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OSCILLATORY BEHAVIOUR OF THE SOLUTIONS OF NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT DEPENDING ON THE UNKNOWN FUNCTION

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Abstract: Sufficient conditions are found for oscillation of the solutions of nonlinear second order differential equations with retarded argument depending on the independent variable as well as on the unknown function.

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1. Introduction

A great number of papers is dedicated to the study of the oscillatory properties of the solutions of functional differential equations with retarded argument. The case when the retarded argument depends on the unknown function is of special interest. Some initial oscillation results for that type of differential equations are obtained in Angelova and Bainov [1], Bainov and Simeonov [2], Bainov et al [3] and Markova and

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Simeonov [5]–[7].

In the present paper sufficient conditions are found for oscillation of the solutions of second order differential equations of the type

$$(r(t)x'(t))' + q(t)f(x(\Delta(t, x(t)))) = 0 \tag{1}$$

in the case when $r \in C(\mathbb{R}_+, (0, +\infty))$ and $\displaystyle\int^\infty \frac{dt}{r(t)} = +\infty$.

2. Preliminary Notes

Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{N} = \{1, 2, 3, \dots\}$.

Introduce the following conditions:

H1. $r \in C(\mathbb{R}_+, (0, +\infty))$ and $\displaystyle\int^\infty \frac{ds}{r(s)} = +\infty$.

H2. $q \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\sup\{q(s) : s \geq t\} > 0$ for $t \in \mathbb{R}_+$.

H3. $\Delta \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$.

H4. There exist $\Delta_* \in C(\mathbb{R}_+, \mathbb{R})$ and $T \in \mathbb{R}_+$ such that

$$\lim_{t \to +\infty} \Delta_*(t) = +\infty \quad \text{and} \quad \Delta_*(t) \leq \Delta(t, x) \leq t \text{ for } t \geq T, \quad x \in \mathbb{R}.$$

H5. $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ for $x \neq 0$ and $f(x)$ is nondecreasing in $\mathbb{R}$.

H6. There exists $K > 0$ such that

$$f(xy) \geq Kf(x)f(y) \quad \text{and} \quad f(-xy) \leq Kf(-x)f(y)$$

for $x > 0$, $y > 0$.

For any $T_0 \in \mathbb{R}_+$ we define $T_{-1} = \inf\{\Delta(t, x) : t \geq T_0, x \in \mathbb{R}\}$.

Definition 1. A function $x(t)$ is called a **solution** of equation (1) in the interval $[T_0, +\infty)$ if $x \in C([T_{-1}, +\infty), \mathbb{R})$, $rx' \in C^1([T_0, +\infty), \mathbb{R})$ and $x(t)$ satisfies (1) for $t \geq T_0$.

Definition 2. The solution $x(t)$ of equation (1) is said to be:

2.1. **regular** if it is defined in some interval $[T_x, +\infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq T_x$;

2.2. **finally positive** if there exists $T \geq 0$ such that $x(t)$ is defined for $t \geq T$ and $x(t) > 0$ for $t \geq T$;

2.3. **finally negative** if there exists $T \geq 0$ such that $x(t)$ is defined for $t \geq T$ and $x(t) < 0$ for $t \geq T$;

2.4. **oscillatory** if it is regular and neither finally positive nor finally negative;

2.5. **nonoscillatory** if it is either finally positive or finally negative.

3. Main Results

In order to prove our main results we need the following two lemmas.

Lemma 1. *Let condition* **H1** *hold,*

$$x \in C([T_0, +\infty), \mathbb{R}), \quad rx' \in C^1([T_0, +\infty), \mathbb{R})$$

and $\sup\{|(r(s)x'(s))'| : s \geq t\} > 0$ *for* $t \geq T_0$.
Then:

1. If $x(t) > 0$ *and* $(r(t)x'(t))' \leq 0$ *for* $t \geq T_0$, *then* $r(t)x'(t)$ *is nonincreasing and* $x'(t) > 0$ *for* $t \geq T_0$.

