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INFINITELY MANY POSITIVE SOLUTIONS OF A NONLINEAR THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract: In this paper, existence criteria for infinitely many positive solutions of the nonlinear three-point boundary value problem

$$\begin{cases} u''(t)+a(t)u'(t)+b(t)u(t)+h(t)f(u)=0, \quad t\in(0,1)\,, \\ u(0)=0, \quad u(1)=\alpha u(\eta) \end{cases}$$

are established by using the Krasnosel'skii's Fixed Point Theorem for operators on cone.

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1. Introduction

Recently, the study of the existence of positive solutions for multi-point boundary value problems has evolved rapidly [3], 2–8. In particular, Ma and Wang [9], and Sun et al [10] considered the existence of one or three positive solutions for more general three-point boundary value problem

$$u''(t)+a(t)u'(t)+b(t)u(t)+h(t)f(u)=0, \quad t\in(0,1), \tag{1.1}$$

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$$u(0) = 0, \quad u(1) = \alpha u(\eta), \quad 0 < \eta < 1 \tag{1.2}$$

under the suitable conditions, where:

(H1) $f \in C([0,\infty),[0,\infty))$.

(H2) $h \in C([0,1],[0,\infty))$ and there exists $x_0 \in [0,1]$ such that $h(x_0) > 0$.

(H3) $a \in C[0,1]$, $b \in C([0,1],(-\infty,0))$.

(H4) $0 < \alpha\phi_1(\eta) < 1$, here ϕ_1 is the unique solution of the linear boundary value problem

$$\begin{cases} \phi_1''(t) + a(t)\phi_1'(t) + b(t)\phi_1(t) = 0, t \in (0,1), \\ \phi_1(0) = 0, \phi_1(1) = 1. \end{cases} \tag{1.3}$$

Their main tool is Krasnosel'skii's Fixed Point Theorem (see [2, 1]) or Leggett-Williams' Fixed Point Theorem (see [6]).

In this paper we will continue to consider the existence of infinitely many positive solutions for the above boundary value problem and our main tool is the following well-known fixed point theorem, which is due to Krasnosel'skii and Guo.

Theorem 1.1. *Let E be a Banach space, and P be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$$

be a completely continuous operator such that either:

(i) $\|Au\| \le \|u\|, \forall u \in P \cap \partial\Omega_1$ *and* $\|Au\| \ge \|u\|, \forall u \in P \cap \partial\Omega_2,$

or

(ii) $\|Au\| \ge \|u\|, \forall u \in P \cap \partial\Omega_1$ *and* $\|Au\| \le \|u\|, \forall u \in P \cap \partial\Omega_2.$

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

2. Main Results

To state the main result of this paper, we need the following lemma, which was established by Ma and Wang, see [9].

Lemma 2.1. *Assume that (H3) holds. Let ϕ_1 and ϕ_2 be the solutions of equation (1.3) and*

$$\begin{cases} \phi_2''(t) + a(t)\phi_2'(t) + b(t)\phi_2(t) = 0, \quad t \in (0,1), \\ \phi_2(0) = 1, \quad \phi_2(1) = 0. \end{cases} \tag{2.1}$$

Then:

(i) ϕ_1 is strictly increasing on $[0,1]$;

(ii) ϕ_2 is strictly decreasing on $[0,1]$.

In view of (H2), $h \in C([0,1],[0,\infty))$ and there exists $x_0 \in [0,1]$ such that $h(x_0) > 0$, and hence we may assume that $x_0 \in (0,1)$. Take $\delta \in (0, \frac{1}{2})$ such that $x_0 \in (\delta, 1-\delta)$.

For convenience, we let

$$G(t,s) = \frac{1}{\phi_1'(0)} \begin{cases} \phi_1(t)\phi_2(s), \ s \geq t, \\ \phi_1(s)\phi_2(t), \ s \leq t, \end{cases}$$

$$D =$$

$$\left[\max_{t\in[0,1]} \int_0^1 G(t,s)p(s)h(s)ds + \frac{\alpha}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)ds\right]^{-1},$$

and

$$C = [1-\alpha\phi_1(\eta)] \left[\int_\delta^{1-\delta} G(\eta,s)p(s)h(s)ds\right]^{-1},$$

where $p(t) = \exp\left(\int_0^t a(s)ds\right)$.

