International Journal of Differential Equations and Applications

Volume 9 No. 2 2004, 127-142

THE L^p-APPROXIMATION OF GENERALIZED BI-AXIALLY SYMMETRIC POTENTIALS

H.S. Kasana[1], D. Kumar[2 §]

[1]Department of Mathematics
Thapar Institute of Engineering and Technology
Patiala, 147 001, INDIA

[2]Department of Mathematics
Addis Ababa University
Addis Ababa, ETHOPIA
e-mail: d_kumar001@rediffmail.com

Abstract: In this paper we deal with growth and approximation of solutions (not necessarily entire) of certain elliptic partial differential equations. These solutions are called generalized bi-axially symmetric potentials (GBSP). We obtain the characterization of q-type and lower q-type of a GBSP having fast rates of growth in terms of ratio of approximation errors in L^p norm.

AMS Subject Classification: 40D10, 30E05

Key Words: generalized bi-axially symmetric potentials, elliptic partial differential equations, index q, GBSP polynomials, approximation error in L^p norm.

1. Introduction

Generalized bi-axially symmetric potentials (GBSP's) are the solutions of elliptic partial differential equation

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+\frac{2\alpha+1}{y}\frac{\partial H}{\partial y}+\frac{2\beta+1}{x}\frac{\partial H}{\partial x}=0, \quad \alpha,\ \beta>-\frac{1}{2}, \tag{1.1}$$

which are even in x and y, cf. Gilbert [1]. A polynomial of degree n

Received: November 19, 2004

§Correspondence author

which is even in x and y is said to be a GBSP polynomial of degree n if it satisfies (1.1). A GBSP H regular about origin can be expanded as

$$H \equiv H(r,\theta) = \sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\alpha,\beta)}(\cos 2\theta), \tag{1.2}$$

where $x = r\cos\theta$, $y = r\sin\theta$ and $P_n^{(\alpha,\beta)}(t)$ are Jacobi polynomials.

Let $D_R = \{(x,y) : x^2+y^2 < R,\ 0 < R < \infty\}$ and $\overline{D}_R$ be the closure of D_R. *A GBSP H is said to be regular in D_R if the series (1.2) converges uniformly on every compact subset of D_R.* Let H_R be the class of all GBSP's regular in $D_{R'}$ for every $R' \le R$ but for no $R' > R$. The functions in the class H_∞ are called entire GBSP's.

McCoy [6] considered the approximation of an entire GBSP H by GBSP polynomials and found the rate of decay of approximation error:

$$E_{n,p}(H,1) = \inf_{g\in\pi_n} \|H-g\|_{1,p}$$

$$= \inf_{g\in\pi_n} \left(\iint\limits_{\overline{D}_1} \mu(x,y)|H(x,y)-g(x,y)|^p dx\ dy \right)^{1/p},$$

in terms of growth parameters associated with the maximum modulus $M(r,H) = \max_\theta |H(r,\theta)|$, where μ is a weight function and $1 \le p < \infty$.

Also, McCoy [7] considered the approximation of pseudo analytic functions, constructed as complex combination of real valued analytic functions to the Stokes-Beltrami system on the disc. These functions include the GBSP's. He obtained some coefficients and Bernstein type growth theorems on the disc in sup norm.

A GBSP H is said to be regular in $\overline{D}_{R_o}$, the closure of D_{R_o} if it is regular in $D_{R'}$ for some $R' > R_o$. Let $\overline{H}_{R_o}$ be the class of all GBSP's regular on $\overline{D}_{R_o}$. For $H \in \overline{H}_{R_o}$, set

$$\|H\|_{R_o} = \max_{(x,y)\in D_{R_o}} |H(x,y)|,$$

and for $1 \le p < \infty$,

$$\|H\|^1_{R_o,p} = \left(\int_0^{2\pi} w(R_o,\theta)|H(R_o,\theta)|^p\ d\theta \right)^{1/p}, \tag{1.3}$$

$$\|H\|^2_{R_o,p} = \left(\int_{\overline{D}_{R_o}} \overline{w}(x,y)|H(x,y)|^p \, dx \, dy \right)^{1/p}, \tag{1.4}$$