2. If $x(t) < 0$ *and* $(r(t)x'(t))' \geq 0$ *for* $t \geq T_0$, *then* $r(t)x'(t)$ *is nondecreasing and* $x'(t) < 0$ *for* $t \geq T_0$.

Proof. 1. Obviously $r(t)x'(t)$ is nonincreasing for $t \geq T_0$. The case $r(t)x'(t) \leq 0$ for some $t \geq T_0$ is impossible. Otherwise, there exist $K > 0$ and $t_1 \geq T_0$ such that $r(t)x'(t) \leq -K$ for $t \geq t_1$ and we obtain a contradiction:

$$0 < x(t) \leq x(t_1) - \int_{t_1}^{t} \frac{K ds}{r(s)} \to -\infty \quad \text{as} \quad t \to +\infty.$$

2. The proof of Assertion 2 is analogous. □

Introduce the function $R(t) = \displaystyle\int_0^t \frac{ds}{r(s)}$, $t \in \mathbb{R}_+$.

Lemma 2. *Let condition* **H1** *hold,*

$$x \in C([T_0, +\infty), \mathbb{R}), \quad rx' \in C^1([T_0, +\infty), \mathbb{R})$$

and $\sup\{|(r(s)x'(s))'| : s \geq t\} > 0$ *for* $t \geq T_0$.

Then:

1. If $x(t) > 0$, $x'(t) > 0$, $(r(t)x'(t))' \leq 0$, $t \geq T_0$, *then for each* $\lambda \in (0, 1)$ *there exists* $T_\lambda \geq T_0$ *such that*

$$x(t) \geq \lambda R(t) r(t) x'(t), \quad t \geq T_\lambda. \tag{2}$$

2. If $x(t) < 0$, $x'(t) < 0$, $(r(t)x'(t))' \geq 0$, $t \geq T_0$, *then for each* $\lambda \in (0, 1)$ *there exists* $T_\lambda \geq T_0$ *such that*

$$x(t) \leq \lambda R(t) r(t) x'(t), \quad t \geq T_\lambda. \tag{3}$$

Proof. 1. By the Cauchy Mean Value Theorem there exists $\xi \in (T_0, t)$ such that

$$x(t) - x(T_0) = \frac{x'(\xi)}{R'(\xi)}(R(t) - R(T_0)). \tag{4}$$

Since $r(t)x'(t)$ is nonincreasing for $t \geq T_0$, then

$$\frac{x'(\xi)}{R'(\xi)} = r(\xi)x'(\xi) \geq r(t)x'(t). \tag{5}$$

Let $\lambda \in (0, 1)$. Since $\lim\limits_{t \to +\infty} R(t) = +\infty$, then there exists $T_\lambda \geq T_0$ such that

$$R(t) - R(T_0) \geq \lambda R(t), \quad t \geq T_\lambda. \tag{6}$$

Hence, using (4)–(6) we obtain (2).

2. The proof of inequality (3) is analogous. □

Theorem 1. *Assume that conditions* **H1**–**H6** *hold,*

$$\int_{+0}^{a} \frac{du}{f(u)} < +\infty, \quad \int_{-0}^{-a} \frac{du}{f(u)} < +\infty, \quad \text{for} \quad a > 0 \tag{7}$$

and

$$\int^{\infty} q(t) f(R(\Delta_*(t)))dt = +\infty. \tag{8}$$

Then every regular solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Assume that $x(t) > 0$ for $t \geq T_1 \geq 0$. From condition **H4** it follows that there exists $T_2 \geq T$ such that $x(\Delta(t, x(t))) > 0$ for $t \geq T_2$. From conditions **H2**, **H5** and equation (1) we have

$$(r(t)x'(t))' \leq 0 \quad \text{and} \quad \sup\{|(r(s)x'(s))'| : s \geq t\} > 0 \quad \text{for} \quad t \geq T_2.$$

