $\alpha_1 = 4$, $\beta_1 = 7$, $\beta_2 = 10$, $\gamma_1 = -6$, $\gamma_2 = -6$, $\gamma_3 = 0$.

Again we have the relations $\alpha_2 = \beta_1$, $\alpha_3 = \gamma_1$, $\beta_3 = \gamma_2$.

So using the above values of ρ, α_i, β_i, γ_i $(i = 1, 2, 3)$ in (29) we have the ratios:

$$\frac{d(x^2+y^2+2xy)}{2(x^2+y^2+2xy)} = \frac{d(2x^2+\frac{25}{2}y^2+18z^2+10xy-30yz-12zx)}{-1(2x^2+\frac{25}{2}y^2+18z^2+10xy-30yz-12zx)}$$
$$= \frac{d(2x^2+5y^2+7xy-6yz-6zx)}{2(2x^2+5y^2+7xy-6yz-6zx)}.$$

These ratios give us the required solution of the PDE (21) as

$$F\Big((x+y)(2x^2+\frac{25}{2}y^2+18z^2+10xy-30yz-12zx),$$
$$(2x^2+5y^2+7xy-6yz-6zx)(x+y)^2\Big) = 0,$$

where F is an arbitrary real-valued function of two real variables.

7.2. Solution of the PDE (24)

The simultaneous ordinary differential equations corresponding to the PDE (24) are

$$\frac{dx}{x-2y+z}=\frac{dy}{2x+y+z}=\frac{dz}{2x+2y+z}. \tag{31}$$

Suppose it is possible to find numbers α_i, β_i, γ_i $(i=1,2,3)$ and ρ such that each ratio of (31) is equal to

$$\begin{aligned}
&\frac{(\alpha_1x+\beta_1y+\gamma_1z)dx+(\alpha_2x+\beta_2y+\gamma_2z)dy+(\alpha_3x+\beta_3y+\gamma_3z)dz}{(\alpha_1x+\beta_1y+\gamma_1z)(x-2y+z)+(\alpha_2x+\beta_2y+\gamma_2z)(2x+y+z)+(\alpha_3x+\beta_3y+\gamma_3z)(2x+2y+z)}\\
&=\frac{(\alpha_1x+\beta_1y+\gamma_1z)dx+(\alpha_2x+\beta_2y+\gamma_2z)dy+(\alpha_3x+\beta_3y+\gamma_3z)dz}{\begin{array}{l}(\alpha_1+2\alpha_2+2\alpha_3)x^2+(-2\beta_1+\beta_2+2\beta_3)y^2+(\gamma_1+\gamma_2+\gamma_3)z^2+(-2\alpha_1+\beta_1+\alpha_2+2\beta_2\\ \quad +2\alpha_3+2\beta_3)xy+(\beta_1-2\gamma_1+\beta_2+\gamma_2+\beta_3+2\gamma_3)yz+(\alpha_1+\gamma_1+\alpha_2+2\gamma_2+\alpha_3+2\gamma_3)zx\end{array}}\\
&=\frac{dD}{\rho d},
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
D&=(\alpha_1+2\alpha_2+2\alpha_3)x^2+(-2\beta_1+\beta_2+2\beta_3)y^2\\
&\quad+(\gamma_1+\gamma_2+\gamma_3)z^2+(-2\alpha_1+\beta_1+\alpha_2+2\beta_2+2\alpha_3+2\beta_3)xy\\
&\quad+(\beta_1-2\gamma_1+\beta_2+\gamma_2+\beta_3+2\gamma_3)yz+(\alpha_1+\gamma_1+\alpha_2+2\gamma_2+\alpha_3+2\gamma_3)zx,
\end{aligned}$$