where the functions w and $\overline{w}$ are positive and integrable (in the sense of Lebesgue) such that $\frac{1}{w}$ and $\frac{1}{\overline{w}}$ are bounded and $\| \ \|^1_{R_o,p}$ and $\| \ \|^2_{R_o,p}$ are L^p-norms on $\overline{H}_{R_o}$. For $H \in \overline{H}_{R_o}$ approximation errors $E^1_{n,p}(H, R_o)$ and $E^2_{n,p}(H, R_o)$ are defined as

$$E^1_{n,p}(H, R_o) = \inf_{g \in \pi_n} \|H - g\|^1_{R_o,p} \tag{1.5}$$

$$E^2_{n,p}(H, R_o) = \inf_{g \in \pi_n} \|H - g\|^2_{R_o,p}, \tag{1.6}$$

where π_n consists of all GBSP polynomials of degree at most $2n$. The concept of index q, the q-order $\rho(q)$ and lower q-order $\lambda(q)$ were introduced by Sato [8] in order to obtain a measure of growth of the maximum modulus when it is rapidly increasing. Thus, let $M(r, H) \to \infty$ as $r \to R$ and for $q = 2, 3, \ldots$, we define

$$\rho_{(q)}(H, R) = \limsup_{r \to R} \frac{\log^{[q]} M(r, H)}{\log(\frac{R}{R-r})},$$

where $\log^{[0]} M(r, H) = M(r, H)$ and

$$\log^{[q-1]} M(r, H) = \log(\log^{[q-2]} M(r, H)).$$

The GBSP $H \in H_R$ is said to have the index q if $\rho_q(H, R) < \infty$ and $\rho_{(q-1)}(H, R) = \infty$. If q is the index of H then $\rho_q(H, R)$ is called the q-order of H. The notions of the index and q-order play a significant role in classifying the rapidly increasing functions analytic in D_R. To compare the growth of two functions analytic in D_R that have same q-orders the distinct growth parameters are used.

We have the following definition.

Definition 1. A GBSP $H \in H_R$, $0 < R < \infty$ having q-order $\rho_q(H, R)(\rho_q(H, R) > 0)$ is said to have q-type $T_q(H, R)$ and lower q-type $t_q(H, R)$ if

$$\begin{matrix} T_q(H,R) \\ t_q(H,R) \end{matrix} = \lim_{r\to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[q-1]} M(r,H)}{\left(\frac{R}{R-r}\right)^{\rho_q(H,R)}},$$

$$0 \le t_q(H,R) \le T_q(H,R) \le \infty.$$

In this paper we study the growth and approximation of solutions (not necessarily entire) of certain elliptic partial differential equations. These solutions are called *generalized bi-axially symmetric potentials* (GBSP's). The GBSP's are taken to be regular in a finite hyperball and influence the growth of their maximum moduli on the rate of decay of their approximation errors in L^p-norm defined by (1.3) and (1.4). The results and methods employed are different from those of McCoy [7]. The text has been divided into three parts. Section 1 consists of introductory exposition of the topic and Section 2 includes some lemmas. Finally, we prove some theorems which characterize the q-type $T_q(H,R)$ and lower q-type $t_q(H,R)$ of a GBSP $H \in H_{R_o}$, $0 \le R_o < \infty$, in terms of ratio of approximation errors $E^i_{n,p}(H,Ro), 0 < Ro < R < \infty, i = 1,2$.

2. Preliminary Results

In this section we give some lemmas as preliminary results which have been used in the sequel.

Lemma 2.1. *Let $H \in H_R$, $R > R_o$. Then there exist GBSP polynomials $g_n \in \pi_n$ such that*

$$\|H - g_n\| \le KM(r,H)(n+1)^{\eta+1/2}(R/r)^{2(n+1)}$$

for all r sufficiently near to R and all sufficiently large values of n. Here K is a constant independent of r and n and $\eta = \max(\alpha,\beta)$.

Proof. The proof of this lemma follows from [4]. □

Lemma 2.2. *Let $H \in \overline{H}_{R_o}$, $R > R_o$. Then there exist GBSP polynomials $g_n \in \pi_n$ such that*

$$E^i_{n,p}(H,R_o) \le K_i(n+1)^{\eta+1/2}(R_o/r)^{2(n+1)}M(r,H)\,, \quad i = 1,2 \qquad (2.1)$$

for all r sufficiently near to R and all sufficiently large values of n. Here K_i is a constant depending on R_o, w, and p only and K_2 a constant depending on R_o, $\overline{w}$ and p.