For the function $G(t,s)$, it follows from Lemma 2.1 that

$$0 \leq G(t,s) \leq G(s,s), \quad (t,s) \in [0,1] \times [0,1] \tag{2.2}$$

and

$$G(t,s) \geq \gamma G(s,s), \quad (t,s) \in [\delta, 1-\delta] \times [0,1], \tag{2.3}$$

where $\gamma = \min\{\phi_1(\delta), \phi_2(1-\delta)\}$.

Our main result is the following theorem.

Theorem 2.1. *Assume that (H1)-(H4) hold and that there exist two positive sequences* $\{a_k\}_{k=1}^{\infty}$ *and* $\{b_k\}_{k=1}^{\infty}$ *such that*

$$a_{k+1} < b_k < a_k, \qquad k = 1, 2, \cdots .$$

Then the boundary value problem (1.1) and (1.2) has infinitely many positive solutions if the following conditions hold:

$$f(u) \leq D a_k, \ u \in [0, a_k], k = 1, 2, \cdots , \tag{2.4}$$

and

$$f(u) \geq C b_k, \ u \in [\gamma b_k, b_k], k = 1, 2, \cdots . \tag{2.5}$$

Proof. Let E be the set $C[0,1]$ of all real continuous functions defined on $[0,1]$ endowed with the usual linear structure and the maximum norm. Set

$$P = \left\{ u \in E : u(t) \geq 0, t \in [0,1], \min_{t \in [\delta, 1-\delta]} u(t) \geq \gamma \|u\| \right\}.$$

Then it is easily seen that P is a cone in E. For $u \in P$, define

$$(Au)(t) = \int_0^1 G(t,s)p(s)h(s)f(u(s))ds$$
$$+ \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)f(u(s))ds, \quad t \in [0,1].$$

It is easy to check that [9] $A : P \to P$ is completely continuous and fixed points of A are solutions of the boundary value problem (1.1) and (1.2).

Let

$$\Omega_{a_k} = \{u \in E : \|u\| < a_k\} \text{ and } \Omega_{b_k} = \{u \in E : \|u\| < b_k\}.$$

Fix k and let $u \in P \cap \partial\Omega_{a_k}$, then for $s \in [0,1]$, we have

$$0 \leq u(s) \leq \|u\| = a_k. \tag{2.6}$$

It follows from (2.6) and (2.4) that

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s)p(s)h(s)f(u(s))ds \\
&\quad + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)f(u(s))ds \\
&\leq Da_k \left[\int_0^1 G(t,s)p(s)h(s)ds \right. \\
&\quad \left. + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)ds\right] \\
&\leq Da_k \max_{t\in[0,1]} \left[\int_0^1 G(t,s)p(s)h(s)ds \right. \\
&\quad \left. + \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)ds\right] \\
&\leq Da_k \left[\max_{t\in[0,1]} \int_0^1 G(t,s)p(s)h(s)ds \right. \\
&\quad \left. + \frac{\alpha}{1-\alpha\phi_1(\eta)} \int_0^1 G(\eta,s)p(s)h(s)ds\right] \\
&= a_k = \|u\|, \quad t \in [0,1].
\end{aligned}$$

So,

$$\|Au\| \leq \|u\|, \quad u \in P \cap \partial\Omega_{a_k}. \tag{2.7}$$

Now let $u \in P \cap \partial\Omega_{b_k}$, then for $s \in [\delta, 1-\delta]$, we have

$$b_k = \|u\| \geq u(s) \geq \min_{s\in[\delta,1-\delta]} u(s) \geq \gamma\|u\| = \gamma b_k. \tag{2.8}$$

It follows from (2.8) and (2.5) that

$$\begin{aligned}(Au)(\eta) &= \frac{1}{1-\alpha\phi_1(\eta)}\int_0^1 G(\eta,s)p(s)h(s)f(u(s))ds \\ &\geq \frac{1}{1-\alpha\phi_1(\eta)}\int_\delta^{1-\delta} G(\eta,s)p(s)h(s)f(u(s))ds \\ &\geq \frac{Cb_k}{1-\alpha\phi_1(\eta)}\int_\delta^{1-\delta} G(\eta,s)p(s)h(s)ds \\ &= b_k = \|u\|.\end{aligned}$$

And so,

$$\|Au\| \geq \|u\|, \quad u \in P \cap \partial\Omega_{b_k}. \tag{2.9}$$

By (2.7), (2.9) and Theorem 1.1 the operator A has a fixed point $u_k \in P \cap (\overline{\Omega}_{a_k} \backslash \Omega_{b_k})$. Since k is arbitrary, the proof is complete. □

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