By Lemma 1 $r(t)x'(t)$ is nonincreasing and $x'(t) > 0$ for $t \geq T_2$. Then $x(\Delta(t, x(t))) \geq x(\Delta_*(t))$ for $t \geq T_3 \geq T_2$. Let $\lambda \in (0, 1)$ be given. By Lemma 2 there exists $T_\lambda \geq T_3$ such that

$$x(t) \geq \lambda R(t) r(t) x'(t), \quad t \geq T_\lambda.$$

Hence

$$x(\Delta_*(t)) \geq \lambda R(\Delta_*(t)) r(\Delta_*(t)) x'(\Delta_*(t)), \quad t \geq T_\lambda. \tag{9}$$

Set $r(t)x'(t) = u(t)$. From (1), (9) and conditions **H5** and **H6** we obtain

$$u'(t) + K^2 f(\lambda) q(t) f(R(\Delta_*(t))) f(u(\Delta_*(t))) \leq 0, \quad t \geq T_\lambda. \tag{10}$$

Since f is nondecreasing in $\mathbb{R}$ and u is nonincreasing for $t \geq T_\lambda$, (10) implies that

$$u'(t) + K^2 f(\lambda) q(t) f(R(\Delta_*(t))) f(u(t)) \leq 0, \quad t \geq T_\lambda. \tag{11}$$

Integrating (11) from T_λ to t we get the inequality

$$K^2 f(\lambda) \int_{T_\lambda}^{t} q(s) f(R(\Delta_*(s)))ds \leq \int_{0}^{r(T_\lambda)x'(T_\lambda)} \frac{du}{f(u)} < +\infty$$

which contradicts (8).

The proof in the case $x(t) < 0$, $t \geq T_1$ is analogous. □

Corollary 1. *Assume that conditions* **H1**–**H5** *hold,*

$$\int^{\infty} q(t)R^{\alpha}(\Delta_*(t))dt = +\infty \tag{12}$$

and

$$\frac{f(u)}{u^{\alpha}} \geq K > 0 \quad \text{for} \quad u \neq 0\,, \tag{13}$$

where α is the ratio of two positive odd integers with $0 < \alpha < 1$.

Then every regular solution of equation (1) is oscillatory.

Proof. The proof is analogous to that of Theorem 1. □

Now consider the differential equation

$$u'(t) + K^2 f(\lambda)q(t)f(R(\Delta_*(t)))f(u(\Delta_*(t))) = 0, \tag{14}$$

where $\lambda \in (0,1)$ and $K > 0$ is as in condition **H6**.

Theorem 2. *Assume that:*

1. Conditions **H1**–**H6** *hold.*

2. There exists $\lambda \in (0,1)$ such that either every bounded regular solution of equation (14) is oscillatory or every nonoscillatory solution $u(t)$ of equation (14) satisfies $\lim\limits_{t\to+\infty} u(t) \neq 0$.

Then every regular solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Assume that $x(t) > 0$ for $t \geq T_1 \geq 0$. As in the proof of Theorem 1 we obtain

$$x(\Delta(t,x(t))) \geq x(\Delta_*(t)) \geq \lambda R(\Delta_*(t))r(\Delta_*(t))x'(\Delta_*(t))$$

for all $t \geq T_\lambda$ and T_λ sufficiently large.

Set $r(t)x'(t) = u(t)$ for $t \geq T_\lambda$. Then

$$\begin{aligned} -u'(t) &= q(t)f(x(\Delta(t,x(t)))) \\ &\geq K^2 f(\lambda)q(t)f(R(\Delta_*(t)))f(u(\Delta_*(t))), \quad t \geq T_\lambda. \end{aligned}$$

Thus, by integration and using the fact that $u(t)$ is a positive nonincreasing function, we have

$$u(t) \geq \int_t^\infty K^2 f(\lambda) q(s) f(R(\Delta_*(s))) f(u(\Delta_*(s))) ds, \quad t \geq T_\lambda.$$

It is easy to check that the hypotheses of [8], Theorem 1 are satisfied. Applying this theorem we conclude that equation (14) has a positive solution $v(t)$ with $\lim\limits_{t\to+\infty} v(t) = 0$ which is a contradiction.