Proof. Using (1.3), (1.4), (1.5), (1.6) and Lemma 2.1 we get the required result. □

Lemma 2.3. *Let $H \in \overline{H}_R$. Then for $n \geq 1$,*

$$|a_n|R_o^{2n}$$

$$\leq \frac{T^{1/p}(2\pi)^{1/v^*}(2n+\alpha+\beta+1)P(n,\alpha,\beta)\Gamma(n+\eta+1)}{\Gamma(\eta+1)\Gamma(n+1)} E^1_{n-1,p}(H,R_o),$$

where

$$P(n,\alpha,\beta,) = \Gamma(n+1)\Gamma(n+\alpha+\beta+1)/(n+\alpha+1)(n+\beta+1).$$

Proof. By (1.5), for $H \in \overline{H}_{R_o}$, there exists a GBSP polynomial $g^*_{n-1} \in \pi_{n-1}$ such that

$$2E^1_{n-1,p}(H,R_o) \geq \|H - g^*_{n-1}\|^1_{R_o,p}$$

$$\geq \frac{1}{T^{1/p}} \left(\int_0^{2\pi} |H(R_o,\theta) - g^*_{n-1}(R_o,\theta)|^p \, d\theta \right)^{1/p}, \quad (2.2)$$

since $1/w$ is bounded and we have $w \geq \frac{1}{T}$, $T > 0$. For $p > 1$ choose v^* such that $1/p + 1/v^* = 1$. Using Hölder's inequality we get

$$\int_0^{2\pi} |H(R_o,\theta) - g^*_{n-1}(R_o,\theta)| d\theta$$

$$\leq \left(\int_0^{2\pi} |H(R_o,\theta) - g^*_{n-1}(R_o,\theta)|^p d\theta \right)^{1/p} \left(\int_0^{2\pi} d\theta \right)^{1/v^*}. \quad (2.3)$$

Combining (2.2) and (2.3), we get

$$2E_{n-1,p}(H,R_o) \geq \frac{1}{2\pi T^{1/p}} \int_0^{2\pi} |H(R_o,\theta) - g^*_{n-1}(R_o,\theta)| d\theta$$

$$= \frac{1}{(2\pi)^{1/v^*} T^{1/p}} \int_0^{\pi/2} |H(R_o,\theta) - g^*_{n-1}(R_o,\theta)| d\theta$$

for $p > 1$, since GBSP's H and g^*_{n-1} are even in x and y. For $p = 1$, (2.4) is obvious with $v^* = 0$. From the orthogonality of Jacobi polynomials [9] and uniform convergence of the series (1.2) on $\overline{D}_{R_o}$, we have

$$\frac{a_n R_o^{2n}}{(2n+\alpha+\beta+1)p(n,\alpha,\beta)}$$

$$= 2\int_0^{\pi/2} H(R_o,\theta) p_n^{(\alpha,\beta)}(\cos 2\theta) \sin^{2\alpha+1}\theta \cos^{2\beta+1}\theta \; d\theta.$$

Thus, for any $g \in \pi_{n-1}$ we have

$$\frac{a_n R_o^{2n} p(n,\alpha,\beta)^{-1}}{(2n+\alpha+\beta+1)}$$

$$= 2\int_0^{\pi/2} (H(R_o,\theta) - g(R_o,\theta)) p_n^{(\alpha,\beta)}(\cos 2\theta) \sin^{2\alpha+1}\theta \cos^{2\beta+1}\theta \; d\theta. \tag{2.4}$$

From [9], we know that

$$\max_{-1\leq t\leq 1} |p_k^{(\alpha,\beta)}(t)| = \frac{\Gamma(k+\eta+1)}{\Gamma(\eta+1)\Gamma(k+1)}, \quad \eta = \max(\alpha,\beta). \tag{2.5}$$