and dD denotes the total derivative of D. We see that (32) holds if

$$\begin{aligned}
\rho\alpha_1&=2\alpha_1+4\alpha_2+4\alpha_3,\\
\rho\beta_1&=-2\alpha_1+\beta_1+\alpha_2+2\beta_2+2\alpha_3+2\beta_3,\\
\rho\gamma_1&=\alpha_1+\gamma_1+\alpha_2+2\gamma_2+\alpha_3+2\gamma_3,\\
\rho\alpha_2&=-2\alpha_1+\beta_1+\alpha_2+2\beta_2+2\alpha_3+2\beta_3,\\
\rho\beta_2&=-4\beta_1+2\beta_2+4\beta_3,\\
\rho\gamma_2&=\beta_1-2\gamma_1+\beta_2+\gamma_2+\beta_3+2\gamma_3,\\
\rho\alpha_3&=\alpha_1+\gamma_1+\alpha_2+2\gamma_2+\alpha_3+2\gamma_3,\\
\rho\beta_3&=\beta_1-2\gamma_1+\beta_2+\gamma_2+\beta_3+2\gamma_3,\\
\rho\gamma_3&=2\gamma_1+2\gamma_2+2\gamma_3.
\end{aligned}$$

The above equations give us a linear homogeneous system of equations

$$\begin{aligned}
(2-\rho)\alpha_1+4\beta_1+4\gamma_1&=0,\\
-2\alpha_1+(2-\rho)\beta_1+2\beta_2+2\gamma_1+2\gamma_2&=0,\\
-4\beta_1+(2-\rho)\beta_2+4\gamma_2&=0,\\
\alpha_1+\beta_1+(2-\rho)\gamma_1+2\gamma_2+2\gamma_3&=0,\\
\beta_1+\beta_2-2\gamma_1+(2-\rho)\gamma_2+2\gamma_3&=0,\\
2\gamma_1+2\gamma_2+(2-\rho)\gamma_3&=0,
\end{aligned} \tag{33}$$

where $\alpha_2 = \beta_1$, $\alpha_3 = \gamma_1$, $\beta_3 = \gamma_2$.

The linear homogeneous system (33) will have a non-trivial solution for $(\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3)$ if the determinant of the coefficient matrix of the system (33) is zero, i.e.

$$\begin{vmatrix} 2-\rho & 4 & 0 & 4 & 0 & 0 \\ -2 & 2-\rho & 2 & 2 & 2 & 0 \\ 0 & -4 & 2-\rho & 0 & 4 & 0 \\ 1 & 1 & 0 & 2-\rho & 2 & 2 \\ 0 & 1 & 1 & -2 & 2-\rho & 2 \\ 0 & 0 & 0 & 2 & 2 & 2-\rho \end{vmatrix} = 0.$$

This leads to the equation

$$(2-\rho)^6 = 0.$$

So all the roots of this equation are equal to 2.

For $\rho = 2$ the system (33) has two linearly independent solutions, namely $(3, -1, 3, 1, -1, 0)$, $(2, 0, 2, 0, 0, -1)$ for $(\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3)$.

Using the values of ρ, α_i, β_i, γ_i $(i = 1, 2, 3)$ in (32) we see that each of the ratios in (32) is equal to

$$\frac{d(3x^2 + 3y^2 - 2xy - 2yz + 2zx)}{2(3x^2 + 3y^2 - 2xy - 2yz + 2zx)} = \frac{d(2x^2 + 2y^2 - z^2)}{2(2x^2 + 2y^2 - z^2)}.$$

Again by (6.2) we have found that each ratios in (6.2) is equal to

$$\frac{d(x - y + z)}{(x - y + z)}.$$

From the above three equal ratios we get the solution of the PDE (24) as

$$F\Big(\frac{2x^2 + 2y^2 - z^2}{(x - y + z)^2},\ \ \frac{3x^2 + 3y^2 - 2xy - 2yz + 2zx}{(x - y + z)^2}\Big) = 0,$$

where F is an arbitrary real-valued function of two real variables.

8. Remarks

The genesis of the method described in §§ 3, 4, 7, lies in finding an equivalent ratio of the ratios in (2.1), in which the numerator is the total differential of the denominetor. The idea may be employed to other types of equations. Further works in this direction are under progress, to be reported in future.

References

[1] E.L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1956.