In the case $x(t) < 0$ for $t \geq T_1$ we reach a contradiction by similar arguments. □

Theorem 3. *Assume that:*

1. Conditions **H1**–**H6** *hold and*

$$\frac{f(x)}{x} \geq \gamma > 0 \quad \text{for} \quad x \neq 0. \tag{15}$$

2. There exists $\lambda \in (0, 1)$ *such that either every bounded solution of the equation*

$$u'(t) + \gamma K^2 f(\lambda) q(t) f(R(\Delta_*(t))) u(\Delta_*(t)) = 0 \tag{16}$$

is oscillatory or every nonoscillatory solution $u(t)$ *of equation (16) satisfies* $\lim\limits_{t\to+\infty} u(t) \neq 0$.

Then every regular solution of equation (1) is oscillatory.

Proof. The proof is similar to that of Theorem 2. □

From Theorem 3 we deduce the following corollary.

Corollary 2. *Assume that:*

1. Conditions **H1**–**H6** *and (15) hold.*

2. One of the following inequalities is fulfilled:

$$\limsup_{t\to+\infty} \int_{\Delta_*(t)}^t q(s) f(R(\Delta_*(s))) ds > \frac{1}{\gamma K^2 f(1)}, \tag{17}$$

or

$$\liminf_{t\to+\infty}\int_{\Delta_*(t)}^{t} q(s)f(R(\Delta_*(s)))ds > \frac{e^{-1}}{\gamma K^2 f(1)}. \tag{18}$$

Then every regular solution of equation (1) is oscillatory.

Proof. Let (17) hold. Then

$$\limsup_{t\to+\infty}\int_{\Delta_*(t)}^{t} \gamma K^2 f(\lambda) q(s)f(R(\Delta_*(s)))ds > 1 \tag{19}$$

for some $\lambda \in (0,1)$ sufficiently close to 1.

From (19) and [4], Theorem 2.1.3 we conclude that all regular solutions of equation (16) are oscillatory. Thus by Theorem 3 all regular solutions of equation (1) are oscillatory.

If (18) holds, then the proof is similar and is based on [4], Theorem 2.1.1. □

References

[1] D.Ts. Angelova, D.D. Bainov, Oscillatory and asymptotic behaviour of the solutions of first-order functional differential equations, *IMA Journal of Applied Mathematics*, **39** (1987), 75-89.

[2] D. Bainov, P. Simeonov, Positive solutions of a superlinear first-order differential equation with delay depending on the unknown function, *Journal of Computational and Applied Mathematics*, **88** (1998), 95-101.

[3] D.D. Bainov, N.T. Markova, P.S. Simeonov, Asymptotic behaviour of the nonoscillatory solutions of differential equations of second order with delay depending on the unknown function, *Journal of Computational and Applied Mathematics*, **91** (1998), 87-96.

[4] G.S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Pure and Applied Mathematics, Volume **110**, Marcel Dekker (1987).

[5] N.T. Markova, P.S. Simeonov, Asymptotic and oscillatory properties of the solutions of differential equations with delay depending on the unknown function, In: *Invited Lectures delivered at the VII-th Int. Colloquium on Differential Equations, August 18-23, 1996, Plovdiv, Bulgaria*, Volume **II**, 71-78.

[6] N.T. Markova, P.S. Simeonov, Oscillatory and asymptotic behaviour of the solutions of first order differential equations with delays depending on the unknown function, In: *Invited Lectures delivered at the VII-th Int. Colloquium on Differential Equations, August 18-23, 1996, Plovdiv, Bulgaria*, Volume **II**, 79-91.

[7] N.T. Markova, P.S. Simeonov, On the asymptotic behaviour of the solutions of a class of differential equations of second order with delay depending on the unknown function, In: *Invited Lectures delivered at the VII-th Int. Colloquium on Differential Equations, August 18-23, 1996, Plovdiv, Bulgaria*, Volume **I**, 89-100.

[8] Ch.G. Philos, On the existence of nonoscillatory solutions tending zero at ∞ for diffrential equations with positive delays, *Arch. Math.*, **36** (1981), 168-178.