Taking, in particular g^*_{n-1}, it follows that

$$\frac{a_n R_o^{2n}}{(2n+\alpha+\beta+1)p(n,\alpha,\beta)}$$

$$\leq \frac{2\Gamma(n+\eta+1)}{\Gamma(\eta+1)\Gamma(n+1)} \int_0^{\pi/2} |H(R_o,\theta) - g^*_{n-1}(R_o,\theta)| d\theta. \tag{2.6}$$

Combining (2.5) and (2.7), the lemma follows. □

Lemma 2.4. *Let $H \in \overline{H}_{R_o}$. Then for $n \geq 1$, we have*

$$|a_n| R_o^{2n+2} \leq \frac{\tilde{T}^{1/p}(\pi R_o^2)^{1/v^*}(2n+2)(2n+\alpha+\beta+1)P(n,\alpha,\beta)\Gamma(n+\eta+1)}{\Gamma(\eta+1)\Gamma(n+1)} \times E^2_{n-1,p}(H,R_o).$$

Proof. By (1.6), for $H \in \overline{H}_{R_o}$, there exists $\tilde{g}_{n-1} \in \pi_{n-1}$ such that

$$\begin{aligned} 2E^2_{n-1,p}(H,R_o) &\geq \|H - \tilde{g}_{n-1}\|^2_{R_o,p} \\ &\geq \frac{1}{\tilde{T}^{1/p}} \left(\int\!\!\int_{\overline{D}_{R_o}} |H(x,y) - \tilde{g}_{n-1}(x,y)|^p \, dx \, dy \right)^{1/p} \\ &\geq \frac{1}{\tilde{T}^{1/p}(\pi R_o^2)^{1/v^*}} \left(\int\!\!\int_{\overline{D}_{R_o}} |H(x,y) - \tilde{g}_{n-1}(x,y)| \, dx \, dy \right)^{1/p}, \end{aligned} \tag{2.7}$$

where $\tilde{w} = 1/\tilde{T}$, $\tilde{T} > 0$ and $(1/p) + (1/v^*) = 1$. From the orthogonality of Jacobi polynomials and uniform convergence of the series (1.2) on $\overline{D}_{R_o}$, we have for $0 \leq r \leq R$,

$$\frac{a_n r^{2n}}{(2n+\alpha+\beta+1)P(n,\alpha,\beta)} = 2\int_0^{\pi/2} (H(r,\theta) - \overline{g}_{n-1}(r,\theta)) P_n^{(\alpha,\beta)}(\cos\theta) \sin^{2\alpha+1}\theta \cos^{2\beta+1}\theta \, d\theta.$$

Using (2.6), we get

$$\frac{a_n r^{2n}}{(2n+\alpha+\beta+1)P(n,\alpha,\beta)}$$

$$\leq \frac{\Gamma(n+\eta+1)}{2\Gamma(n+1)\Gamma)\eta+1)} \int_0^{2\pi} |H(r,\theta) - \tilde{g}_{n-1}(r,\theta)| \; d\theta.$$

Since H and $\tilde{g}_{n-1}$ are even in x and y. Multiplying both sides of the above inequality by $r\,dr$ and integrating from 0 to R_o, we get

$$\frac{a_n r^{2n+2}(2n+2)^{-1}}{(2n+\alpha+\beta+1)P(n,\alpha,\beta)}$$

$$\leq \frac{\Gamma(n+\eta+1)}{2\Gamma(n+1)\Gamma(\eta+1)} \int\int_{\overline{D}_{R_o}} |H(x,y) - \tilde{g}_{n-1}(x,y)| \; dx \; dy. \quad (2.8)$$

Combining (2.8) and (2.9) we obtain the required result. □

Lemma 2.5. *Let $H \in H_R$, $0 < R < \infty (R > R_o)$. Then*

$$M(r,H) \leq |a_o| + \frac{T^{1/p}(2\pi)^{1/v^*}}{\Gamma(n+1)} M(r,h),$$

where

$$h(u) = \sum_{n=1}^{\infty} (2n+\alpha+\beta+1)P(n,\alpha,\beta)\frac{\Gamma(n+\eta+1)}{\Gamma(n+1)}$$

$$\times \, E^1_{n-1,p}(H,R_o)(\frac{u}{R_o})^{2n}.$$

Proof. Using (2.6) and Lemma 2.3 we get

$$|\sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\alpha,\beta)}(\cos 2\theta)| \leq |a_o| + \sum_{n=1}^{\infty} |a_n| r^{2n} \frac{\Gamma(n+\eta+1)}{\Gamma(n+1)\Gamma(\eta+1)}$$

$$\leq |a_o| + \frac{T^{1/p}(2\pi)^{1/v^*}}{\Gamma(n+1)} \sum_{n=0}^{\infty} E^1_{n-1,p}(H,R_o)(\frac{r}{R_o})^{2n}$$

$$\times \, (2n+\alpha+\beta+1)P(n,\alpha,\beta)\frac{\Gamma(n+\eta+1)}{\Gamma(n+1)},$$

which agrees with the desired result. □

Lemma 2.6. *Let $H \in H_R$, $0 < R < \infty$. Then*

$$M(r,H) \leq |a_o| + \frac{\tilde{T}^{1/p}(\pi)^{1/v^*}}{\Gamma(n+1)} R_o^{2(1-v^*)/v^*} M(r,h^*),$$

where

$$h^*(u) = \sum_{n=1}^{\infty}(2n+2)(2n+\alpha+\beta+1)P(n,\alpha,\beta)\frac{\Gamma(n+\eta+1)}{\Gamma(n+1)}$$

$$\times E^2_{n-1,p}(H,R_o)\left(\frac{u}{R_o}\right)^{2n}.$$

Proof. Using Lemma 2.4 the proof has the same analysis as that of Lemma 2.5. □

Lemma 2.7. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$. Then the function $f(z)$ is of q-order and q-type $T(q)$ if and only if*

$$T(q) = B(q)\ V(q),$$

where

$$V(q) = \limsup_{n\to\infty}(\log^{[q-2]} n)\left(\frac{\log^+ |a_n| R^n}{n}\right)^{\rho(q)+A(q)},$$

$B(q) = (\rho+1)^{\rho+1}/\rho^{\rho}$, $A(q) = 1$ *for* $q = 2$; $B(q) = 1$, $A(q) = 0$ *for* $q = 3, 4 \ldots.$

Proof. The lemma can be proved by simple manipulation of the results in [2] and [3]. □

Lemma 2.8. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$ and have q-order $\rho(q)$ ($\rho(q) > 0$) and q-type $T(q)$. If $\psi(n) = |a_n/a_{n+1}|$ forms a nondecreasing sequence of n for all $n > n_o$, then*

$$\limsup_{n\to\infty} \log^{[q-2]} n\,(\log|a_n/a_{n-1}|R)^{\rho(q)+A(q)} = B(q)T(q).$$

Here $B(q)$ and $A(q)$ have the same meaning as in Lemma 2.7.

Proof. Denoting $\rho(q) + A(q)$ by $\rho^*(q)$, we let

$$\limsup_{n\to\infty} \log^{[q-2]} n \left(\log\left|\frac{a_n}{a_{n-1}}\right| R\right)^{\rho^*(q)} = Q.$$

First consider $0 \le Q < \infty$. For given $\epsilon > 0$ and $n > n_1(\epsilon)$, we have

$$\log\left|\frac{a_n}{a_{n-1}}\right| R < \left(\frac{Q+\epsilon}{\log^{[q-2]} n}\right)^{1/\rho^*(q)}.$$

Writing above inequality for $n = N+1, N+2, \ldots, k$ and adding, we obtain

$$\begin{aligned} \log\left|\frac{a_k}{a_N}\right| R^{k-N} &< \sum_{n=N+1}^{k} \left((Q+\epsilon)/\log^{[q-2]} n\right)^{1/\rho^*(q)} \\ &< (k-N)\left(\frac{Q+\epsilon}{\log^{[q-2]} n}\right)^{1/\rho^*(q)}. \end{aligned}$$

Hence for all large k,

$$\log^+ |a_k|\, R^k < O(1) + \left(1 + o(1))k \log^{[q-2]} k\right)^{1/\rho^*(q)},$$

or

$$\left(\frac{\log^+ |a_k| R^k}{k}\right)^{\rho^*(q)} < \frac{Q+\epsilon}{\log^{[q-2]} k)} + o(1),$$

or

$$\limsup_{k\to\infty} \log^{[q-2]} k \left[\frac{\log |a_k| R^k}{k}\right]^{\rho^*(q)} \le Q, \tag{2.9}$$

which implies (in view of Lemma 2.7), $T(q)/B(q) \le Q$. In (2.11) equality holds. To prove this let us assume that the left hand side expression of (2.11) equal to $V(q)$ such that $V(q) < Q$. Then, for arbitrary $\epsilon > 0$ we have for all large values of $n > N(\epsilon)$,

$$\log^+ |a_n| R^n < n((V(q)+\epsilon)\log^{[q-2]} n)^{1/\rho^*(q)}. \tag{2.10}$$

Now we use the assumption that $\psi(n) = \log|a_n/a_{n-1}|$ is a nondecreasing function of n. Then for large n,

$$\begin{aligned}\log\left|\frac{a_N}{a_n}\right| &= \log\left|\frac{a_N}{a_{N+1}}\right| + \ldots + \log\left|\frac{a_{n-1}}{a_n}\right| \\ &\leq (n-N)\psi(n-1)\end{aligned}$$

or

$$\log^+|a_n|R^n > \log|a_N| + (n-N)\log^+\left|\frac{a_n}{a_{n-1}}\right| + n\log R. \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$\limsup_{n\to\infty}\log^{[q-2]} n\left[\frac{\log^+|a_n|R^n}{n}\right]^{\rho(q)+A(q)}$$

$$\geq \limsup_{n\to\infty}\log^{[q-2]} n\left(\log^+\left|\frac{a_n}{a_{n-1}}\right| R\right)^{\rho(q)+A(q)}.$$

But this contradicts our assumption, and hence the lemma holds true. □

3. Main Results

Theorem 3.1. *Let $H \in H_R$, $0 < R < \infty (R > R_o)$ and have q-order $\rho_q(H,R) > 0$ and q-type $T_q(H,R)$. Then*

$$G'(q) = B(q,H)\; T_q(H,R), \tag{3.1}$$

where, for $i = 1, 2$,

$$G'(q) = \limsup_{n\to\infty}(\log^{[q-2]} n)$$

$$\times\left(\log^+\frac{E^i_{n,p}(H,R_o)}{E^i_{n-1,p}(H,R_o)}\left(\frac{R}{R_o}\right)^2\right)^{\rho_q(H,R)+A(q,H)}. \tag{3.2}$$

Proof. By the definition of $G'(q)$, for given $\epsilon > 0$ and $n > n_o(\epsilon)$, we have

$$(\log^{[q-2]} n)\left(\log^{+}\frac{E^i_{n,p}(H,R_o)}{E^i_{n-1,p}(H,R_o)}\left(\frac{R}{R_o}\right)^2\right)^{\rho'(q)} < G'(q)+\epsilon\,,$$

or

$$\log^{+}\frac{E^i_{n,p}(H,R_o)}{E^i_{n-1,p}(H,R_o)}\left(\frac{R}{R_o}\right)^2 < \left(\frac{G'(q)+\epsilon}{\log^{[q-2]} n}\right)^{1/\rho'(q)}\,,$$

where $\rho'(q) = \rho_q(H,R) + A(q,H)$. Writing the above inequality for $n = N+1, N+2, \dots, k$ and adding

$$\log^{+}\frac{E^i_{k,p}(H,R_o)}{E^i_{N,p}(H,R_o)}\left(\frac{R}{R_o}\right)^{2(k-N)} < \sum_{n=N+1}^{k}\left(\frac{G'(q)+\epsilon}{\log^{[q-2]} n}\right)^{1/\rho'(q)}$$

$$< (k-N)\left(\frac{G'(q)+\epsilon}{\log^{[q-2]} k}\right)^{1/\rho'(q)}\,,$$

or

$$\log^{+} E^i_{n,p}(H,R_o)\left(\frac{R}{R_o}\right)^{2(k-N)}$$

$$< O(1) + (1+o(1))k\left(\frac{G'(q)+\epsilon}{\log^{[q-2]} k}\right)^{1/\rho'(q)}\,,$$

or

$$\left(\frac{\log^{+} E^i_{k,p}(H,R_o)}{k}\left(\frac{R}{R_o}\right)^{2k}\right)^{\rho'(q)} < \frac{G'+\epsilon}{\log^{[q-2]} k} + O(1)\,,$$

or

$$\limsup_{k\to\infty}\log^{[q-2]} k\left(\frac{\log^{+} E^i_{k,p}(H,R_o)}{k}\left(\frac{R}{R_o}\right)^{2k}\right)^{\rho_q(H,R)+A(q,H)}$$

$$\le G'(q).$$

Using Theorem 1 of [5], we get $T_q(H,R)\ B(q,H) \leq G'(q)$. This inequality is obviously true if $G'(q) = \infty$. To prove the reverse inequality we use Lemma 2.5 for $i = 1$ and Lemma 2.6 for $i = 2$ and apply Lemma 2.7 and Lemma 2.8 to the functions $h(u)$ and $h^*(u)$. $\square$

Theorem 3.2. *Let $H \in H_R$, $0 < R < \infty (R > R_o)$ and have q-order $\rho_q(H,R) > 0$ and lower q-type $t_q(H,R)$. Let $\{n_k\}$ be an increasing sequence of natural numbers. Then*

$$B(q,H)\ t_q(H,R) \geq \liminf_{k\to\infty} \log^{[q-2]} n_{k-1} \times \left(\frac{\log^+ \frac{E^i_{n_k,p}(H,R_o)}{E^i_{n_{k-1},p}(H,R_o)}}{n_k - n_{k-1}} \left(\frac{R}{R_o}\right)^{2(n_k-n_{k-1})} \right)^{\rho'(q)}. \tag{3.3}$$

Proof. Let

$$\liminf_{k\to\infty} \log^{[q-2]} n_{k-1} \times \left(\frac{\log^+ \frac{E^i_{n_k,p}(H,R_o)}{E^i_{n_{k-1},p}(H,R_o)}}{n_k - n_{k-1}} \left(\frac{R}{R_o}\right)^{2(n_k-n_{k-1})} \right)^{\rho'(q)} = C.$$

The inequality (3.3) obviously holds if $C = 0$. Hence we assume that $0 < C < \infty$. Then, for given $\epsilon > 0$ and sufficiently large values of k, we have

$$\frac{\log^+ \frac{E^i_{n_k,p}(H,R_o)}{E^i_{n_{k-1},p}(H,R_o)}}{n_k - n_{k-1}} \left(\frac{R}{R_o}\right)^{2(n_k-n_{k-1})} > \left(\frac{C-\epsilon}{\log^{[q-2]} n_{k-1}}\right)^{1/\rho'(q)},$$

or

$$\log^+ \frac{E^i_{n_k,p}(H,R_o)}{E^i_{n_{k-1},p}(H,R_o)} \left(\frac{R}{R_o}\right)^{2(n_k-n_{k-1})} > (n_k - n_{k-1}) \left(\frac{C-\epsilon}{\log^{[q-2]} n_{k-1}}\right)^{1/\rho'(q)}.$$

Writing the above inequality for $k = N+1, N+2, \ldots, j$ and adding, we get

$$\log^{+} \frac{E^{i}_{n_j,p}(H,R_o)}{E^{i}_{n_N,p}(H,R_o)} \left(\frac{R}{R_o}\right)^{2(n_j-n_N)}$$

$$> \sum_{k=N+1}^{j} (n_k - n_{k-1}) \left(\frac{C-\epsilon}{\log^{[q-2]} n_{k-1}}\right)^{1/\rho'(q)} . \quad (3.4)$$

To estimate the right hand side we put $f(t) = ((C-\epsilon)/\log^{[q-2]} t)^{1/\rho'(q)}$ and $n(t) = n_j$ for $n_{j-1} < t \leq n_j$. Hence right hand side of (3.4) can be written as

$$\begin{aligned}
& \sum_{k=N+1}^{j} F(n_{k-1})(n_k - n_{k-1}) \\
= \ & (n_{N+1} - n_N)F(n_N) + (n_N - n_{N-1})F(n_{N-1}) \\
& + \cdots + (n_j - n_{j-1})F(n_{j-1}) \\
= \ & n_j F(n_{j-1}) - n_{j-1}\{F(n_{j-1}) - F(n_{j-2})\} \cdots \\
& - n_{N+1}\{F(n_{N+1}) - F(n_N)\} - n_N F(n_N) \\
= \ & n_j F(n_{j-1}) - \sum_{k=N+1}^{j} n_{k-1}\{F(n_{k-1}) - F(n_{k-2})\} - n_N F(n_N) \\
= \ & n_j F(n_{j-1}) - \int_{n_N}^{n_{J-1}} n(t)\ dF(t)\ -\ n_N F(n_N) \\
= \ & n_j F(n_{j-1}) + \frac{1}{\rho'(q)} \int_{n_N}^{n_{j-1}} \frac{n(t)}{t \prod_{m=1}^{q-2} \log^{[m]} t}\, dF(t) + O(1).
\end{aligned}$$

Since $n(t)/t \geq 1$, on substituting the above expression in (3.4),

$$\log^{+} E_{n_j}(H,R_o)(\frac{R}{R_o})^{2(n_j-n_N)}$$

$$
\begin{aligned}
&> \quad n_j F(n_{j-1}) + \frac{(n_{j-1} - n_N)F(n_{j-1})}{\rho'(q) \prod_{m=1}^{q-2} \log^{[m]} n_{j-1}} + O(1) \\
&= \quad F(n_{j-1})n_j \left[1 + \frac{n_{j-1}(1+o(1))}{\rho'(q) n_j \prod_{m=1}^{q-2} \log^{[m]} n_{j-1}} \right] + O(1) \\
&= \quad \left(\frac{C-\epsilon}{\log^{[q-2]} n_{j-1}} \right)^{1/\rho'(q)} n_j(1+o(1)) + O(1),
\end{aligned}
$$

for all large j. Since $\frac{n_{j-1}}{n_j} < 1$, it follows that

$$
\frac{\log^+ E_{n_j,p}(H,R_o)}{n_j} \left(\frac{R}{R_o} \right)^{2n_j} > \left(\frac{C-\epsilon}{\log^{[q-2]} n_{j-1}} \right)^{\frac{1}{\rho^*(q)}} n_j \; + \; o(1).
$$

Proceeding to limits, we get

$$
\liminf_{j\to\infty} \log^{[q-2]} n_{j-1} \left[\frac{\log^+ E_{n_j,p}(H,R_o)}{n_j} \left(\frac{R}{R_o} \right)^{2n_j} \right]^{\rho^*(q)} \geq C.
$$

Combining Theorem 2 [5] and above inequality the result (3.3) is obtained. □

References

[1] R.P. Gilbert, Integral operator methods in biaxially symmetric potential theory, *Contrib. Diff, Eqns.*, **2** (1963), 441-456.

[2] K. Gopal, G.P. Kapoor, Coefficient characterization for functions analytic in the unit disc having fast rates of growth, *Bull. Math. Soc. Sci. Math. R.S. Roumaine*, N.S., **25**, No. 1 (1981), 367-380.

[3] G.P. Kapoor, K. Gopal, On the coefficients of functions analytic in unit disc having fast rates of growth, *Annali di Mat. Pura ed Applicata*, **61** (1979), 337-349.

[4] G.P. Kapoor, A. Nautiyal, Growth and approximation of generalized bi-axially symmetric potentials, *Indian J. Pure Appl. Math.*, **19** (1980), 464-476.

[5] H.S. Kasana, D. Kumar, Approximation of generalized bi-axially symmetric potentials with fast rates of growth, *Acta Mathematica Scientia*, Wuhan-China, **15**, No. 4 (1995), 458-467.

[6] P.A. McCoy, Best L^p approximation of generalized bi-axially symmetric potentials, *Proc. Amer. Math. Soc.*, **79** (1980), 435-440.

[7] P.A. McCoy, Approximation of pseudo analytic functions, Complex Variables, *Theory and Application*, **6** (1986), 123-133.

[8] D. Sato, On the rate of growth of entire functions of fast growth, *Bull. Amer. Math. Soc.*, **69** (1963), 411-414.

[9] G. Szegö, *Orthogonal Polynomials, Colloquium Publications*, Amer. Math. Soc. Providence, **23**, R.I. (1967).